Temporal Griffiths Phases

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(Dated: May 18, 2011)

Disorder is an unavoidable ingredient of real systems. Spatial disorder generates *Griffiths phases* (GPs) which, in analogy to critical points, are characterized by a slow relaxation of the order parameter and divergences of quantities such as the susceptibility. However, these singularities appear in an extended region of the parameter space and not just at a (critical) point, i.e. there is *generic scale invariance*. Here, we study the effects of temporal disorder, focusing on systems with absorbing states. We show that for dimensions $d \geq 2$ there are *Temporal Griffiths phases (TGPs)* characterized by generic power-law spatial scaling and generic divergences of the susceptibility. TGPs turn out to be a counterpart of GPs, but with space and time playing reversed roles. TGPs constitute a unifying concept, shedding light on the non-trivial effects of temporal disorder.

PACS numbers: 05.40.Ca,05.10.Gg,05.70.Fh,05.70.Jk

Quenched disorder affects the behavior of particle systems, altering critical properties and introducing new universality classes. Non-magnetic impurities in magnetic systems or defects in type-I superconductors are typical examples of this [1]. Moreover, novel phases, with phenomenology unheard-of in pure systems, can be induced by spatial disorder. This is the case of *Griffiths phases* (GPs) appearing in classical, quantum, and non-equilibrium disordered systems [2–4]. GPs, which are of relevance in condensed matter physics as well as in other contexts [4, 5], are regions of the phase space –actually a sub-region of the disordered phase –characterized by extended singularities of the thermodynamic potentials and, as a consequence, generic divergences of magnitudes such as the susceptibility [2]. Furthermore, GPs are characterized by an anomalously slow (power-law or stretched exponential) relaxation to zero of the order parameter (and of other time-dependent quantities) which contrasts with the fast (exponential) decay typical of pure systems. Such an anomalous relaxation in the disordered phase occurs owing to the presence of rare regions where the disorder is such that the system is locally in its ordered phase and, hence, a potential barrier has to be overcome for it to relax. The convolution of different, exponentially rare, sizes with exponentially large decaying times gives rise to an overall slowing down of the system's dynamics, which typically becomes algebraic in time –with continuously varying exponents –and logarithmic at the critical point (see below for more details) [3, 4]. Divergences in the potentials together with slow relaxation are two features strongly reminiscent of criticality and its concomitant scale-invariance. However, in GPs these traits appear not just at a critical point but in a broad extended region, providing a robust mechanism to justify some cases of scale-invariance in Nature [6].

The modeling of some problems in physics, chemistry or ecology requires parameters to be disordered in *time* rather than in *space* [7]. This is the case of magnetic systems under a fluctuating external field [1], or of ecological populations under changing environmental conditions [8]. In general, temporal fluctuations in the parameters can shift critical points [9] and affect universal features both in equilibrium [10] and non-equilibrium systems [11]. In a pioneering work, Leigh showed that, in one-variable (mean-field) models of stochastic populations, environmental noise changes the system mean lifetime (time to reach the absorbing state) from exponential to a power-law in system-size [8, 12]. This result inspired us to systematically explore the role of temporal disorder in spatially extended systems (beyond mean-field) and to study if rare temporal regions induce new phases analogous to spatial-disorder induced GPs. Do *temporal Griffiths phases* exist? If so, which properties do they have? Do they exhibit any type of generic scale invariance?

To tackle these questions, we start by analyzing a specific model with absorbing states: the contact process (CP) [13], in the presence of temporal disorder. In the CP, each site of a d-dimensional lattice can be either occupied $z(\mathbf{x}) = 1$ (active) or vacant $z(\mathbf{x}) = 0$. At each time step, an active site is randomly chosen and, with probability b, it converts into active a nearest neighboring site (provided it was empty), while with probability $1-b$ it is declared empty. Time, t, is then increased by $1/N(t)$, where $N(t)$ is the total number of active sites. The "pure" CP is critical only at some dimension-dependent value $b_{c,pure}(d)$ separating an active from an absorbing phase (see [13] and the schematic diagram in Fig.1). This phase transition, occurring at $b_{c,pure} \approx 0.767$, 0.622, and 0.5 for $d = 1$, $d = 2$ and $d = \infty$, respectively, lies in the very robust *directed percolation* universality class [13]. For the spatially-disordered case, b is replaced by $b(\mathbf{x})$; in this case, a GP emerges between $b_{c,pure}$ and the critical point of this quenched version, $b_{q,c} > b_{c,pure}$ (see Fig. 1).

Temporal disorder is implemented by allowing b to be a time-dependent random variable, $b \to b(t)$, for all $\mathbf{x}$. In the simplest (uncorrelated) case, b takes a random value extracted at each Monte Carlo step (i.e. whenever the integer value of t increases) from some distribution of mean b_0 and width σ. Correlated fluctuations can also be implemented, by allowing b to obey an Ornstein-Uhlenbeck dynamics [12, 14]. This temporally-disordered contact process (**TD-CP**) is similar to the Jensen's model in [11]. Following the instantaneous value of b, the system shifts between the tendencies to be active ($b(t) > b_{c,pure}$) or absorbing ($b(t) < b_{c,pure}$), provided that the disorder distribution is broad enough (see Fig.1 for a schematic diagram). Owing to fluctuations, any finite system is, however, condemned to end up in the absorbing state, either for fixed and changing b [14]. However, the mean lifetime $\tau(N)$ grows exponentially with system size N (Arrhenius law [14]) in the pure system case, making it stable in the thermodynamic limit. Instead, in the TD-CP, as $b(t)$ can be adverse ($b(t) < b_{c,pure}$) for arbitrarily long time intervals, $\tau(N)$ is expected to be significantly reduced. But, does $\tau(N)$ still diverge for $N \to \infty$? i.e. does a truly stable active phase exist?

Let us first report on numerical simulations of the TD-CP performed in dimensions $d = 1$, $d = 2$, and $d \to \infty$ (for which we consider a fully connected network (FCN)). $b(t)$ is independently extracted at each Monte Carlo step from a homogeneous distribution $b \in [b_0 - \sigma, b_0 + \sigma]$. We performed both, homogeneous initial density experiments (with all sites initially active) and spreading ones (starting from a single active seed) [13]. In the first set of experiments we measured the average value, over many realizations, of the density of activity $\rho(t)$ as a function of time and the mean lifetime $\tau(N)$ vs system size N. In the second set, we measured standard quantities such as the survival probability as a function of time, $P_s(t)$ [13]. Searching for power-laws of the form $\rho(t) \sim t^{-\theta}$ and $P_s(t) \sim t^{-\delta}$, we determined the critical point location $b_{0,c} = 0.907(2)$, $0.656(1)$, and $0.500(1)$, and the exponent $\delta \approx 0.10(5)$, $0.126(2)$, and 0.5, for dimensions $d = 1$, $d = 2$, and the FCN respectively (in all cases $\theta \approx \delta$; $\sigma = 0.55$ for $d = 1$ and $\sigma = 0.4$ otherwise). For a fixed value of σ, the shift $b_{0,c} - b_{c,pure}$ is larger in $d = 1$ than in $d = 2$, and vanishes in $d = \infty$. Except for the mean-field value, and in agreement with previous findings [11], these critical exponents are non universal, as they decrease upon increasing the noise amplitude σ. Remarkably, in $d = 2$ and $d = \infty$, $\tau(N)$ scales at criticality as $\tau \sim (\ln N)^{z'}$ with $z'(d = 2) = 5.18(5)$ (inset of Fig.2) and $z'(d = \infty) = 3.49(5)$ for $\sigma = 0.2$. The values of $z'(d)$ do not seem universal either, as they decrease with increasing σ. Instead, in $d = 1$ we observe standard power-law scaling $\tau \sim N^{1.55(1)}$. Furthermore, in $d = 2$ and $d = \infty$ (but, again, not in $d = 1$), we find a whole region within the active phase ($b > b_{0,c}$) in which $\tau(N)$ grows generically as a power-law with continuously varying exponent ζ, $\tau(N) \sim N^{\zeta}$, with $\zeta \to 0$ as $b_0 \to b_{0,c}^{+}$ (observe, in Fig.2, the slight downward curvature in the $\log\tau - \log N$ curves at criticality, reflecting the asymptotic logarithmic behavior). Let us remark that obtaining data for larger sizes and deeper into the active phase, where the surviving times are huge, becomes excessively expensive. Hence, estimating with accuracy the upper limit of the algebraic scaling region is prohibitive. We have also measured $\tau(N)$ in the absorbing state; as in the pure model case, it scales as $\tau(N) \sim \ln(N)$ in all dimensions. In summary, while the behavior of $\tau(N)$ in $d = 1$ is similar to that of pure systems, in $d = 2$ and $d \to \infty$, we found **(i)** logarithmic scaling at criticality and **(ii)** an extended region with algebraic scaling.

Let us now present analytical calculations for the high-dimensional limit (FCN). Given that, at every single step, the change on the global density ρ is $\pm 1/N$, one can map its dynamics into a random walk in the interval $[0, 1]$, with jumps $\pm 1/N$ occurring with probabili-

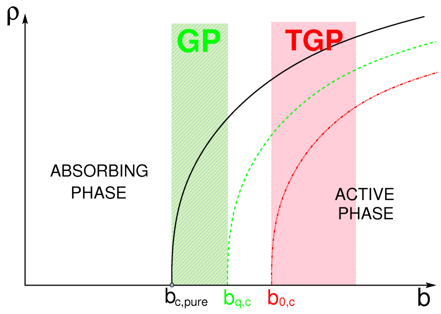

FIG. 1: Schematic phase diagram for the pure contact process (CP) (solid line) the CP with quenched disorder (dashed line), and the CP with temporal disorder (dot-dashed line). For the second, a Griffiths phase appears within the absorbing region, while for the third a Temporal Griffiths phase appears (for $d > 1$) within the active region. The actual locations of the critical points may depend on noise intensity and dimension.

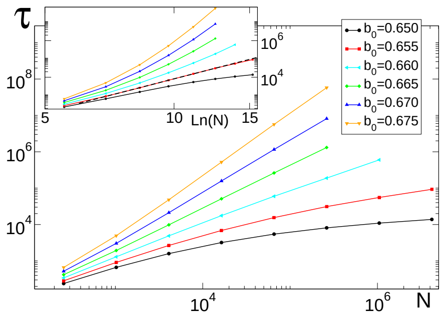

FIG. 2: Main: Log-log plot of the lifetime τ as a function of system size N for the TD-CP in $d = 2$ for various b_0 and $\sigma = 0.4$. There is a finite region, $b_0 \in [0.656, 0.675]$ with generic algebraic scaling of $\tau(N)$ and continuously varying exponents. Inset: log-log plot of $\tau(N)$ vs $\ln(N)$; from the fit at criticality (dashed line) we estimate $\tau \sim (\ln N)^{5.18(5)}$.

ties $b(t)\,\rho(1-\rho)$ and $[1-b(t)]\,\rho$ respectively. The Master Equation for this process is easily written [14], and by performing a $1/N$ expansion one readily obtains a Fokker-Planck equation whose (Ito) Langevin equivalent is (up to leading order),

$$\dot{\rho}(t) = a\rho - b\rho^2 + \alpha\sqrt{\rho}\,\eta(t) + \sigma\rho\,\xi(t), \qquad (1)$$

with $a = 2b_0 - 1 + \sigma^2/2$, $b = b_0$, $\alpha = 1/\sqrt{N}$, and the noise $\xi(t) = 2(b(t) - b_0)/\sigma$. Observe the presence of both, a *demographic* noise, proportional to $\sqrt{\rho}$ which vanishes in the $N \to \infty$ limit and an external or environmental noise, linear in ρ [9, 15]. Generalizing to any spatial dimension one can show that the corresponding Langevin equation is just that of the directed percolation universality class [13] with a fluctuating linear-term parameter:

$$\dot{\rho} = (a + \sigma\xi(t))\rho - b\rho^2 + \nabla^2\rho(\mathbf{x},t) + \gamma\sqrt{\rho}\,\eta(\mathbf{x},t), \quad (2)$$

where γ is a constant. For the sake of generality, we have numerically integrated Eq. (2) in $d = 1$, $d = 2$ and in a FCN, by using the scheme in [16]. We reproduced all the findings above proving that our conclusions are robust, and apply to any model in the directed percolation class not just the TD-CP.

In the case of uncorrelated white noise (as used in the numerics above) the quasi-stationary solution of Eq. (1) is $P(\rho) \sim \rho^{-1}\left(1 + \sigma^2\rho/\alpha^2\right)^{2\frac{b\alpha^2 + a\sigma^2}{\sigma^4} - 1} \exp(-2b\rho/\sigma^2)$ which in the large N limit can be approximated by $P(\rho) \sim \rho^{2(a/\sigma^2 - 1)} e^{-2b\rho/\sigma^2}$, exhibiting an a/σ^2-dependent singularity at $\rho = 0$. In this limit, the singularity is not integrable for $a < a_c = \sigma^2/2$ ($b < b_{0,c}(FCN) = 0.5$ for all σ), and the only solution is a delta distribution at $\rho = 0$, i.e. the system is absorbing. Instead, for finite N, there is a $1/\rho$ singularity for any value of a and the only possible steady state is the absorbing one (as occurs for any finite system with demographic noise [13]). Defining $z = \ln\rho$, Eq. (1) becomes $\dot{z} = \tilde{a} - b\exp(z) + \sigma\xi(t)$, with $\tilde{a} = a - \sigma^2/2$, which describes a random walker trapped in a potential $V(z) = -\tilde{a}z + b\exp(z)$. It exhibits the three following regimes: **(i) Active phase** ($\tilde{a} > 0$): the time required for the active state to fluctuate and reach the vicinity of the absorbing state (which is approximately $\rho = 1/N$, i.e. $z = -\ln(N)$) and, eventually, die, is exponential in the height of the potential [14], $\tau(N) \sim \exp\left[V(-\ln N)/(\sigma^2/2)\right] \sim \exp(2\tilde{a}\ln N/\sigma^2) \sim N^{2\tilde{a}/\sigma^2}$. This is, $\tau(N)$ exhibits generic algebraic scaling with continuously varying exponents[8]. Hence, the active phase is truly stable when $N \to \infty$. **(ii) Critical point** ($\tilde{a} = 0$): For sufficiently small values of z we have a free random walk (no potential barrier to be overcome) which covers a typical distance $\sqrt{\tau}$ in time τ; equating this distance to $z = \ln N$, the time to die scales logarithmically, $\tau \sim [\ln(N)]^2$. **(iii) Absorbing phase** ($\tilde{a} < 0$): z decays linearly in time and, hence, the time needed to reach $z = -\ln N$ scales as $\tau \sim \ln N$. These predictions are in excellent agreement with the corresponding numerical results for the FCN.

Result **(i)** can be recovered by using the path-integral representation of Eq. (1) [13]. The most probable path to the absorbing state can be easily calculated in semi-classical approximation. τ is simply the inverse of the probability weight associated with such a path. By using this formalism, Kamenev et al. [12] have recently investigated in an interesting work the effect of correlated temporal disorder on a one-variable birth-death process. They conclude that, in the case of interest here (short-time correlated noise), $\tau(N)$ grows exponentially with N for weak noise amplitudes and algebraically in the strong external-noise limit. Our result differs slightly from this: given that the demographic noise amplitude in Eq. (1) vanishes in the large N-limit, the strong-noise limit does not need to be invoked to obtain algebraic scaling.

In order to extend our conclusions to finite dimensions and to make a parallelism between the reported broad regions of generic algebraic scaling –that we call *temporal Griffiths phases* (TGPs) –and standard GPs, let us sketch the main properties of GPs for the contact process equipped with quenched disorder, i.e. $b \to b(\mathbf{x})$ [17]. In the quenched CP, rare regions with $b(\mathbf{x}) > b_{c,pure}$ and arbitrary size s appear with probability $\exp(-\alpha s)$, where α is a disorder-dependent constant. Such regions are locally active and, hence, activity survives on them until a coherent fluctuation kills it. This occurs at a characteristic time $t_c(s) \sim \exp(\beta s)$ where β is a constant, as given by the Arrhenius law [14]. Hence, the time-decay of the survival probability of a homogeneous initial condition is given by the convolution $P_s(t) \propto \int ds \exp(-\alpha s)\exp(-t/t_c(s))$, and the leading contribution in saddle point approximation comes from size $s^*(t) = (1/\beta)\ln(\beta t/\alpha)$, implying $P_s(t) \propto t^{-\alpha/\beta}$ (right at the critical point, the exponent vanishes, and there is "activated scaling", characterized by a logarithmic decay $P_s(t) \sim (\ln t)^{-\theta'}$ [4]). Similar expressions apply to the time decay of other quantities such as the activity density, as well as to many different systems with quenched disorder [4].

Thus, some analogies between GPs and TGPs are: *i)* In GPs disorder is "quenched in space"; in TGPs it is "quenched in time". *ii)* In GPs rare (locally active) regions exist even if the overall state is absorbing; in TGPs rare (temporarily absorbing) time-intervals exist even if the overall state is active; i.e. the roles of active/absorbing phases are exchanged. *iii)* In GPs the probability for a (rare) active region of size s to occur is $\exp(-\alpha s)$; in TGPs (rare) time intervals of length T are absorbing with probability $\exp(-\alpha T)$; hence, the typical time to observe them is $\tau \sim \exp(\alpha T)$. *iv)* In GPs, as we just argued, the leading contribution of the decay at time t comes from a rare region of size $s^* \sim \ln(t\beta/\alpha)/\beta$; this combined with *(iii)* leads to a *generic power-law decay*

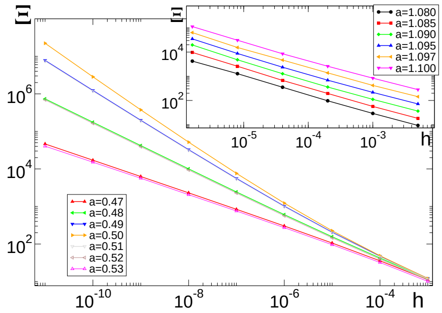

FIG. 3: Log-log plot of the susceptibility, Ξ, as a function of the field h, obtained by integrating Eq. (1) (main plot) and Eq. (2) for $d = 2$ (inset), for different values of a.

in time, $t^{-\alpha/\beta}$. In TGPs, the time required to reach the absorbing state in an absorbing time-interval, is given by $\exp(-\beta t^*) \sim 1/N$, or $t^* \sim \ln(N)/\beta$. Equating t^* with T in (*iii*), one obtains a *generic algebraic decay in system size*; $\tau(N) \sim N^{\alpha/\beta}$. In conclusion, TGPs are analogous to GPs by exchanging the roles of space and time. These heuristic arguments seem to be valid even for finite dimensional systems (down to $d = 2$). The reason why a TGP phase is not observed in $d = 1$ is not completely clear to us. Presently we are developing a semiclassical approximation, analogous to that in [12], but for the spatialy explicit Eq. (2) (see [18]), in order to have a more precise understanding of low-dimensional cases.

To further delve into the GP/TGP analogy, and given that GPs exhibit generic divergences of the susceptibility [2, 3], we have measured numerically the susceptibility, defined as $\Xi = \partial\rho_{st}(h)/\partial h|_{h\to 0}$ where ρ_{st} is the average value of the activity in the steady state after introducing an external field h coupled to the system's dynamics, for Eq. (1) and for Eq. (2) in $d = 2$. Fig.3 shows that Ξ measured from Eq. (1) (which perfectly agrees with that in Monte Carlo simulations of the TD-CP on FCN) diverges all along the TGP. Fig.3-inset shows generic divergences also for $d = 2$; however, given that very small values of h cannot be reached in this case, it is difficult to elucidate numerically whether such a divergence is for real in the $h \to 0$ limit or it just a transient effect.

To analytically understand these findings, we take Eq. (1) in the $N \to \infty$ limit and include an external field h. A new term $\exp\left[-2h/(\rho\sigma^2)\right]$, exhibiting an essential singularity at $\rho = 0$, appears in the stationary solution (see above). It is a matter of algebra to verify analytically that $\partial\rho_{st}(h)/\partial h$ diverges algebraically as $h^{-1+2|a-a_c|/\sigma^2}$, in an extended interval $a \in [0, \sigma^2]$ around the critical point [15]. If, in this calculation, we replace α by γ which does not vanish when $N \to \infty$ and mimics what happens in finite dimensions (see Eq.(2)), the parameter-dependent singularities are replaced by the usual ρ^{-1} absorbing state singularity. This suggests that the generic divergence of the susceptibility is a transient effect in the presence of non-vanishing demographic noise. In that case, the strong external-noise limit needs to be taken for the generic divergence to survive. Going beyond mean-field, it can be proved by using simple field-theoretical arguments (similarly to [15]) that Eq. (2) with $\alpha = 0$ exhibits generic divergences of the susceptibility in a broad interval even in finite spatial dimensions.

In summary, systems with absorbing states and fluctuating external conditions exhibit a region in the active phase —the "temporal Griffiths phase"—such that the mean lifetime scales generically as a power-law (with continuously varying exponents) of system size and logarithmically at criticality. This occurs not only in mean field [8, 12] but also in extended systems as long as $d \geq 2$. TGPs have deep analogies with standard GPs, but the roles of space and time are reversed: in GPs (TGPs) spatial (temporal) disorder leads to generic algebraic scaling as a function of time (size). Moreover, as GPs, TGPs exhibit (at least in the strong noise limit) generic divergences of magnitudes such as the susceptibility and of stationary distribution functions.

TGPs could be measured in the experimental realizations of the directed percolation class with liquid-crystals [19] by introducing externally changing fields, and could appear in many other systems such as in bistable Ising-like models with randomly changing conditions. We hope this work will stimulate new research along these lines.

We acknowledge the MICINN(FEDER) (FIS2007-60327 and FIS2009-08451), DARPA grant HR0011-09-1-055 and J. de Andalucía P09-FQM4682 for support.

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