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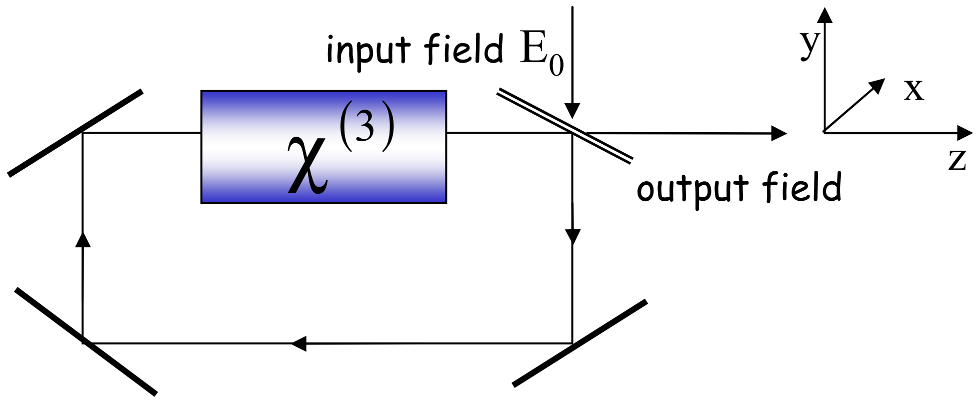
HEXAGONAL PATTERN CORRELATIONS IN A KERR MEDIUM

Damià Gomila and Pere Colet

The authors acknowledge to the European Comission through QSTRUCT TMR Network project

Self-focusing Kerr medium in a ring cavity



$$\vec{E}(\vec{x},z,t) = \vec{E}(\vec{x},t)\, e^{i(k_0 z - w_0 t)}$$

$$\vec{x} = (x, y)$$

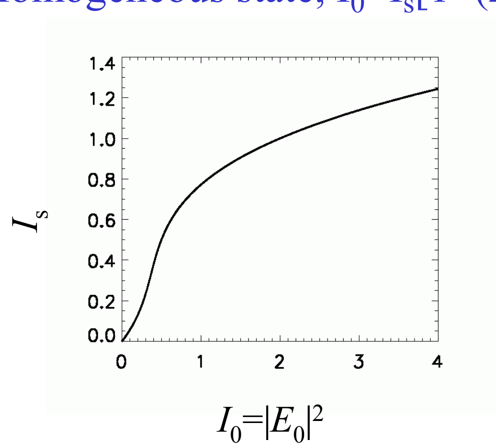
field envelope

Time evolution of field envelope

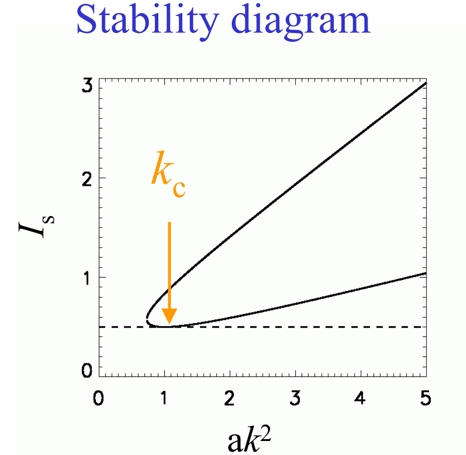
$$\frac{\partial E}{\partial t} = -(1 + i\theta)E + ia\nabla^2 E + E_0 + i2|E|^2 E$$

θ : cavity detuning, E_0 : input field, ∇^2 : transverse Laplacian, a : strength of diffraction

Homogeneous state, $I_0 = I_s[1+(2I_s - \theta)^2]$

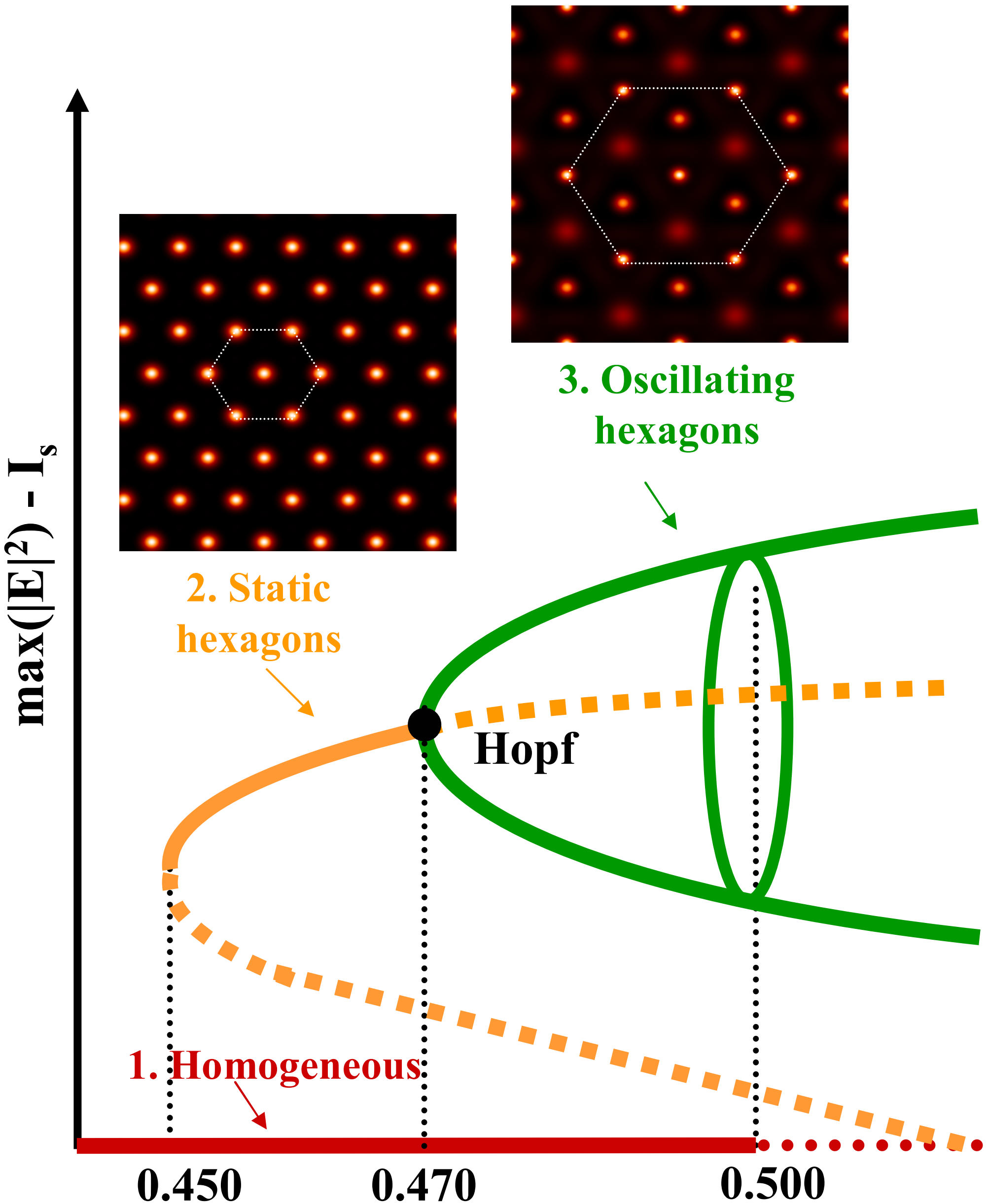


Stability diagram

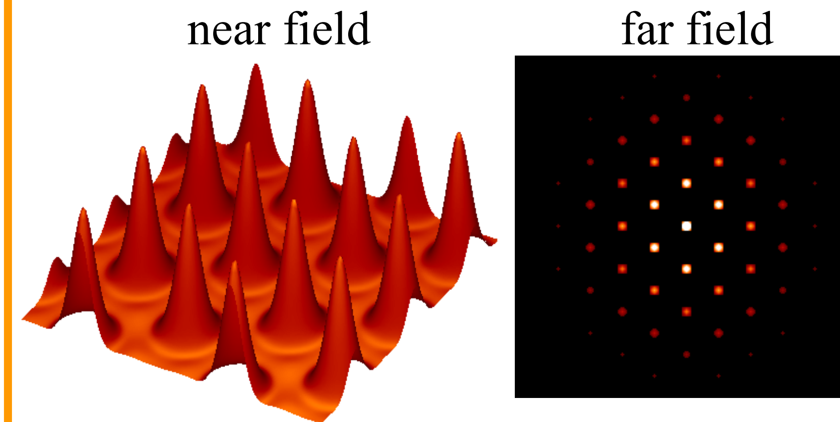


L. A. Lugiato & R. Lefever, Phys. Rev. Lett. **58**, 2209, (1987).

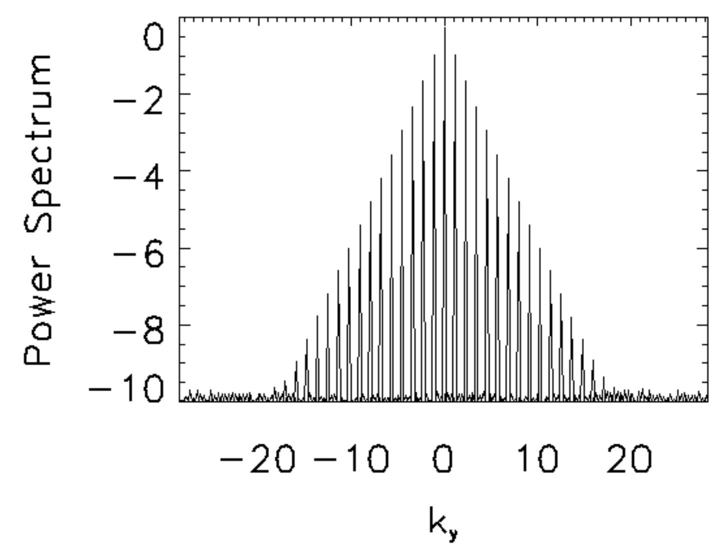
Spatio-temporal regimes



Static hexagons



Hexagonal pattern peaks are **highly focused** $\Rightarrow$ Far-field has not only six mode but **many higher harmonics**.



Fluctuations and Correlations

In order to model fluctuations we add a white Gaussian noise ξ :

$$<\xi(\mathbf{x},t)\, \xi^*(\mathbf{x}',t')>=2\epsilon\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t')$$

$$\frac{\partial E}{\partial t} = -(1+i\theta)E + ia\nabla^2 E + E_0 + i2|E|^2 E + \xi(\vec{x},t)$$

We study correlations of the far field intensity and field fluctuations:

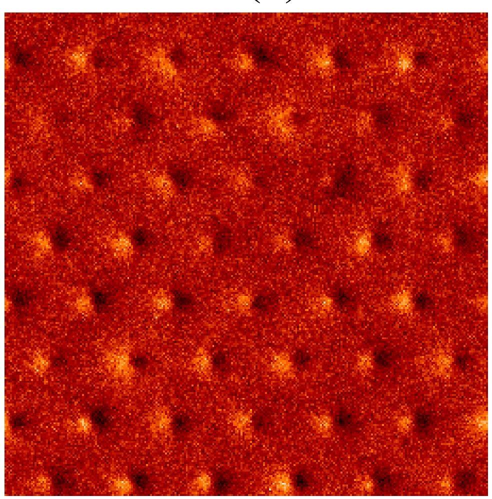
$$\mathbf{C_1(k,k') = <\delta I(k)\, \delta I(k')>}$$

$$\mathbf{C_2(k,k') = <\delta E(k)\, \delta E^*(k')>}$$

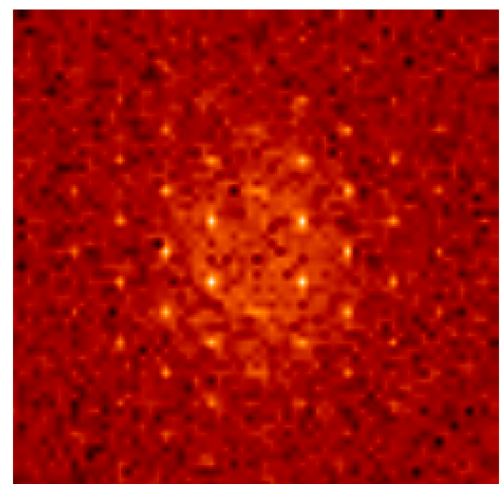
where $\delta I(\mathbf{k}) = I(\mathbf{k}) - <I(\mathbf{k})>$, $\delta E(\mathbf{k}) = E(\mathbf{k}) - <E(\mathbf{k})>$, and $I(\mathbf{k})=|E(\mathbf{k})|^2$. Angular brackets stand for average over noise realizations.

Pattern fluctuations

$\delta E(\mathbf{x})$

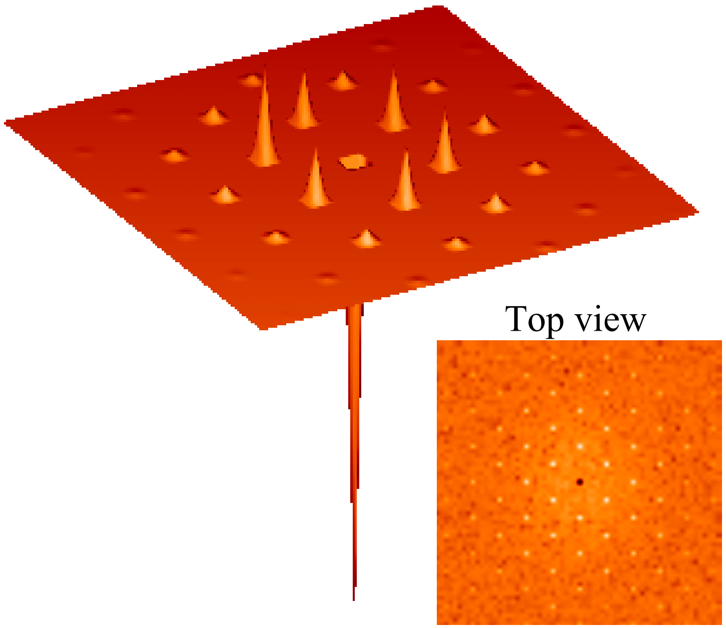


$\delta E(\mathbf{k})$

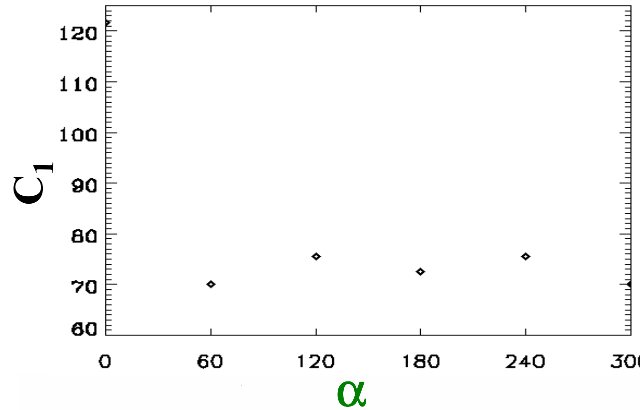


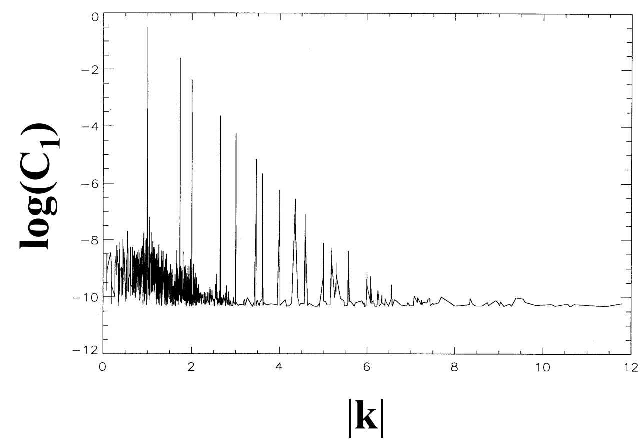
Correlations of the intensity fluctuations

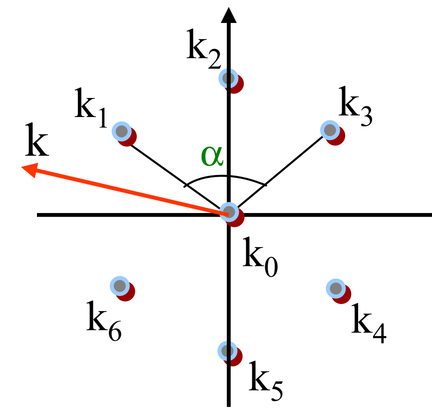
$C_1(\mathbf{k},\mathbf{k}_1)$



Intensity correlations versus α angle







* Strong correlations between the intensity fluctuations of the fundamental pattern modes (momentum conservation).

* Strong anticorrelation between the intensity fluctuations of the homogeneous mode and the pattern modes (energy conservation).

* Maximum correlation between intensity fluctuations of fundamental harmonics separated by 120^o :

$$C_1(\alpha=120^o) > C_1(\alpha=180^o) > C_1(\alpha=60^o)$$

* There are significative correlations between the intensity fluctuations of one the fundamental and higher order modes: $C_1(k,k_1)$ decays exponentially with $|k|$

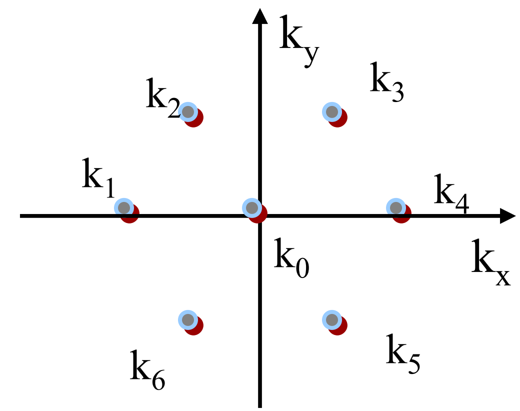
Quantum Correlations

Four-wave mixing interaction

$$H_{int} = \frac{hg_0}{2} \iint dx\, dy\, [A^+(x,y)]^2 [A(x,y)]^2$$

Homogeneous + six mode hexagonal pattern

$$A(x,y) = \frac{1}{b} \sum_n a_n \exp[i\vec{k}_n \vec{x}], \quad n = 0,\ldots,6$$

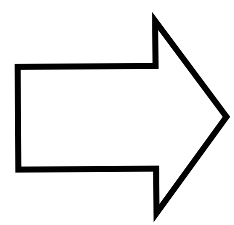


$$[H_{int}, N_i + N_{i+1} - N_{i+3} - N_{i+4}] = 0$$

$$P_y = \frac{1}{2}\hbar k_t [N_2 + N_3 - N_5 - N_6] = 0$$

$$P_x = \frac{1}{2}\hbar k_t [N_4 - N_5 + \frac{1}{2}(N_3 + N_5 - N_2 - N_6)] = 0$$

Momentum conservation



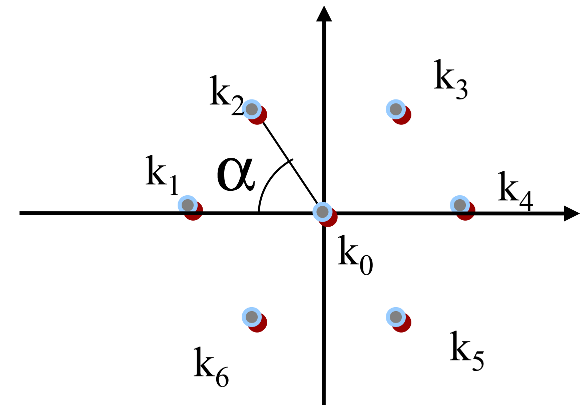
CORRELATIONS

* G. Grynberg and L.A. Lugiato, Opt. Comm. 101, 69-73 (1993)

Structure of correlations

Microscopic mechanism for correlations:

correlation inequalities can be deduced from momentum conservation relations



$$P_y = \frac{1}{2}\hbar k_t[N_2 + N_3 - N_5 - N_6] = 0$$

$$P_x = \frac{1}{2}\hbar k_t[N_4 - N_5 + \frac{1}{2}(N_3 + N_5 - N_2 - N_6)] = 0$$

$$<(\delta I(k_2)+\delta I(k_3)-\delta I(k_5))\delta I(k_6)> \; > \; <(\delta I(k_2)+\delta I(k_3)-\delta I(k_5))\delta I(k_4)>$$

$$<(\delta I(k_2)+\delta I(k_3)-\delta I(k_5))\delta I(k_6)> \; > \; <(\delta I(k_2)+\delta I(k_3)-\delta I(k_5))\delta I(k_1)>$$

Symmetry: correlations only depend on α

$$<\delta I(k_3)\,\delta I(k_4)> = <\delta I(k_5)\,\delta I(k_4)>, \quad <\delta I(k_2)\,\delta I(k_6)> = <\delta I(k_2)\,\delta I(k_4)>$$

$$<\delta I(k_3)\,\delta I(k_1)> = <\delta I(k_5)\,\delta I(k_1)>, \quad <\delta I(k_5)\,\delta I(k_6)> = <\delta I(k_2)\,\delta I(k_1)>$$

$$\mathbf{C_1(\alpha=180^o) > C_1(\alpha=60^o)}$$

$$\mathbf{C_1(\alpha=120^o) + C_1(\alpha=180^o) > 2C_1(\alpha=60^o)}$$

Pattern Stability Analysis

The hexagonal pattern can be expanded in Fourier modes: $E_h(\vec{x}) = \sum_{n=1}^{N} a_n e^{i\vec{k}_n \cdot (\vec{x}-\vec{x}_0)}$ (N=169)

Taking $E = E_h + \delta E$, and linearizing:

$$\partial_t \delta E(\vec{x}) = -(1+i\theta)\,\delta E(\vec{x}) + ia\nabla^2 \delta E(\vec{x}) + i2(2\,|E_h(\vec{x})|^2\,\delta E(\vec{x}) + E_h^2(\vec{x})\delta E^*(\vec{x}))$$

This eigenvalue problem is a linear differential equation with periodic coefficients, so a general bounded solution can be found under a Floquet form *:

$$\delta E(\vec{x},\vec{q}) = M_+(\vec{x})e^{i\vec{q}\vec{x}} + M_-(\vec{x})e^{-i\vec{q}\vec{x}} \text{, where } M_\pm(\vec{x}) = M_\pm(\vec{x} + \frac{\vec{k}_n}{2\pi}\lambda_0)$$

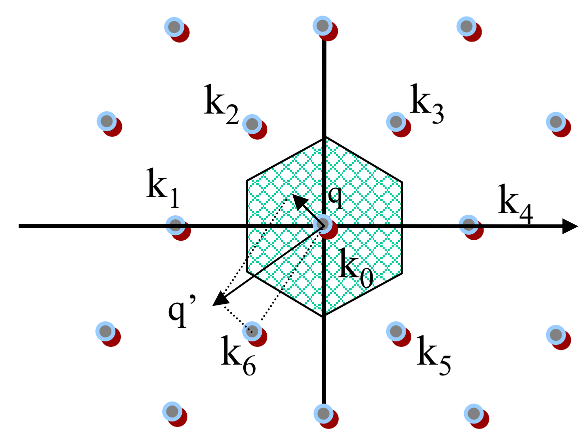
$$\delta E(\vec{x},\vec{q}) = \sum_{n=1}^{N} (\delta a_{n,\vec{q}} e^{i(\vec{k}_n+\vec{q})\vec{x}} + \delta a_{n,-\vec{q}} e^{i(\vec{k}_n-\vec{q})\vec{x}})$$

$$\partial_t \delta a_{l,\vec{q}} = [-(1+i\theta) - ia\,|\vec{k}_l + \vec{q}\,|^2]\,\delta a_{l,\vec{q}} + i2\sum_{n,m}\{2a_n a_m^* \delta a_{l-n+m,\vec{q}} + a_n a_m [\delta a_{-l+n+m,-\vec{q}}]^*\}$$

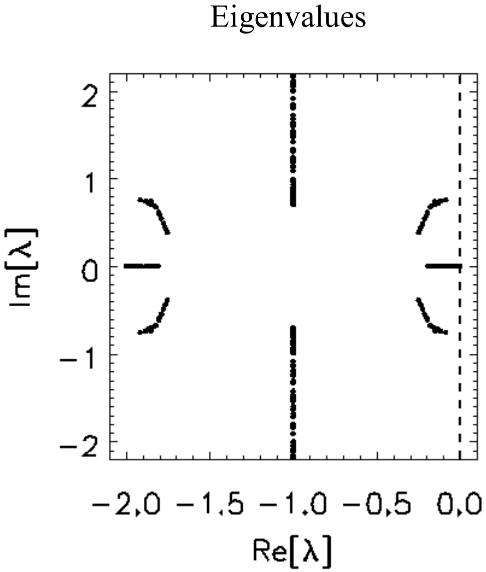
For each q we have a set of N coupled linear equations

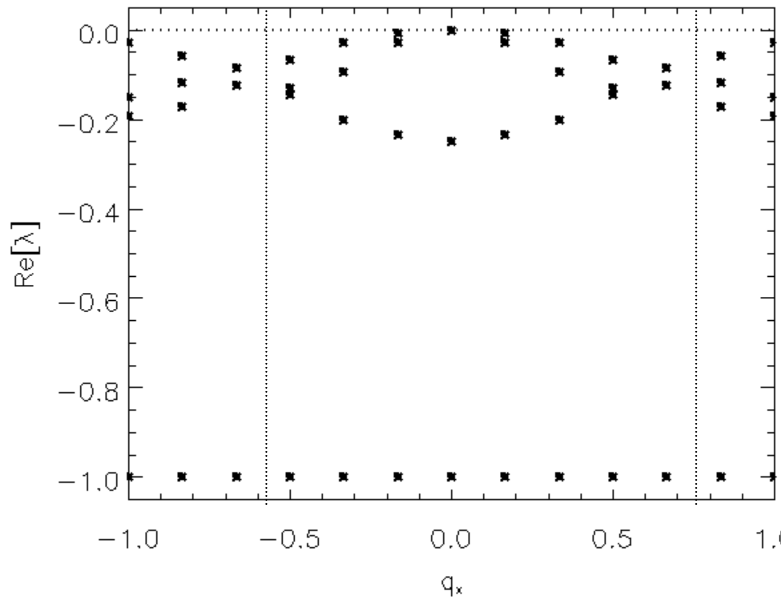
$\Rightarrow$ N eigenvalues σ_i^q and N eigenmodes v_i^q.

Moving q over half 1st Brillouin zone one considers all possible perturbations. Any q'outside is equivalent to a q in the 1st Brillouin zone by translation with a pattern wavevector.

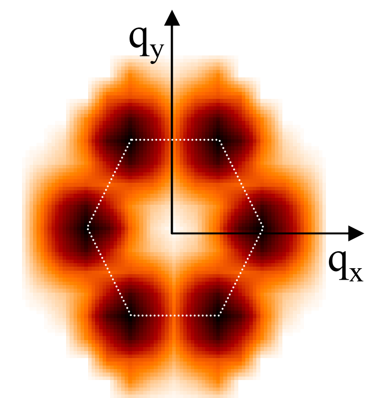


* P. Coullet and G. Iooss, PRL, **64**, 866 (1990)





Top view of Re[λ] as a function of q

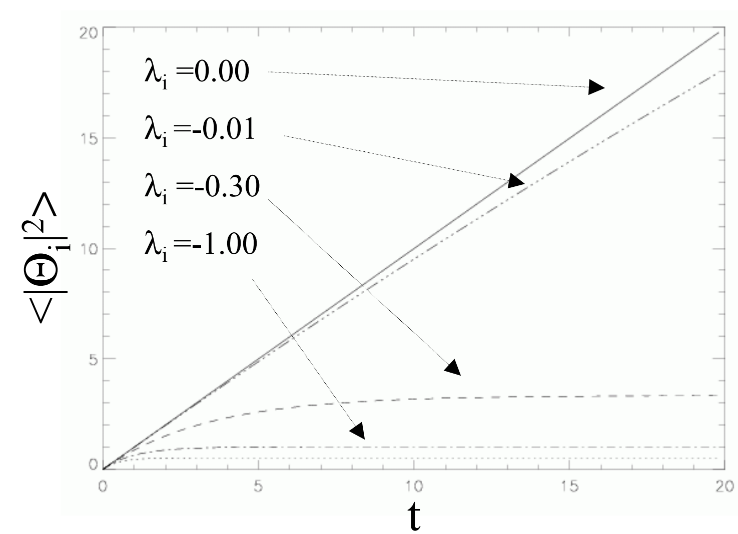


The amplitude Θ_i of each eigenmode v_i associated to the eigenvalue λ_i follows an Ornstein-Uhlenbeck process:
$\partial_i \Theta_i = \lambda_i \Theta_i + \eta_i$, where η_i is the noise in the diagonal basis.

$$\left\langle \eta_i(t)\eta_j(t')\right\rangle = \frac{\varepsilon}{2}\delta(t-t')\sum_{k=1}^{N} C_{ik}^{-1} C_{jk}^{-1*}$$

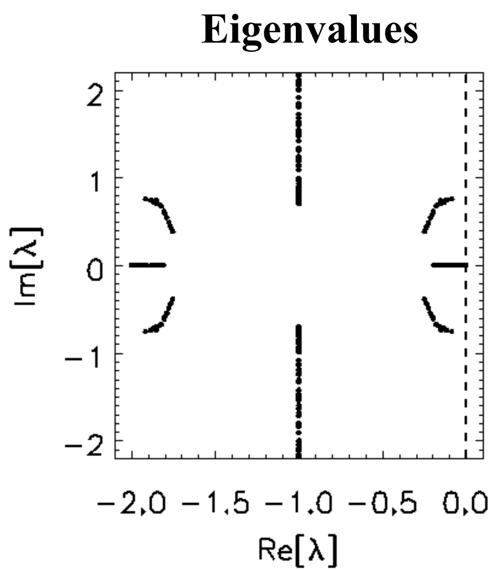
$$\delta E(\vec{x},\vec{q}) = \sum_{i=1}^{N} \Theta_i \vec{v}_i \quad , \quad \delta E(\vec{x}) = \iint \delta E(\vec{x},\vec{q})\, \mathrm{d}\vec{q}$$

$$\left\langle |\Theta_i|^2 \right\rangle = \frac{\varepsilon}{-8\,\mathrm{Re}[\lambda_i]}\left(1 - e^{2\,\mathrm{Re}[\lambda_i]t}\right)\sum_{j}^{N} |C_{ij}^{-1}|^2$$



C_{ij} is the eigenvectors matrix

Calculation of correlations



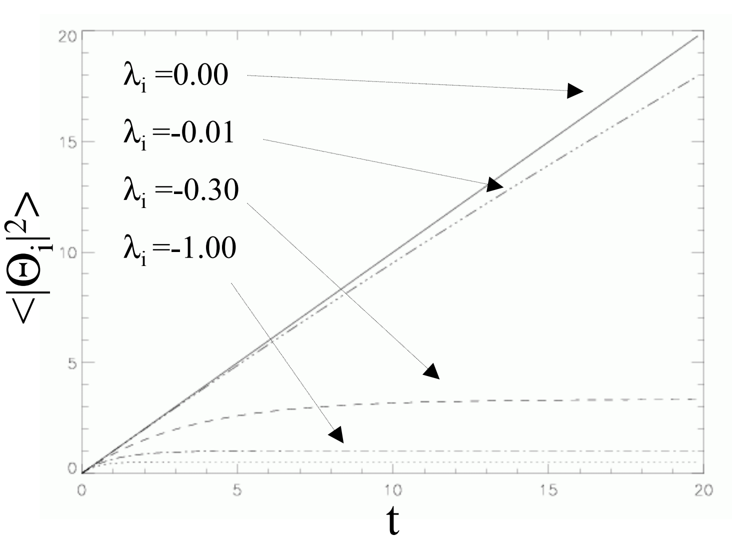
The field fluctuation are linear combinations of eigenmodes

$$\delta E(\vec{x},\vec{q}) = \sum_{i=1}^{N} \Theta_i^q \vec{\mathrm{v}}_i^q \quad , \quad \delta E(\vec{x}) = \iint \delta E(\vec{x},\vec{q})\,\mathrm{d}\vec{q}$$

The amplitude Θ_i^q of each eigenmode v_i^q associated to the eigenvalue λ_i^q follows an Ornstein-Uhlenbeck process: $\partial_i \Theta_i^q = \lambda_i^q \Theta_i^q + \eta_i$, where η_i is the noise ξ in the diagonal basis (C_{ij} are the eigenvectors matrix coefficients):

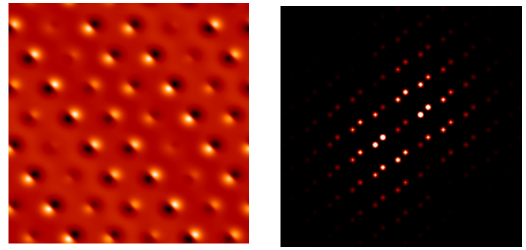
$$\langle \eta_i(t)\eta_j(t') \rangle = \frac{\varepsilon}{2}\delta(t-t')\sum_{k=1}^{N} C_{ik}^{-1} C_{jk}^{-1*}$$

$$\langle |\Theta_i|^2 \rangle = \frac{\varepsilon}{-8\,\mathrm{Re}[\lambda_i]}(1-e^{2\,\mathrm{Re}[\lambda_i]t})\sum_{j}^{N} |C_{ij}^{-1}|^2$$



Re[v(x)] **|v(k)|**

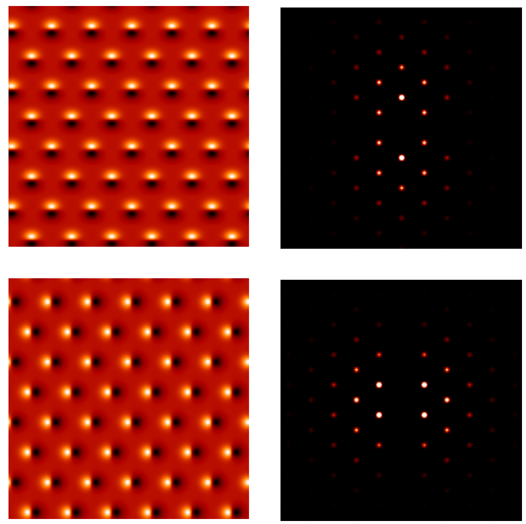
Low damped mode (λ=-0.01, **q**=(0.1,0.1))



Goldstone modes (λ=0, **q**=(0,0)) associated to translational invariance of the pattern.

$\nabla_y E_h$

$\nabla_x E_h$

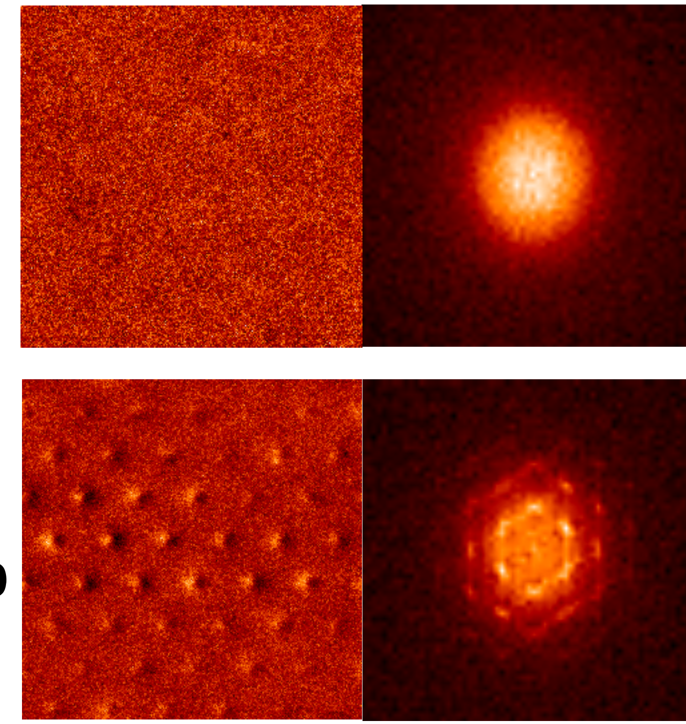


FIELD FLUCTUATIONS

$\mathrm{Re}[\delta E(\mathbf{x})]$ $<|\delta E(\mathbf{k})|^2>$

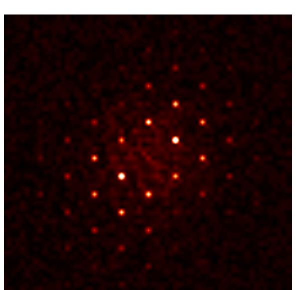
t = 2

t = 100



FIELD CORRELATIONS

$|C_2(\mathbf{k},\mathbf{k_3})|$



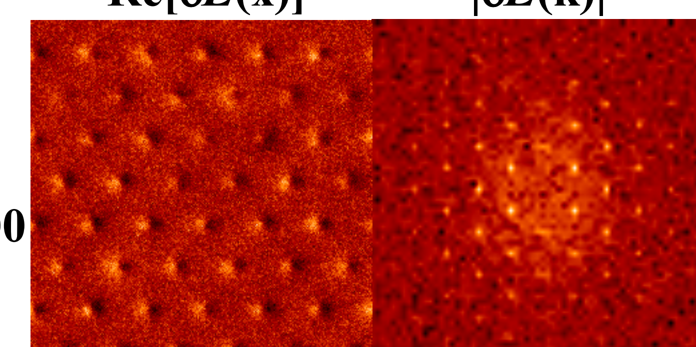
* The highest **correlation** is between the **field fluctuations** of opposite peaks.

*Correlations may be larger with higher harmonics than with some fundamental modes.

* The homogeneous mode is almost uncorrelated

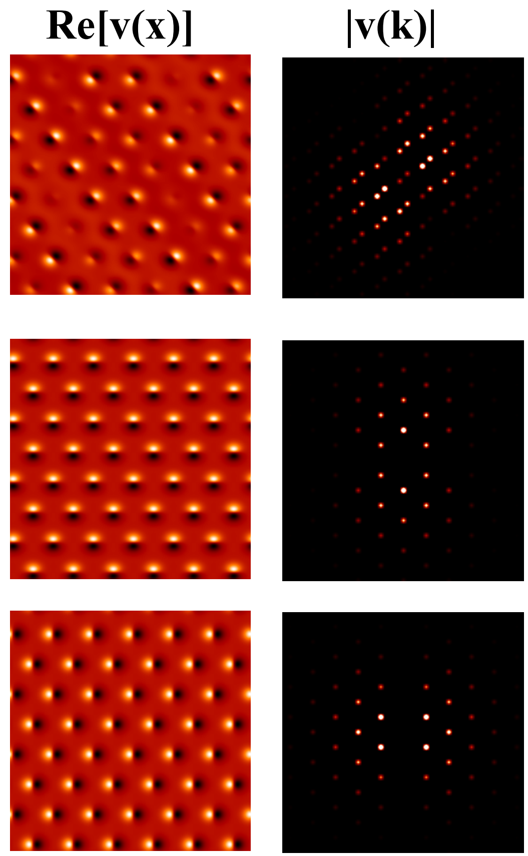
$\mathrm{Re}[\delta E(\mathbf{x})]$ $|\delta E(\mathbf{k})|$

t = 2000



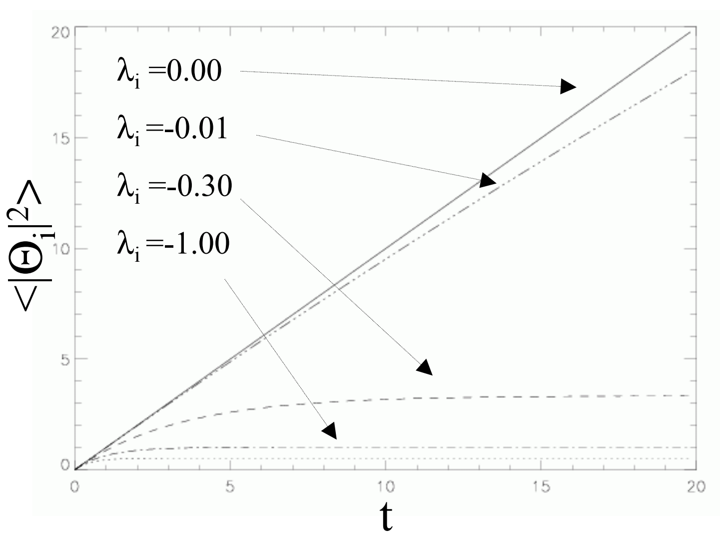
Low damped mode
(λ=-0.01, **q**=(0.1,0.1))

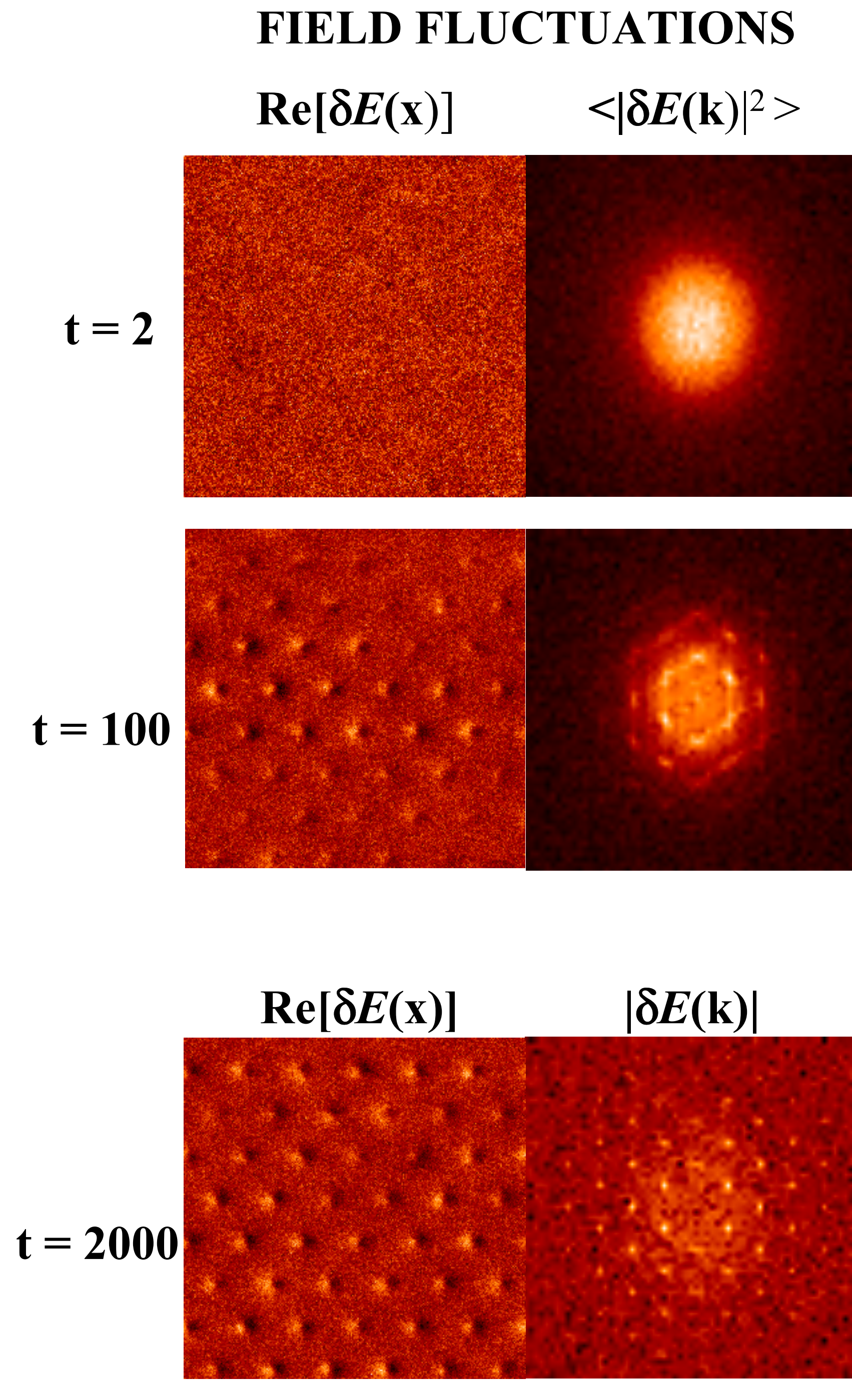
Goldstone modes (λ=0, **q**=(0,0)) associated to translational invariance of the pattern



$\nabla_y E_h$

$\nabla_x E_h$




FIELD FLUCTUATIONS
Re[δE(x)]
<|δE(k)|² >
t = 2
t = 100
Re[δE(x)]
|δE(k)|
t = 2000

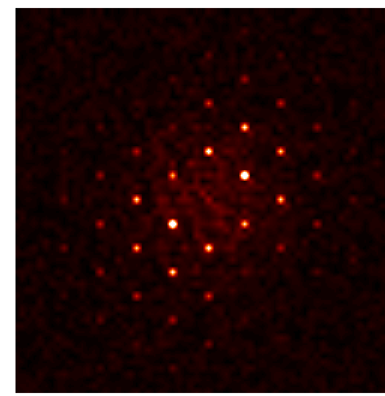
FIELD CORRELATIONS

* The highest **correlation** is between the **field fluctuations** of opposite peaks.

*Correlations may be larger with higher harmonics than with some fundamental modes.

* The homogeneous mode is almost uncorrelated

$|C_2(\mathbf{k},\mathbf{k_3})|$



[illegible]

Correlations of intensity fluctuations

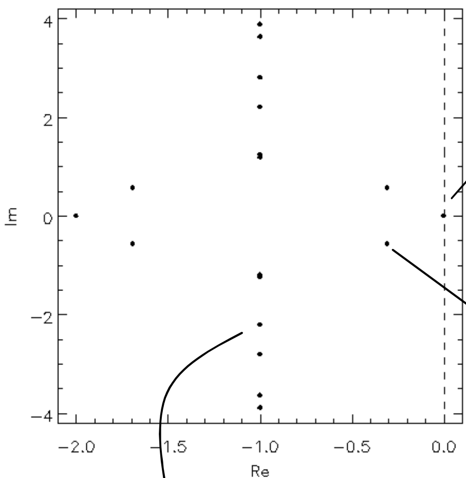
$$\delta I(\vec{k}) = 2\,\mathrm{Re}[E_h^*(\vec{k})\delta E(\vec{k})]$$

$$E_h(\vec{k}) = \sum_{n=1}^{N} (2\pi)^2 a_n \delta(\vec{k}_n - \vec{k})$$

δI (**k**) is only important for **k** being one of the pattern modes $\mathbf{k}_n$.

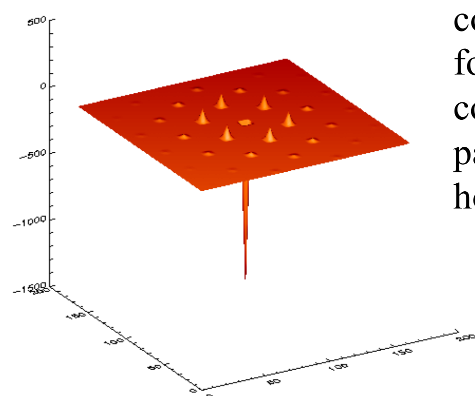
For any other $\mathbf{k}=\mathbf{k}_n+\mathbf{q}$, $\delta I(\mathbf{k})=0$ because $E_h(\mathbf{k})=0$.

Eigenvalues for **q**=0 perturbations

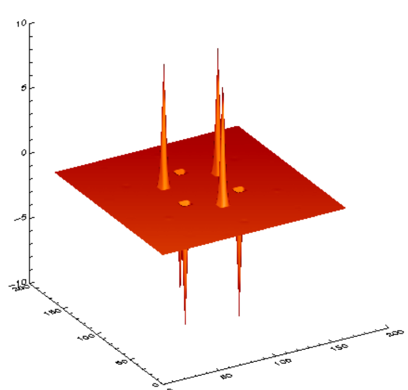


Goldstone modes v^G , $\lambda^G=0$. Despite they are the most excited by noise, they **do not contribute** to intensity fluctuations:

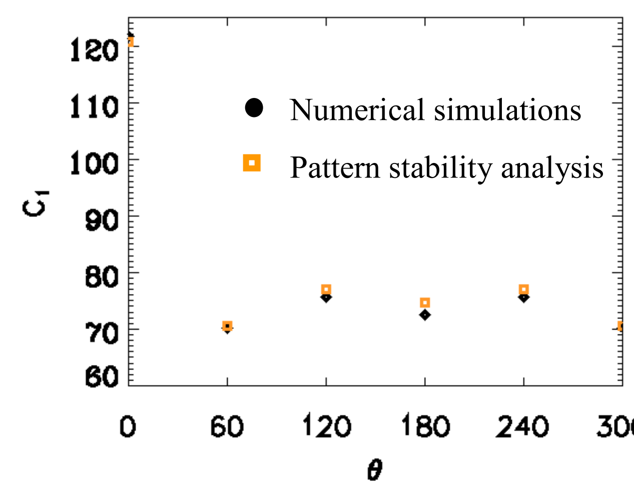
$$\vec{\mathrm{v}}^G(\vec{x}) \propto \nabla E_h(\vec{x}) \qquad \Rightarrow \qquad \mathrm{Re}[E_h^*(\vec{k})\vec{\mathrm{v}}^G(\vec{k})] = 0$$



The eigenmodes associated to these two complex conjugate eigenvalues are the most important ($\lambda\approx$-0.2) for the intensity correlations. They induce strong correlations between the intensity fluctuations of the pattern modes and the anticorrelation with the homogeneous mode.



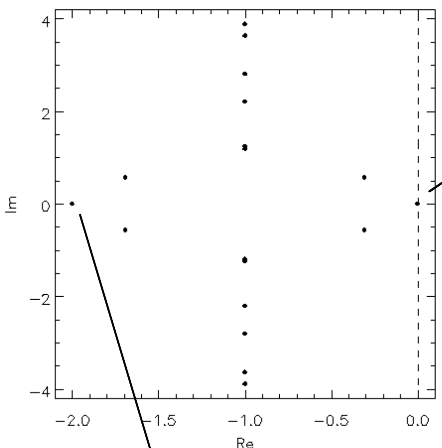
Eigenmodes associated to these eigenvalues (λ=-1.0) induce the differences between the correlations of the intensity fluctuations of the fundamental modes.



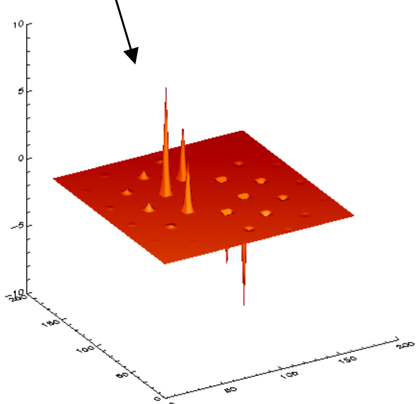
MOMENTUM CONSERVATION

Due to **momentum conservation**, the fluctuations on the quantity $P_x = \sum_n k_x \delta I(\vec{k}_n)$, which is the total transverse momentum in the x direction, are expected to be small (the same for P_y), and just the opposite for the fluctuations on $\mathbf{x}_0$, which fix the origin of the hexagons in the near field, due to **translational invariance** of the problem in the transverse direction *.

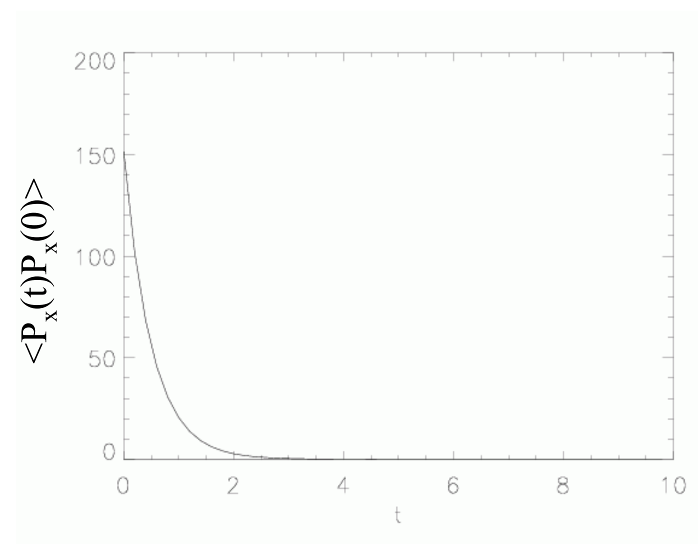
Eigenvalues for **q**=0 perturbations



The **Goldstone modes**, which are associated to the **translational invariance**, induce **great** (λ=0) **fluctuations** on $\mathbf{x}_0$. The excitation by noise of the Goldstone modes move the pattern in the transverse direction.



P_x is zero for the intensity fluctuations associated to all the eigenmodes except for two, which are the **most damped** (λ=-2). The excitation by noise of these two eigenmodes induces the fluctuations on P_x. They are small because of the strong damping of these eigenmodes.



* G. Grynberg and L.A. Lugiato, Opt. Comm. 101, 69-73 (1993)

Conclusions

- **Field fluctuations**:
  - can be understood in terms of noise excitation of **low damped eigenmodes**
  - for long times field fluctuations are dominated by **Goldstone modes (λ=0)**. They induces **great fluctuations** on $\mathbf{x}_0$ (origin of the hexagons in the near field), as it is expected from the **translational invariance** of the problem.

- **Intensity fluctuations**:
  - are not originated by Goldstone modes. They are explained by the noise excitation of the damped eigenmodes of the fluctuations for q=0. They became stationary for relatively short times.
  - The structure of the **correlations** is:
    - Strong anticorrelation between the intensity fluctuations of the homogeneous and the pattern modes (energy conservation).
    - The correlations between the intensity fluctuations of the pattern modes:
      - are maxima between fundamental harmonics separated by 120°

$$C_1(\alpha=120^o) > C_1(\alpha=180^o) > C_1(\alpha=60^o)$$

      - correlation of fundamental modes with higher harmonics decay exponentially with k

- The fluctuations of the **total transverse momentum** are originated by **the most damped eigenmodes (λ=-2)**, so they are **small**, as it is expected from **momentum conservation**.

- Extension: **Quantum fluctuations**