Chapter 3

VIABILITY AND RESILIENCE IN THE DYNAMICS OF LANGUAGE COMPETITION

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Abstract This chapter describes the a first case study applying viability based resilience, presented in Chapter 2, on an individual based model (IBM). A particularly sensitive issue is to derive macroscopic descriptions from these IBMs, involving a limited number of variables. Indeed, as explained in Chapter 7, this is the condition to make tractable the computation of the viability kernel and resilience values. The chapter introduces Individual Based Models (IBMs) of language competition, and explores, through computer simulations, the pattern dynamics of these models and the qualitative role of the prestige and volatility parameters. Then it proposes several approaches inspired by physics to perform the derivation of macroscopic descriptions that are able to capture key aspects of the phenomena observed in the IBMs. Finally, it presents an explicit calculation of viability and resilience based on a macroscopic description.

1. Introduction

The study of language dynamics has been addressed from at least three different perspectives: language evolution (or how the structure of language evolves), language cognition (or the way in which the human brain processes linguistic knowledge), and language competition (or the dynamics of language use in multilingual communities). The last is the approach followed in this chapter in which, therefore, we focus on problems of social interactions. We aim to contribute to the study of the complex phenomenon of language survival (viability), thoroughly studied in linguistics, from the perspective of pattern resilience.

The fact that 97% of the people in the world speak about 4% of the extant languages and that 50% of the 6,000 languages are expected to die in the current century (Crystal, 2000) has made scholars struggle to identify and study the factors that may determine the survival or disappearance of most of our linguistic heritage as well as the mechanisms that could be implemented so as to revitalize an endangered language (cf., for example, Fishman, 1991, Grenoble and Whaley, 1998; Grenoble and Whaley, 2006, Nettle and Romaine, 2000, Hinton and Hale, 2001, Bradley and Bradley, 2002, UNESCO, 2003, Wölck, 2004, Tsunoda, 2005). A motivation for many scholars is to understand how language endangerment occurs so as to avoid it or put it to an end. Such has been the aim of works such as Fishman's Reversing Language Shift (Fishman, 1991), where he proposes two four-stage steps at each of which social and political actions should be taken so as to revitalize an endangered language. The first step consists of reversing language shift to attain diglossia and the second one addresses reversing language shift to transcend diglossia. Here we are concerned with this second step which is mathematically related to viability theory. Fishman (1991) has been the inspiration for a number of authors interested in language planning policies who, in the last decades, have tried to determine the degree of endangerment of a given language, as well as to measure the extent to which a given language may be revitalized stemming from any of these stages (see for example, Manley, 2008, who assesses the role of micro-prestige among speakers of Quechua in Cuzco).

Language survival has also been subject of study of UNESCO (2003), who created an Ad Hoc Expert Group on Endangered Languages, so as to identify languages on a path toward extinction. The degree to which a language is actually bound to disappear may be assessed following different scales that account for each of the factors that have usually been identified as determinant in the vitality of a language. UNESCO (2003) has isolated nine of these factors (related to population, intergenerational transmission, and linguistic policies, among others). Although none of them can properly be considered in isolation to determine the viability of a language (they must be interconnected so as to thoroughly ascertain how endangered a given language is), in this work we focus on the role of two of such factors:

i) governmental and institutional language attitudes and policies, including official status and use, and

ii) community members' attitudes toward their own language.

Thus we have considered a class of models of socially interacting agents to describe language competition, featuring two parameters associated with these two factors. Factor (i) is taken into account by a parameter measuring the prestige of the language. In fact, the prestige of a language has been considered as one of the main factors affecting language competition since Labov's *Sociolinguistic Patterns* (Labov, 1972). It measures the status associated to a language due to individual and social advantages related to the use of that language, being higher according to its presence in education, religion, administration and the media. Factor (ii) is taken into account by a volatility parameter, a property which is not so often discussed in the linguistic literature as prestige is. These two parameters were already considered by Abrams and Strogatz (Abrams and Strogatz, 2003). For the parameter values that they explored in connection with Spanish-Quechua, Scottish-English and Welsh-English competitions, the prediction is that one of the languages eventually disappears. But reality provides myriads of counter-examples to this, in many cases achieved by active linguistic policies (e.g. French and Flemish in Belgium, Spanish and Catalan in Catalonia, etc.). We will describe combined ranges of prestige and volatility that make language coexistence viable.

We try to assess the viability and resilience of the language diversity in the line of Chapter 2 and (Martin, 2004). Resilience is seen as the capacity of a system to restore its properties of interest, which it may have lost after some perturbations. In language competition, these perturbations may be achieved by different situations, such as a military conquest (as was the case of Spanish in Latin America, in which the native languages were strongly threatened by the language of the conquerors), or massive immigration (e.g. the arrival of thousands of Spanish-speaking people to Catalan-speaking areas in Catalonia in the 1960s, as a result of industrial development). In our model, a language will be considered resilient if, after a perturbation like any of these, adequate political actions can restore its viability and, therefore, guarantee its survival.

This chapter intends to be a contribution to the understanding of the mechanisms underlying processes of social interaction at work in the dynamics of language competition, as well as the consequences of these mechanisms as regards language survival or extinction and the viability of language coexistence. We proceed in three steps. In the first step (Section 2) we introduce Individual Based Models (IBMs) of language competition, and explore, through computer simulations, the pattern dynamics of these models and the qualitative role of the prestige and volatility parameters. In the second step (Section 3) we discuss the derivation of macroscopic descriptions of these models, and we discuss how the macroscopic descriptions capture key aspects of the phenomena observed in the IBMs. Finally, in Section 4 we present an explicit calculation of viability and resilience based on a macroscopic description. A summary of conclusions is given in Section 5, while technical mathematical details of the micro-macro connection are contained in the Appendix.

2. IBMs for language competition

In this Section we present two Individual Based Models (IBMs) for language competition: the Abrams-Strogatz model (AS) and its extension allowing for bilingual agents, the Minett-Wang model (MW). After introducing the corresponding transition probabilities and the parameters of the model, we give a qualitative description of the role played by these parameters.

2.1 The Abrams-Strogatz model

The microscopic version (i.e. individual based) (Stauffer et al., 2007) of the AS-model (Abrams and Strogatz, 2003) is a two-state model proposed for the competition between two languages. An agent i sits in a node within a social network of N individuals and has k_i neighbours. A neighbour means here another agent with which agent i has a social interaction. Agents can be in either of two states: X, agent using language X (monolingual X); or Y, agent using language Y (monolingual Y).

The state of an agent evolves according to the following dynamical rules: starting from a given initial condition, at each step we choose one agent i at random and we compute the local densities for each of the language states in the neighbourhood of node i, $\sigma_{i,l}$ (l=X, Y). The agent changes its state according to the following transition probabilities:

$$p_{i,X\to Y} = (1-S)(\sigma_{i,Y})^a \quad , \qquad p_{i,Y\to X} = S(\sigma_{i,X})^a \tag{3.1}$$

Equations (3.1) give the probabilities for an agent i to change from community X to Y, or vice versa. They depend on the local densities ($\sigma_{i,X}$, $\sigma_{i,Y}$) and on two parameters: the *prestige* of the language X, $0 \leq S \leq 1$; and the *volatility*, $a \geq 0$. Prestige is a language property measuring the different status between the two languages, that is, the more prestigious language is the one which gives an agent more possibilities in the social and personal spheres. The case of socially equivalent languages corresponds to $S = 1/2$ (language X is more prestigious for $S > 1/2$). The volatility is a parameter characterizing social dynamics which gives shape to the functional form of the transition probabilities. The case a=1 is the neutral situation of random imitation of a neighbour, where the transition probabilities depend linearly on the local densities. A high volatility regime regime exists for $a < 1$, with a probability of changing language state above the neutral case, and therefore agents change their state rather frequently. A low volatility regime exists for $a > 1$, with a probability of changing language state below the neutral case, where agents have a larger resistance to change their state. In this way, the volatility parameter gives a measure of the degree of accommodation or resistance of the agents to change their language use.

2.2 The Bilinguals Minett-Wang model

We consider here an extension of the AS-model proposed by Minett and Wang[1], which takes into account the presence of a third possible state Z associated with bilingual agents using[2] both languages, X and Y. There are three local densities to compute for each node i: $\sigma_{i,l}$ $(l = X, Y, Z)$. The agent changes its state according to the following transition probabilities:

$$p_{i,X\to Z} = (1-S)(\sigma_{i,Y})^a \quad , \qquad p_{i,Y\to Z} = S(\sigma_{i,X})^a. \tag{3.2}$$
$$p_{i,Z\to Y} = (1-S)(1-\sigma_{i,X})^a \quad , \qquad p_{i,Z\to X} = S(1-\sigma_{i,Y})^a. \tag{3.3}$$

Equations (3.2) give the probabilities for changing from a monolingual community, X or Y, to the bilingual community Z, while equations (3.3) give the probabilities for an agent to move from the Z community towards the X or Y communities. Notice that the latter depend on the local density of agents using the language to be adopted, including bilinguals. It is important to stress that a change from state X to state Y or vice versa, always implies an intermediate step through the Z-state ($p_{i,X\to Y} = p_{i,Y\to X} = 0$).

2.3 Qualitative dynamics of IBMs

An implementation of these two IBMs in a two-dimensional regular network with four neighbours per node has been performed by designing a Java Applet in which one can tune the parameters described above, set different initial conditions, and see the simulations in real time[3]. The following descriptive overview of the models with different parameter settings gives insights on the emergent complex behavior of these models, including issues of linguistic domain growth, linguistic boundaries, language coexistence, survival and extinction, and the role of bilingual agents.

- Neutral volatility. ($a = 1$)

[1]Notice that this extension was proposed in a working paper in 2005 (see also Wang and Minett, 2005). The final version of the paper (Minett and Wang, 2008) differs slightly on the transition probabilities. However, we analyse here their initial proposal.

[2]Notice that we consider *use* of a language rather than *competence*. In this way, learning processes are out of reach of the present model. Effectively, the situation is such as if all agents were competent in both languages.

[3]An applet can be found at: `http://ifisc.uib.es/eng/lines/complex/APPLET\_LANGDYN.html`

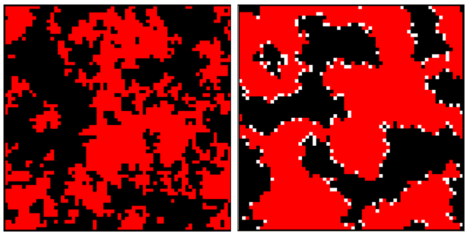

Figure 3.1: Snapshots showing the formation of domains in the AS-model (Left) and the MW-model (Right) starting from an initial random distribution of states of the agents. Neutral volatility ($a = 1$) and socially equivalent languages ($S = 0.5$). $N = 64^2$ agents. Snapshots at time $t = 200$. Notice that in the MW-model, bilingual agents do not form domains, but they place themselves at the interfaces between monolingual domains. Grey: monolingual X; black: monolingual Y; white: bilingual Z.

In the case of socially equivalent languages ($S = 0.5$), we observe in both models (AS and MW) a formation and growth of monolingual domains (see Figure 3.1). However, the growth of these linguistic domains and the motion of linguistic boundaries has been shown to be due to different mechanisms: interfacial noise (AS-model) and curvature reduction (MW-model) (Castelló et al., 2006; Vazquez et al., 2010). Notice that the bilingual agents never form domains but, instead, they place themselves at the boundaries between monolingual ones. Finally, one of the two languages takes over the system. Due to the equivalent prestige, this happens for each of the languages with equal probability.

The well known general role of prestige is clear when $S \neq 0.5$: the most prestigious language dominates, causing the extinction of the other. The near extinction of Old Catalan in Alghero in its competition with modern Italian is a representative example of this situation. One can also see that changing the value of S when a language is in its way to extinction can lead to its recovery. A possible example of this situation is the recovery in recent times of the use of Quechua in its competition with Spanish in Peru.

- Low volatility regime. ($a > 1$)

  When volatility is low, i.e. agents have larger inertia to change the language they are currently using, both models display a similar growth of monolingual domains (see Figure 3.2). These domains evolve smoothly and slowly (curvature-driven like), and the times for extinction increase: language death becomes a slower process.

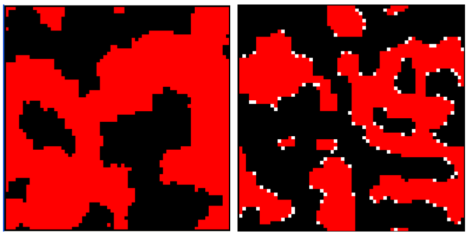

Figure 3.2: Snapshots showing the formation of domains in the AS-model (Left) and the MW-model (Right). Low volatility ($a = 3$) and socially equivalent languages ($S = 0.5$). $N = 64^2$ agents. Snapshots at time $t = 350$. Notice that the boundaries are flatter, due to the increase of curvature driving. Grey: monolingual X; black: monolingual Y; white: bilingual Z.

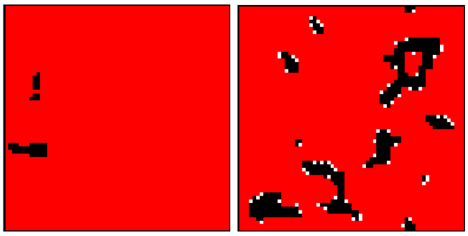

Figure 3.3: Snapshots in the AS-model (Left) and the MW-model (Right). Low volatility ($a = 3$) and socially non-equivalent languages ($S = 0.6$). $N = 64^2$ agents. Snapshots at time $t = 225$. Notice that in the AS-model the less prestigious language is just about to get extinct (around 1% of the population), while in the MW-model the minority language represents still more than 10% of the population. Grey: monolingual X; black: monolingual Y; white: bilingual Z.

For socially asymmetric languages, low volatility delays the effect of prestige difference, so that an endangered language can persist for a long time. In comparison to the AS-model, it is interesting to notice that bilingual individuals slow down further the extinction of the less prestigious language (see Figure 3.3). An example of this situation can be the competition between Galician-Spanish in Galicia (NW of the Iberian Peninsula), where the low volatility of the Galician speakers seems to be preventing a more effective result of current linguistic policies (Monteagudo and Lorenzo, 2005), but there are reasons to think that it also prevented Galician from endangerment in the past (Ayestaran Aranaz and Justo de la Cueva, 1974).

- High volatility regime. ($a < 1$)

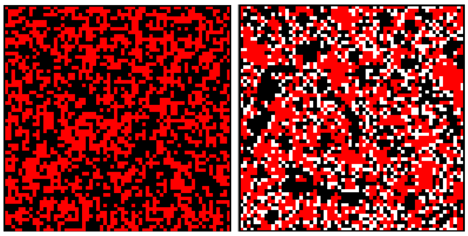

Figure 3.4: Snapshots showing the coexistence regime in the AS-model (Left) and the MW-model (Right). High volatility ($a = 0.1$) and socially equivalent languages ($S = 0.5$). $N = 64^2$ agents. Snapshots at time $t = 200$. Notice that agents do not form linguistic domains, but are completely mixed. Grey: monolingual X; black: monolingual Y; white: bilingual Z.

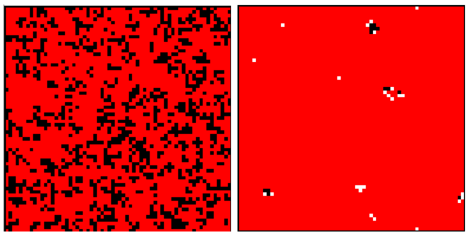

Figure 3.5: Snapshots in the AS-model (Left) and the MW-model (Right). High volatility ($a = 0.1$) and socially non-equivalent languages ($S = 0.6$). $N = 64^2$ agents. Snapshots at time $t = 40$. Notice that in the MW-model the less prestigious language is just about to get extinct, while in the AS-model coexistence is possible. However, this language becomes the one spoken only by a minority. Grey: monolingual X; black: monolingual Y; white: bilingual Z.

In the case in which volatility is high and for socially equivalent languages ($S = 0.5$), language domains cease to be formed and agents in different states are mixed throughout the population: this scenario leads to a long lived dynamical coexistence of the two languages in both models, with the two languages having the same proportion of speakers and also the survival of a large number of bilingual agents in the MW-model. (see Figure 3.4). The high frequency of changes in the language used by the agents makes possible a linguistic interpretation of this phenomenon as *code-switching*: all agents in the lattice shift languages so often that they can be considered to do so even within one single speech exchange. Examples of this sociolinguistic situation in which agents tend to develop a linguistic variety in which they merge both languages in their speech are the case of *Yanito* spoken in Gibraltar (UK colony on the south of the Iberian Peninsula) and the use of *Spanglish* in certain areas of the USA.

The situation is different when languages with different prestige are considered in a situation of high volatility ($S \neq 0.5$; see Figure 3.5). For a relatively small difference in prestige between the two languages ($S = 0.6$), bilingual agents in the MW-model cause a fast extinction of the less prestigious language, while in the absence of bilingual agents (AS-model) both languages coexist for long times (although the majority uses the more prestigious language, around 70% of the population). When the prestige difference becomes larger ($S \geq 0.7$), the less prestigious language dies out in both models rather fast (but still slower when there are no bilingual agents (AS-model)).

In summary, numerical simulations of the AS and WM models show that depending on the volatility of individuals and the relative difference on prestige between both languages, the population can either remain indefinitely in a *coexistence* state with a finite fraction of speakers in each of the two languages, or it can reach a *dominance/extinction* state in which one of the two languages takes over the whole population. Our results make clear that prestige is very important, but it is not the whole story, volatility being a very important social parameter in language competition. For example, when a language gets extinct, this happens much faster in the high volatility regime than in the low volatility regime (compare Figure 3.3 and Figure 3.5). Generally speaking, high volatility is good for the coexistence of languages of similar prestige. However, when a language is situated in a low prestige position, low volatility of the agents gives larger times before extinction, and in this way, enough time to try to enhance its prestige. This delay in the path to extinction is reinforced by the presence of bilingual agents (MW-model). At the point in which social equivalence is achieved, if the volatility is increased, a situation of coexistence for both languages becomes viable and can be maintained indefinitely.

We finally note that our analysis is here based on an underlying regular two-dimensional network of interactions. This set-up accounts for the important ingredient of local interactions, but other dynamical phenomenology such as the existence of metastable states (coexistence for finite but long times) appears in more complex social networks with community or mesoscale structure (Castelló et al., 2007; Toivonen et al., 2008). In these topologies, the agents form language communities which are correlated with the network structure.

3. Macroscopic descriptions of IBMs of language competition

A macroscopic description of the IBMs can be given in terms of the dynamical evolution of the global densities x and y of X and Y speakers respectively (with the density of bilingual agents being $z = 1-x-y$), and of the time dependence of the density of pairs of neighbours in a different state ρ. The quantity ρ describes the linguistic boundaries. The case $x = 1$ or $x = 0$ together with $\rho = 0$, corresponds to the *dominance* or *extinction*, respectively, of language X, with all individuals using the same language; while $0 < x < 1$ and $\rho > 0$ indicates that both languages are present in the system (*coexistence*). The aim is to derive, from the IBMs, macroscopic equations for the time evolution of x, y and ρ, and to analyze their predictions for the evolution of the system. These equations depend on the underlying network of interactions, and we consider here the AS-model in different network topologies: we start from the case of a highly connected society with no social structure (fully connected network), that corresponds to the simplified assumption of a "well mixed" population, widely used in population dynamics and language dynamics (Stauffer et al., 2007). To account for the local effects of social interaction among individuals we consider a complex network of interactions. We will see that the results depend on the particular properties of the network under consideration, reflected in the statistical properties of the distribution of the number of links per node of the network. In order to make further contact with the analysis of IBMs of the previous section, and to account for processes of linguistic domain growth, we also consider an approximate description of the dynamics in a regular two dimensional network by means of a continuous space-dependent field for the density of X speakers. Finally, the effect of bilingual agents, as considered in the MW-model, is discussed in the case of a fully connected network.

3.1 The Abrams-Strogatz model

Mean field description of fully connected networks. We consider a network with N nodes in which each node has a connection (link) to any other node. In a time step $\delta t = 1/N$, a node i with state $X(Y)$ is chosen with probability $x(y)$. Then, according to the transitions (3.1), i switches its state with probability:

$$\begin{aligned} p_{i,X\to Y} &= (1-S)y^a, \\ p_{i,Y\to X} &= Sx^a, \end{aligned} \tag{3.4}$$

where, in this fully connected network, the densities of neighbours of i with states $X(Y)$ are equal to the global densities x and y of nodes in states X and Y, respectively. With these switching probabilities it is easy to obtain that (see

Appendix 3.A.1):

$$\frac{dx}{dt} = x(1-x)\left[Sx^{a-1} - (1-S)(1-x)^{a-1}\right]. \tag{3.5}$$

Equation (3.5) describes the evolution of the density of X-speakers in a very large population ($N \gg 1$), neglecting finite size fluctuations. The density ρ is not an independent quantity in a fully connected network. It can be obtained as the ratio between total number of links between nodes in different state and the number of links in the network. For large N it becomes

$$\rho(t) = 2x(t)\left[1 - x(t)\right]. \tag{3.6}$$

Equation (3.5) has three stationary solutions

$$x = 0, \quad x^*(a,S) = \frac{(1-S)^{\frac{1}{a-1}}}{(1-S)^{\frac{1}{a-1}} + S^{\frac{1}{a-1}}} \quad \text{and} \quad x = 1.$$

The solutions $x = 1$ and $x = 0$ correspond to the complete dominance of X and Y speakers respectively, while $x^*(a,S)$ is a solution corresponding to a coexistence of X and Y speakers with relative fractions x^* and $1 - x^*$, respectively, that depend on a and S. Because we are looking for a long term stationary state, the corresponding solution must be stable under small perturbations (variations in the densities x and y). If the perturbation dies out we say that the solution is stable, otherwise if it grows in time then the solution is unstable. The stability of each stationary solution depends on the values of the parameters a and S, as we summarize in Figure 3.6. For $a < 1$, the coexistence solution $x^*(a,S)$ is stable. In this coexistence region, and given that large values of S favour X speakers, when $S > 0.5$ then $x^* > 0.5$ and vice-versa. For $a > 1$, the stable solutions are those of dominance ($x = 1$) and extinction ($x = 0$). In summary, the fact that agents are highly volatile for $a < 1$, favours language coexistence, and on the contrary, in the low volatility regime $a > 1$, the final state is one of dominance/extinction.

We note that for neutral volatility $a = 1$, the Abrams-Strogatz model becomes equivalent to the biased voter model (Vazquez and Eguiluz, 2008; see Appendix 3.A.1). In finite systems, the ultimate state is always the dominance of one language. If $S > 1/2$ ($S < 1/2$), language X (Y) dominates, while for $S = 1/2$, the probability that a given language dominates equals the initial fraction of speakers of that language.

Complex networks. In real life, most individuals in a large society interact only with a small number of acquaintances. Therefore, we consider a network of N nodes, with a given degree distribution P_k, representing the fraction of individuals connected to k neighbours, such that $\sum_k P_k = 1$. In order to develop

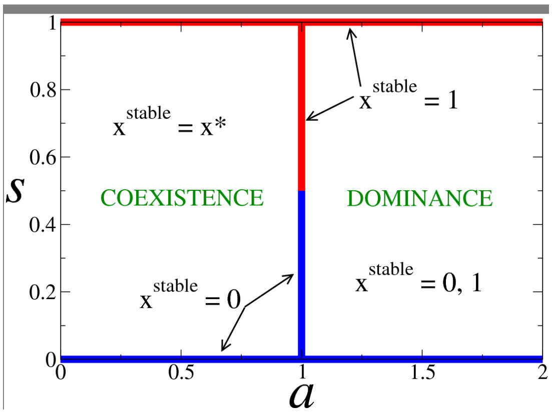

Figure 3.6: Coexistence and dominance regions of the Abrams-Strogatz model in a fully-connected network. For values of the volatility parameter $a > 1$, the stable solutions are those of language dominance, i.e., all individuals using language X ($x = 1$) or all using language Y ($x = 0$), whereas for $a < 1$ both languages coexist, with a relative fraction of speakers that depends on a and the prestige S. In the extreme case $S = 1$ ($S = 0$), only language switchings towards X (Y) are allowed, and thus only one dominance state is stable, independent of a.

a mathematical approach that is analytically tractable, we assume that the network has no degree correlations, as it happens for instance in Erdös-Renyi and scale-free networks (Albert and Barabasi, 2002)[4]. Therefore, we can see the system as composed by a collection of nodes characterized only by its degree k (number of neighbours) and state X or Y, so that nodes with the same degree and state are considered to be indistinguishable. In a time step $\delta t = 1/N$, a node i with degree k and state X (Y) is chosen with probability $P_k\, x$ ($P_k\,(1-x)$), and then, according to transitions (3.1), i switches its state with probability

$$\begin{aligned} P(X \to Y) &= (1-S)\,(n_y/k)^a\,, \\ P(Y \to X) &= S\,(n_x/k)^a\,, \end{aligned} \tag{3.7}$$

where we denote by $n_y(n_x)$ the number of neighbours of i in the opposite state $Y(X)$ $(0 \leq n_x, n_y \leq k)$.

[4]This approximation is called an *homogeneous pair approximation*. Even that our approach can be improved further in scale-free networks by using an *heterogeneous pair approximation*, we have decided to stick to an homogeneous pair approximation because it is possible, in some cases, to obtain analytical expressions for the time evolution of the different densities, and the results are in quite good agreement with the simulations in the IBMs.

Using these switching probabilities one can write down coupled equations for x and ρ. In general, these are complicated equations (see Appendix 3.A.1). As an example, the equations obtained for neutral volatility $a = 1$ are:

$$\frac{dx}{dt} = (2S-1)\frac{\rho}{2} \tag{3.8}$$

$$\frac{d\rho}{dt} = \frac{\rho}{\mu}\left\{\mu - 2 + \frac{(\mu-1)\,[S-1+(1-2S)x]\,\rho}{x(1-x)}\right\}. \tag{3.9}$$

Local effects are included in these equations through the parameter μ that measures the average number of neighbours of a node in the network: $\mu = \sum_k kP_k$. Eqs. (3.8-3.9) have three stationary solutions: the extinction and dominance solutions ($x = 0, 1$) and an extra coexistence solution $x = x^*$. One can verify numerically that these three solutions exist generally for different types of networks. Numerical integration of the general equations for different values of a and S allows us to obtain these stationary solutions and analyze their stability. In Figure 3.7 we plot the resulting stability diagram on the (a, S) plane for a degree-regular random network, that is, a network in which each node is randomly connected to a fixed number of μ neighbours (Continuous lines: $\mu = 3$, dashed lines: $\mu = 10$). The solution $x = 1$ is stable (unstable) for values of S above (below) the curve V_1^μ (solid curve), and correspondingly, the solution $x = 0$ is stable (unstable) for S below (above) the curve V_0^μ (dashed curve), while x^* is stable in the region where both $x = 0, 1$ are unstable. These results define 4 regions in the parameter space: i) Coexistence, when x^* is stable, ii) Dominance, when only $x = 1$ is stable, iii) Extinction[5], when only $x = 0$ is stable, iv) Extinction/dominance when both $x = 0$ and $x = 1$ are stable. In regions ii) and iii) the dominant solution is fully determined by prestige, while in region iv) a solution is chosen also depending on initial conditions.

The case of a fully-connected network, summarized in Figure 3.6, is recovered in the limit in which the number of links per node approaches the total number of nodes, $\mu \to N$. In this limit, the curves V_0^μ and V_1^μ approach the step functions in Figure 3.6. In comparison with that case, we observe that local effects (finite number of neighbours) give rise to the new regions ii) and iii) in which only the most prestigious language is stable. These regions become larger as μ becomes smaller. As a result, region i) gets shrunk, so that language coexistence is found to be harder to achieve in social networks with low connectivity. This is probably due to the fact that, in networks with low degree and for non-equivalent languages ($S \neq 0$), agents using the more prestigious language reduce their interaction with the minority using the endangered

[5]Regions ii) and iii) refer of course to language X. From the point of view of language Y, these regions are naturally reversed.

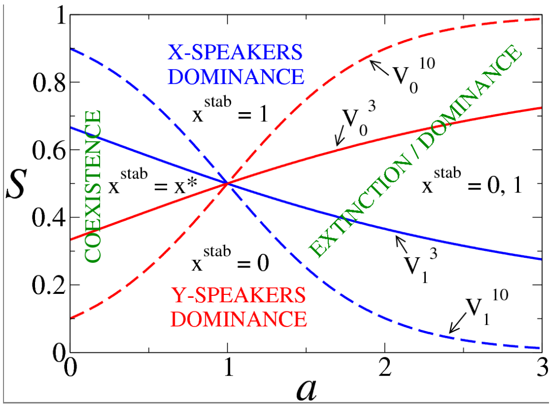

Figure 3.7: Stability diagram for the Abrams-Strogatz model on a degree-regular random network. Continuous lines correspond to $\mu = 3$ and dashed lines to $\mu = 10$. In the coexistence region the system is composed of agents using language X or Y, while in the dominance region, users of either one or the other have become extinct, depending on the initial state. We observe that the region of coexistence is reduced, compared to the model on a fully-connected network (Figure 3.6), and that there are also two single-dominance regions where the same language always dominates.

one (in comparison to the case of a fully connected network). This allows for the formation of domains which eventually grow and make it possible for the prestigious language to ultimately dominate the system. In summary, local interactions are predicted to prevent language coexistence and to reinforce the role of prestige.

Regular two dimensional network. The behaviour of the Abrams-Strogatz model on a regular two dimensional network (square lattice), as described qualitatively in Section 2, is different from its behaviour in fully connected or complex networks. There are two main reasons for these differences. First, the local interactions have an essential role, which is not accounted for in a mean field approximation appropriate for fully connected networks. And second, correlations between second, third and higher order nearest-neighbours are important in lattices and were neglected in our previous discussion of complex networks. Such correlations are essential in the formation and growth of the spatial domains discussed in Section 2. In order to characterize such phenomena, one needs to go beyond the mean field and pair approximations used above. We report here the results of a different approach (Vazquez and López, 2008) based on the derivation of a macroscopic equation for a continu-

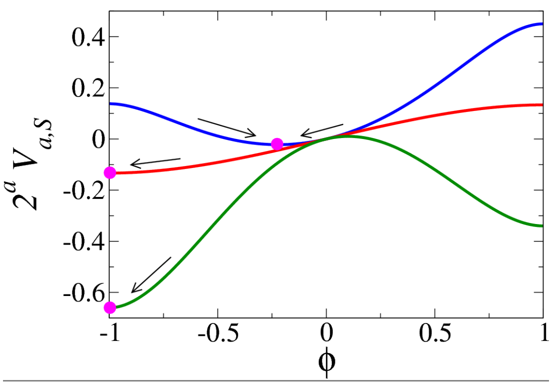

Figure 3.8: Ginzburg-Landau potential for the Abrams-Strogatz model with prestige $S = 0.6$ and values of volatility $a = 0.1, 1.0$ and 3.0 (from top to bottom). Potentials are multiplied by the factor 2^a to show them in the same scale. Arrows show the direction of the field towards the stationary solution (solid circles). For $a = 0.1$ the minimum is around $\phi \simeq -0.25$, indicating that the system relaxes towards a partially ordered stationary state (coexistence), while for $a = 1.0$ and 3.0, it reaches the complete ordered state $\phi = -1$ (dominance).

ous field $\phi_{\mathbf{r}}(t)$ accounting for a space and time coarse grained evolution of the density of users of a language. At the spatial point $\mathbf{r}$, ϕ varies continuously ($-1 < \phi < 1$)) so that $\phi = -1$ corresponds to local dominance of language X, $\phi = 0$ corresponds to coexistence with equal strengths of local populations of users of language X and Y and $\phi = +1$ corresponds to local dominance of language Y.

It can be shown (see Appendix 3.A.2) that the time evolution of $\phi_{\mathbf{r}}(t)$ can be written in the form of a time dependent Ginzburg-Landau equation

$$\frac{\partial \phi_{\mathbf{r}}(t)}{\partial t} = D(\phi_{\mathbf{r}})\Delta\phi_{\mathbf{r}} - \frac{\partial V_{a,S}(\phi_{\mathbf{r}})}{\partial \phi_{\mathbf{r}}}. \tag{3.10}$$

This can be thought of as a reaction-diffusion equation. The diffusion term, with diffusion coefficient $D(\phi_{\mathbf{r}})$ in front of the Laplacian operator Δ, accounts for local spatial coupling, while the reaction term accounts for global overdamped motion in the potential $V_{a,S}(\phi_{\mathbf{r}})$. The form of this potential gives a basic understanding of the qualitative role of the prestige and volatility parameters previously discussed in the simulations reported in Section 2.

From the general form of $V_{a,S}(\phi_{\mathbf{r}})$ shown in Figure 3.8, the dominant effect of prestige becomes clear: For the asymmetric prestige case $S \neq 1/2$ the ordering dynamics is strongly determined by S. When $a > 1$ (low volatility), $V_{a,S}$ has the shape of a double-well potential with minima at $\phi = \pm 1$, and with

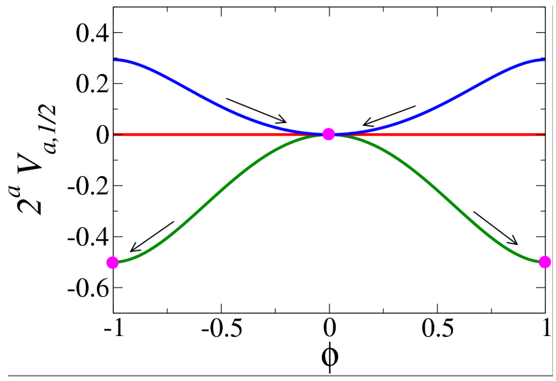

Figure 3.9: Ginzburg-Landau potential Eq. (3.11) for the symmetric case $S = 1/2$ of the Abrams-Strogatz model, with volatility values $a = 0.1, 1.0$ and 3.0 (from top to bottom). For $a = 0.1$ (high volatility) the system relaxes to a state of coexistence with the same fraction of users of language X and Y uniformly distributed over the space, corresponding to the minimum of the potential at $\phi = 0$, while for $a = 3.0$ (low volatility) it reaches a dominance/extinction state, described by the minima of the field at $|\phi| = 1$.

a well deeper than the other. Thus the system is quickly driven by dominant prestige towards the lowest minimum, reaching the dominance state in a rather short time. This is the situation seen in the left panel of Figure 3.3. On the other hand, for $a < 1$ (high volatility) there is a minimum at $|\phi| < 1$, thus the system relaxes to a state of language coexistence with unequal number of users of language X and Y as the one seen in the left panel of Figure 3.5.

In the case of symmetric prestige $S = 1/2$ the explicit form of the potential is (see Figure 3.9)

$$V_{a,1/2}(\phi_{\mathbf{r}}) = 2^{-a}(a-1)\left\{-\frac{\phi_{\mathbf{r}}^2}{2} + [6-(a-2)(a-3)]\frac{\phi_{\mathbf{r}}^4}{24} + (a-2)(a-3)\frac{\phi_{\mathbf{r}}^6}{36}\right\}. \tag{3.11}$$

In this case, with a neutral role of prestige, the role of the volatility parameter becomes even more clear: when $a < 1$ (high volatility) the system relaxes to the minimum of the potential at $\phi = 0$. In this minimum, the average field in a small region around a given point $\mathbf{r}$ is zero, indicating that the system remains in a coexistence state with the same average number of users of language X and Y randomly distributed in space. This is the situation observed in the left panel of Figure 3.4 after the system has reached a stationary active configuration. For $a > 1$ (low volatility) the potential has two wells with minima at $\phi = \pm 1$ corresponding to the states of dominance or extinction, but with the same depth. Thus there is no preference for any of the two states, and either minimum of the potential is achieved through the formation and growth of linguistic domains: small domains tend to shrink and disappear while large domains tend to grow, reducing the curvature of the linguistic boundaries (Castelló et al., 2006). This situation is the one observed in the simulation in the left panel of Figure 3.2. For the special case $a = 1$ (neutral volatility) the potential is flat: $V_{1,1/2} = 0$. There is still growth of linguistic domains but now the motion of linguistic boundaries is driven by noise, as observed in the left panel of Figure 3.1.

3.2 The Bilinguals Minett-Wang model

We consider now the main effect of introducing bilingual agents by studying the MW-model in the simplest case of a fully connected network. In these networks, local densities of neighbours in the different states agree with their global densities. Thus, using the transition probabilities Eqs. (3.2, 3.3), the rate equations for the global densities x and y can be written as

$$\begin{aligned} \frac{dx}{dt} &= Sz(1-y)^a - (1-S)xy^a, \\ \frac{dy}{dt} &= (1-S)z(1-x)^a - Syx^a, \end{aligned} \qquad (3.12)$$

where the global density of bilingual agents is $z = 1-y-x$. The mathematical analysis of these equations is more conveniently done choosing $m = y - x$ and z as independent variables.

One can verify that the points $(m = \pm 1, z = 0)$ in the (m,z) plane are two stationary solutions corresponding to extinction/dominance. But there is also a third non-trivial stationary solution of coexistence, that for the symmetric case $S = 1/2$ occurs for $m = 0$ and a particular value of $z = z^*$. As in the Abrams-Strogatz model, we expect that for a given S, a transition appears at some value a_c of the volatility parameter, where the stability of the stationary solutions changes. By doing a small perturbation around the coexistence solution $(0, z^*)$ in the z direction, one finds that this solution is stable for all values of a. Instead, the stability in the m direction changes at some value a_c which is found to be determined by

$$a_c \ln\left(\frac{1-a_c}{a_c}\right) = \ln\left(\frac{2a_c - 1}{1-a_c}\right), \qquad (3.13)$$

whose solution is $a_c \simeq 0.631$. Then, as in the Abrams-Strogatz model, the (a,S) plane is divided into two regions, but the value of the neutral volatility $a = 1$ is here replaced by a_c. In the high volatility region $a < a_c$, the coexistence solution $(0, z^*)$ is stable, while in the low volatility region $a > a_c$, the stable solutions $(\pm 1, 0)$ indicate the ultimate dominance of one of the languages and the extinction of the other. Since the transition value $a_c \simeq 0.631$ is smaller than the value $a_c = 1$ for the Abrams-Strogatz transition, the region for coexistence is reduced. This has a striking consequence. Suppose that there is population with no bilingual agents, and characterized by a volatility $a = 0.8$, that allows the stable coexistence of the two languages. If now the behaviour of the individuals is changed, so that there exist bilingual agents, language coexistence is lost and finally the system approaches a state with complete dominance of one language. In other words, bilingual agents hinder language coexistence.

4. Viability and resilience of languages in the Abrams-Strogatz model

In the previous section, we showed that the mean field description for the AS-model (see Eq.(3.5)) predicts (i) the extinction of one of the language for $a \geq 1$, whatever the value of prestige S is, and (ii) the safe coexistence of a bilingual society when $a < 1$, except when one language is already extinct. Nevertheless, Abrams-Strogatz's paper finished with the following remarks (Abrams and Strogatz, 2003):

> Contrary to the model's stark prediction, bilingual societies do, in fact, exist. [...] The example of Quebec French demonstrates that language decline can be slowed by strategies such as policy-making, education and advertising, in essence increasing the status of an endangered language. An extension to [the model] that incorporates such control on s through active feedback does indeed show stabilization of a bilingual fixed point.

In this section, we give evidence of this statement by introducing the institutional capacity to modify the prestige of one language. We consider three values of the volatility parameter: $a = 0.2$, 1 and 2. Note that in the case $a = 0.2$ (in general for $a < 1$), the fixed point corresponding to coexistence of the two languages is stable, and thus no control parameter on S needs to be included to stabilize a bilingual fixed point. However, when the difference in the prestige of the two languages is very large, the fixed point might lay outside the constraint set, leading to a situation of coexistence with one of the languages close to extinction (a situation that we may want to avoid).

4.1 Language viability

Stating the viability problem. In general, when defining the viability constraint set in the case of language competition, in order to characterize a language as endangered, the fraction of people speaking it is not enough: other crucial aspects include the point at which children no longer learn the language as their mother tongue, and the increase of the average age of speakers (in an endangered language, eventually only older generations speak the language). However, these factors are out of the scope of the current approach, and we will assume in this work, as a first approximation, that a fraction of speakers below a critical value becomes an endangered situation. Building up from this point, in the Abrams-Strogatz model, we want to determine all the pair values of density of speakers and language prestige which allow for the coexistence of the two languages. The viability constraint set is defined by setting minimal and maximal thresholds on the density of speakers. Below the minimal threshold, $\underline{x}$, or above the maximal threshold, $\bar{x}$, we consider that language X, or Y respectively, is endangered, meaning that the system is not viable. We set $\bar{x} = 1 - \underline{x}$ such that there is no need to consider explicitly language Y: if the density of speakers x of language X is outside the constraint set, so does the density of speakers of language Y, $1 - x$.

As advocated in Abrams and Strogatz (2003), we introduce prestige S as the control variable. The enhancement of the prestige of an endangered language can be triggered by political actions such as the increase of the prestige, wealth and legitimate power of its speakers within the dominant community, the strong presence of the language in the educational system, the possibility that the speakers can write their language down, and the use of electronic technology by its speakers (Crystal, 2000). The computation of the viability kernel for the Abrams-Strogatz model will allow to answer questions like: for a given density of speakers, are there action policies performed in favour of the endangered language that will keep the safe coexistence of the two languages? If the answer is yes, which are convenient policies? The main advantage of using viability theory is that it provides general tools and methods to determine the set of initial densities of speakers for which it is possible to control the system so that the coexistence is ensured.

We suppose here that the prestige can take any value $S \in [0,1]$ but the action on the prestige is not immediate: the time variation of the prestige $\frac{dS}{dt}$ is bounded by a constant denoted c. This bound reflects that changes in prestige take time: to reach a prestige value S_1 starting from an initial prestige $S_0 < S_1$, the stakeholder will have to anticipate at least $\frac{S_1 - S_0}{c\Delta t}$ time steps, where c is the maximum change per unit time Δt. The viability problem consists of defining a function u of time, which maintains the dynamical system:

$$\begin{cases} \dfrac{dx}{dt} = x(1-x)\left[Sx^{a-1} - (1-S)(1-x)^{a-1}\right] \\ \dfrac{dS}{dt} = u \\ u \in [-c, +c] \ ; \ c \in [0,1] \end{cases} \tag{3.14}$$

inside the viability constraint set K:

$$K = [\underline{x}, \bar{x}] \times [0,1]. \tag{3.15}$$

Notice that, for simplicity, we illustrate the application of the viability theory using the Abrams-Strogatz model on a fully connected network. The first step is to determine the viability kernel $Viab(K)$, defined by all couples (x,S) that are solutions of the system, Eq. (3.14), for which there exists at least one control function keeping the system indefinitely inside the viability constraint set defined by Eq. (3.15).

Computation of the viability kernel. We assume that the critical threshold of the density of speakers is 20% of the size of the whole population. Therefore, the viability constraint set is $K = [0.2, 0.8] \times [0, 1]$. The theoretical boundaries of the viability kernel can be computed analytically (Chapel et al., 2010). In addition to the theoretical boundaries, we approximate the viability kernel using the algorithm described in Chapter 7, that considers the dynamics in discrete time Δt. Figure 3.10 shows the analytical and approximated viability kernels of the system for $a = 0.2$, 1, and 2. The line corresponding to the fixed points of the dynamics has been obtained using Eq. (3.5).

We set the maximal change of prestige per unit time $c = 0.1$, which means that the time variation of the prestige cannot be higher than 10% in a time step. The figure shows how for states with a low X or Y-speakers density, the prestige associated with this language must be strong enough to maintain viability. In situations where the density of one language is high, smaller values of its associated prestige also give rise to viable situations. On the contrary, non-viable states correspond to situations where the density of one language and its associated prestige are low at the same time. In this case, if the actions in favour of this language come too late, its density of speakers will get below the critical threshold 20% while the other will spread through the majority of the population (above 80%). As a increases, the viability kernel shrinks. Indeed, the higher the parameter a, the more rarely agents change their language (low volatility regime). The impact of the change on the prestige is then lower as a increases, which means that when a language is close to the boundary of the viability kernel, even with the maximal government action, the effect on the density of speakers will be too slow to avoid leaving the viability constraint set. On the contrary, as a decreases, agents are likely to change their language (high volatility regime) and to restore coexistence. Note that for $a = 0.2$, the viability kernel is not the whole constraint set: non-viable states reach a stable fixed point located outside K.

Determining heavy viable trajectories. The control procedure models an action to enhance the prestige of an endangered language, and we assume that such an action is costly. Therefore, if among different possible action policies to maintain language coexistence, doing nothing keeps the system in a viable situation, we assume that this strategy will be chosen in order to reduce costs. In other words, we suppose that, if several situations with $-c \leq u \leq c$ lead to viable situations, the best choice is $u = 0$. The principle of the control algorithm is described in detail in Chapter 7. Figure 3.11 presents some examples

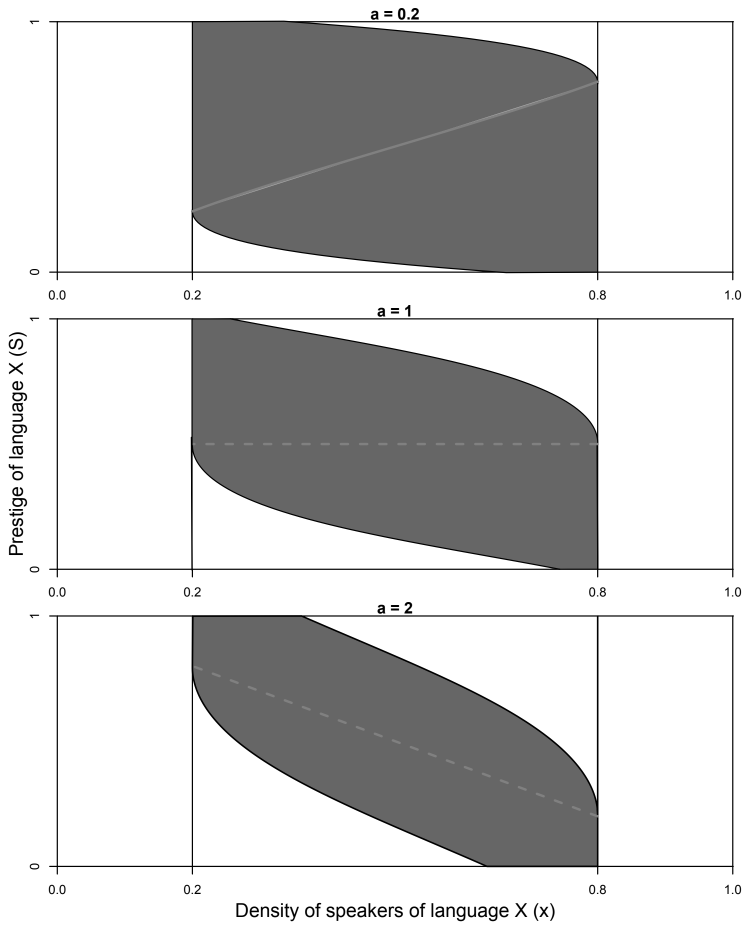

Figure 3.10: Viability kernel for the Abrams and Strogatz Model, with $c = 0.1$ and $\Delta t = 0.05$. From top to bottom: $a = 0.2$, 1, and 2. The continuous black lines represent the theoretical curves of the viability kernel, and the area in grey the approximation. The continuous grey line represents stable fixed points and the dotted light grey line unstable fixed points.

of trajectories for three different values of a, and the time evolution of the control ($c = 0.1$), during 750 time steps. For $a < 1$, there exist stable fixed points corresponding to coexistence of the two languages and the dynamics settles there, keeping $u = 0$ along the trajectory. For $a \geq 1$, there are no stable fixed points inside the viability kernel, and the control procedure must be applied at each time step. As long as the trajectory is far away from the kernel's boundary, the control is kept to zero ; when it approaches the boundary, the control that brings the system away from the boundary corresponds to the maximum value of the control with the appropriate sign, $\pm c$.

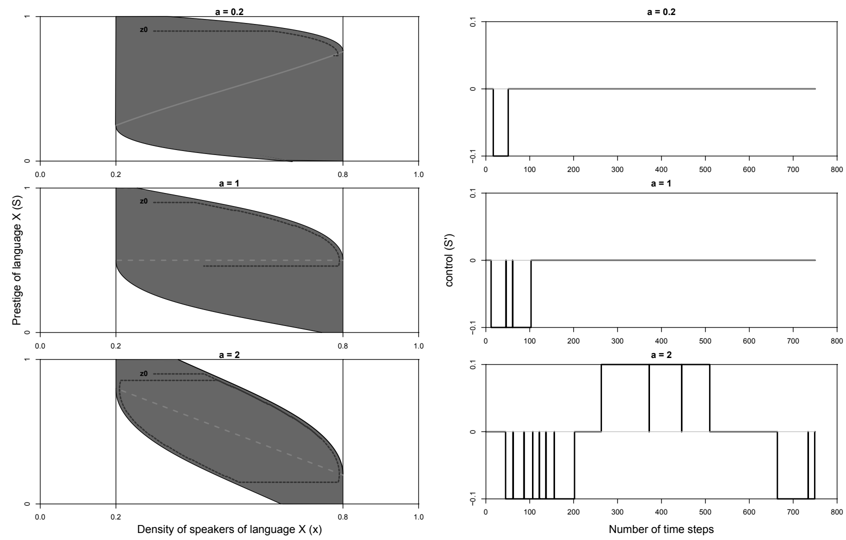

Figure 3.11: (Left side) Examples of trajectories (in dotted dark grey) starting from an initial state $z_0 = (x_0, S_0)$ for three values of a ($a = 0.2$, $a = 1$, $a = 2$) and (right side) evolution of the control, with $c = 0.1$. The continuous light grey line represents stable fixed points and the dotted light grey line unstable fixed points.

4.2 Language Resilience

In the previous subsection, we studied the viability of the AS-model, supposing that one language is endangered when its density of speakers goes below a critical value. However, being endangered does not necessarily mean that the language will disappear. In this section, we are interested in how to maintain or restore coexistence of the two languages when the system is in danger, meaning that a disturbance pulls it outside the viability constraint set.

As we pointed out in the introduction, resilience is the capacity of a system to restore its properties of interest, lost after disturbances. In this section, we define resilience of system Eqs. (3.14) and (3.15) by considering its capacity to return into its viability kernel when a perturbation pulls it out from it, following Martin's definition of resilience (Martin, 2004).

Stating the Resilience Problem. We are interested in situations of crisis, which take place when the system leaves the viability constraint set. We distinguish two types of states located outside the viability kernel:

- States for which there exists at least one evolution driving back the system to the viability kernel after leaving the constraint set are called resilient. The system is resilient to a perturbation which leads it into a resilient state;
- States for which irrespective of the control policy applied, the system remains outside the viability kernel, are called non-resilient. The system is not resilient to perturbations leading the system into a non-resilient state.

For states located inside the viability kernel, the resilience is infinite. Martin (2004) also introduces the notion of cost of restoration. This cost measures the distance between the evolution of the state of the system and the property of interest (i.e. being inside the viability kernel). Its definition must fulfil three conditions. First, the cost of an action which keeps the property of interest indefinitely is zero: maintaining this property may lead to some action update, but they are not taken into account in the cost computation. Second, when the property of interest cannot be restored, the cost of restoration is infinite. Third, when the property can be restored, the cost is finite. It is often defined by the minimum time the system is outside the viability kernel or the minimal deficit accumulated along the trajectory. Then, the resilience is the inverse of the restoration cost of the properties of interest lost after disturbances. The trajectory starting from (x, S) with a minimal cost defines the sequence of "best" action policies to perform, and thus defines the resilience value. Resilience values can be approximated numerically using the algorithm described in chapter 7. In the context of language competition, the use of viability theory provides a measure of the cost associated to a policy action which will favour an endangered language.

Determining the Resilience basin. All the states can undergo a disturbance. For instance, immigration: people speaking language X are exiled to another country, hence the density of X-speakers reduces dramatically in the home country, and increases in the destination country. Another perturbation to the system can be due to an abrupt change in the prestige of a language because of political actions such as invasion, occupation, etc. The states resulting from disturbances might bring the system outside the constraint set, leading to situations where the density of speakers is lower than the minimal threshold or higher than the maximal threshold. Thus, we consider now the set of all the possible situations $H = [0, 1] \times [0, 1]$, and we study the resilience of the system in H.

First, we determine the set of states of infinite resilience, that are the states located inside the viability kernel of the system defined by Eq. (3.14) associated to constraint set defined by Eq. (3.15). It corresponds to the dark grey area on Figure 3.12. Then, we look for all the states for which at least one evolution drives the system back to the viability kernel after spending a finite time in the critical area $H \backslash K$ (where $E \backslash F$ is the complementary set of the set F in the set E). These are the resilient states, represented in light grey in Figure 3.12. Note that states located in $K \backslash Viab(K)$ can have a finite resilience: when coming back towards $Viab(K)$, the trajectory leaves the constraint set and reaches $Viab(K)$ after spending some time in the critical area. The states that, irrespective of the applied policy, remain outside the viability kernel are in the white zone. For these states, the desired level of language coexistence is impossible and resilience is zero (given the assumed value of c, which limits the effect of action).

In Figure 3.12, we show the resilient and non-resilient states for $a = 0.2$, 1, and 2. For small values of a, all the states are resilient, except $x = 0$ and $x = 1$, irrespective of the value of S. As we pointed out previously, the fixed point corresponding to coexistence is stable for $a < 1$. Therefore, the desired level of coexistence for the two languages is ensured or can be reached, irrespective of their initial density of speakers and their prestige, except when a perturbation leads to a situation where one language is already extinct. For $a = 1$, nearly for all the initial density of speakers and prestige, reaching the desired level of languages coexistence is possible, except if the initial state represents a large density of speakers of language X that has, at the same time, high prestige (language Y becomes extinct, irrespective of the action applied) or vice versa. For $a > 1$, the set of resilient states becomes smaller, as can be seen in Figure 3.12. The larger the value of a, the smaller the set of resilient states is. Indeed, as mentioned before for the shrinking of the viability kernel, a high value of a means that agents rarely change their language and the effects of increasing or decreasing the prestige of a language become less effective.

Computing Resilience Values. There exist several ways of defining a cost of restoration, depending on the situation and the point of view. As we pointed out previously, the resilience value is then defined as the inverse of its restoration cost. On the one hand, if the time needed to restore viability is the only ingredient under consideration, the cost value is then the time the system is outside the viability kernel. The cost function C_1 that associates to a state x the minimal cost of restoration among all the trajectories starting from z is defined by:

$$\begin{array}{l} C_1(x) = \min_{z(.)} \left(\int_0^{+\infty} \chi_{z(t) \notin Viab(K)} dt \right) \\ \text{and } \chi_{z(t) \notin Viab(K)} = 1 \text{ when } z(t) \notin Viab(K) \text{ and } 0 \text{ otherwise,} \end{array} \quad (3.16)$$

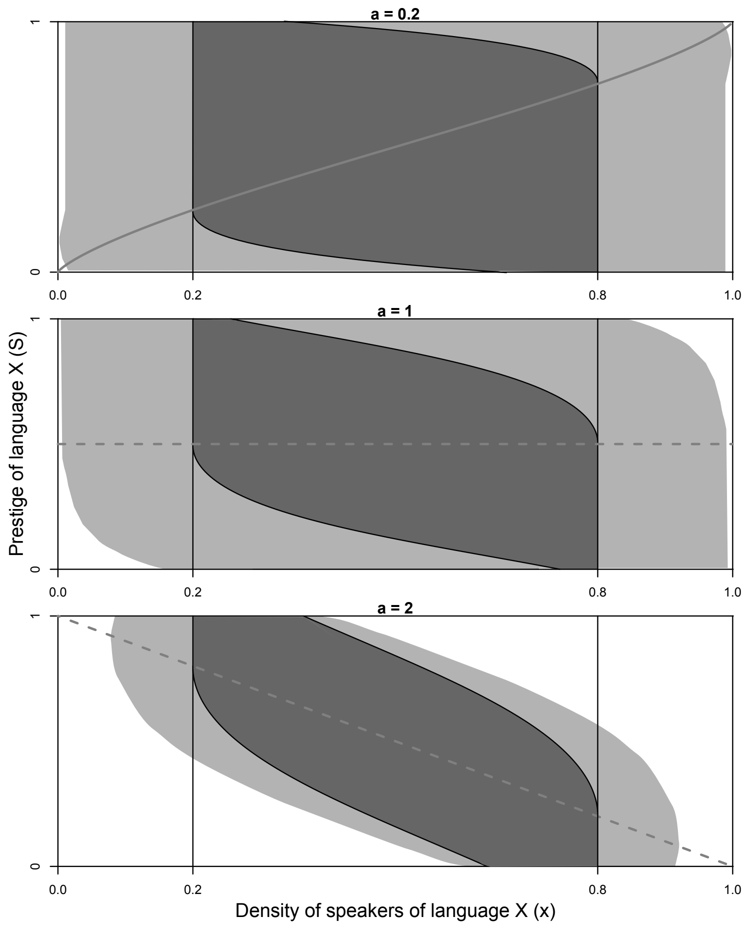

Figure 3.12: Resilient (in grey) and non resilient states (in white) in the model associated to the AS-dynamics (3.14) with constraint set (3.15), for three values of a: $a = 0.2$, $a = 1$, $a = 2$. Viability kernel is in dark grey.

where z represents the state (x, S), $z(t)$ is the state at time t and $z(.)$ is the trajectory starting from this state. In this way, the cost value is zero when the system is inside the viability kernel. On the other hand, if the cost also depends on how far the system is from reaching the constraint set, the cost function is composed of two terms: the first one that accounts for the time the system is not viable, and the second one, representing the distance to the viability constraint set. In this way, the cost function C_2 associates the time of restoration and the measure of the density of speakers above or below the thresholds of the viability constraint set:

$$C_2(x) = \min_{z(.)} \left(\int_0^{+\infty} \chi_{z(t)\notin Viab(K)} dt + c_2 d(z(t),K)\chi_{z(t)\notin K} dt\right) \\ \text{and } \chi_{z(t)\notin K} = 1 \text{ when } z(t) \notin K \text{ and 0 otherwise,} \tag{3.17}$$

where $d(z(t),K) = \max(\underline{x} - x(t), x(t) - \overline{x})$ measures the distance between the density $x(t)$ at time t and the density thresholds. Equation (3.17) takes into account that the cost of restoration of a state near extinction is larger than the one for states located near the boundary of K. The parameter c_2 reflects the relative weight of each cost, fixing the cost of being far from K relatively to the time spent outside the viability kernel.

Figure 3.13 compares resilience basins for the Abrams-Strogatz model for different values of a, and for the two cost functions defined above (with an arbitrary cost parameter $c_2 = 20$ for the second cost function). The difference of cost between two iso-cost curves is 4.8, and therefore the difference in resilience is $\frac{1}{4.8} \approx 0.2$ (the 4.8 value is arbitrary and is linked to the parametrization of the algorithm in chapter 7). The darker the line, the lower the cost value is (and hence the higher the resilience value). In the white area, cost is infinite, meaning that restoring coexistence of both languages is impossible. For $a = 0.2$, the maximal cost of restoration is equal to 4.8 for cost function C_1 defined by Eq. (3.16) and 19.2 for the cost C_2 defined by Eq. (3.17). The cost associated to the function defined by Eq. (3.17) is bigger than the one associated with Eq. (3.16) because it introduces an additional part (the distance to viability) on the final cost. For $a = 1$, the maximal cost of restoration is more important (14.4 for Eq. (3.16) and 62.4 for Eq. (3.17). For $a = 2$, the resilient zone is smaller and the costs of restoration are larger (24 for Eq. (3.16) and 67.2 for Eq. (3.17)). This means that for higher values of a, where the resilient set is smaller, the cost of restoration is larger: there are less resilient situations and the action policies to be performed in order to restore viability are the most costly.

Determining Action Policies to Restore Viability at Minimum Cost. Computing resilience values is instrumental to define action policies that drive back the system inside the viability kernel. Here, we use an optimal controller instead of a heavy controller: we do not look for one action policy that keeps the system in a resilient state, but we define a sequence of actions that allows the system to return to the viability kernel at the lowest cost of restoration. It can be shown (see chapter 7) that choosing the action that decreases the cost at each step (or increases the resilience), minimizes the whole cost of restoration. Hence, theoretically this approach also provides means to compute resilient policies, which minimize the cost of restoration along the trajectory.

Figure 3.14 displays some trajectories starting from resilient states for $a =$ 0.2, 1 and 2. Considering the cost C_2 of Eq. (3.17), the controller produces

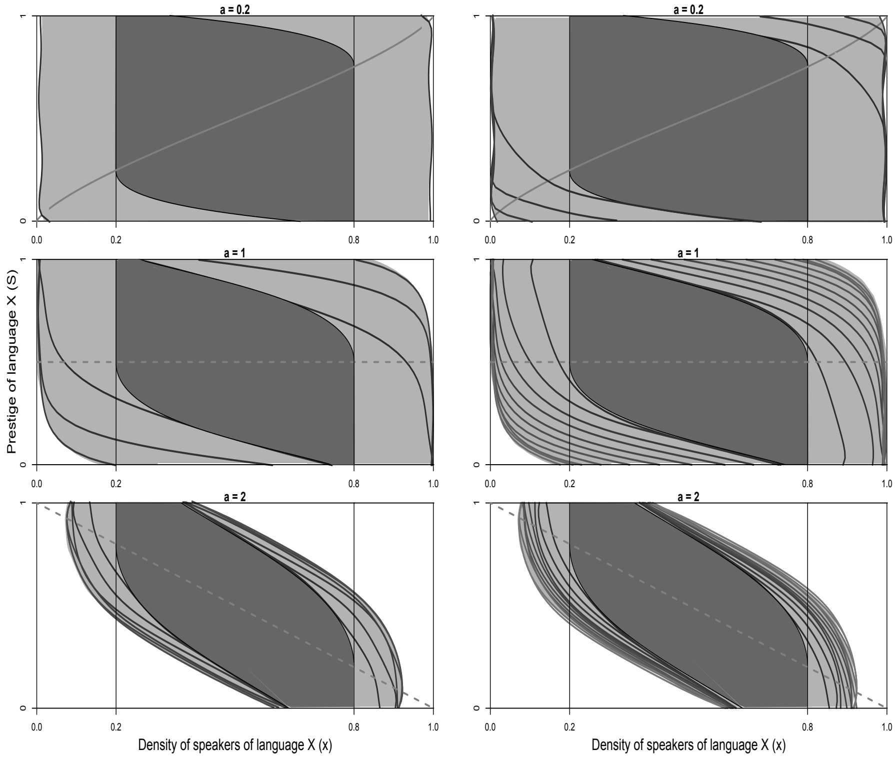

Figure 3.13: Resilience basins of the Abrams and Strogatz model. In dark grey, the viability kernel; between the level lines (light grey area), the cost of restoration is finite (one level line corresponds to a cost of 4.8 and the lighter the line, the higher the cost); in the white area, the cost is infinite and the resilience null. (Left side) cost function defined by Eq. (3.16) (Right side) cost function defined by Eq. (3.17).

a trajectory that avoids situations where the density of speakers is too small or too large, because these are the most costly. Notice that for $a = 0.2$, the trajectory first reaches the equilibrium line outside K, but in order to bring the system inside the viability kernel, the control function is chosen so that it does not get stuck on this fixed point. The procedure leads the system to a second fixed point, located this time inside the viability kernel. Even if the starting point is located inside K but outside the viability kernel (see for example case $a = 1$), the trajectory crosses the viability constraint set before going back to $Viab(K)$, as it is not possible by definition for these states to directly reach the viability kernel.

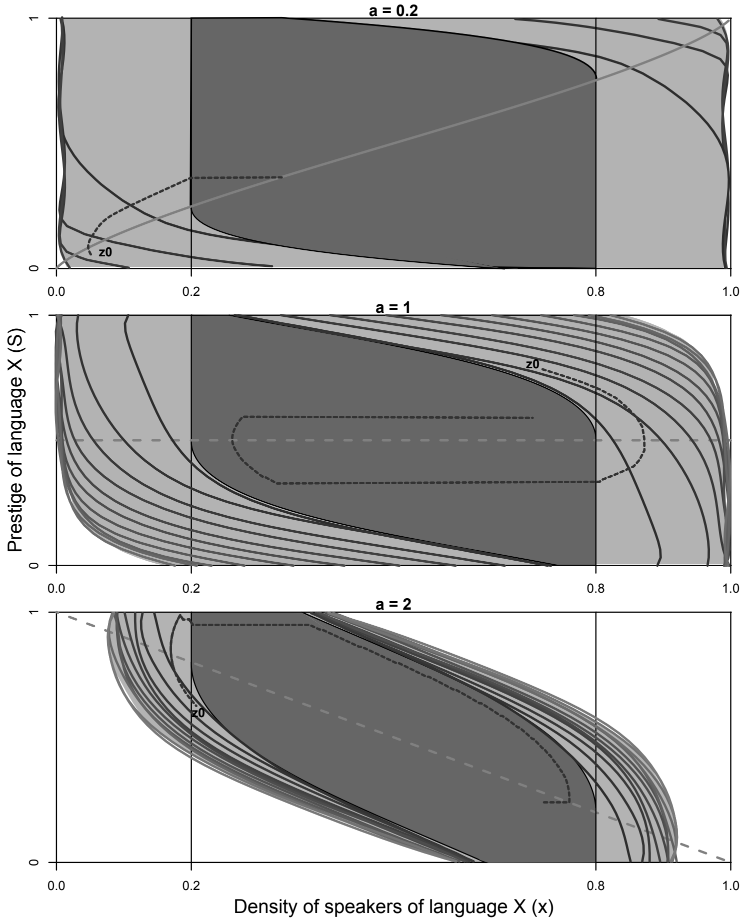

Figure 3.14: Examples of trajectories (in dotted dark grey) starting from a point z_0 during 750 time steps, that allow the system to restore its viability at the minimal cost of restoration, using cost function (3.17). The continuous grey line represents stable fixed points and the dotted light grey line unstable fixed points. Note that for an initial state $z_0 = (x_0, S_0)$ located inside K but outside $Viab(K)$, the trajectory crosses the viability constraint set boundaries before reaching $Viab(K)$.

5. Conclusions

In this chapter, we apply the global approach developed in the PATRES project to the particular case of language competition models. We start with individual based models where explicitly represented agents interact and change their practice of language according to the behaviour of their neighbours. With systmatic simulations, we explore the richness of dynamical patterns that such models can produce, in particular in the case when the network of interactions is a regular grid. We found that the parameter called 'volatility', expressing the propensity of agents to change the language they are currently using, is particularly sensitive: the dynamical patterns can change dramatically when the volatility goes through a critical value. Then we derive macroscopic descriptions of these dynamics, which capture the main features of these patterns. We consider different networks of interactions between the agents for the simplest model (AS), and we propose specific approaches to make this derivation. This step is important to get a better understanding of the patterns. It is also a necessary step in the global approach for computing viability and resilience. Indeed, as explained in more details in chapter 7, the tool computing viability and resilience requires that the dynamical system is described with a small set of differential equations involving a limited number of variables. We illustrate such a computation on the simplest model of language competition the AS model in the case of total connection. In this setting, we suppose that the prestige of the language can be modified to some extent, as the result of some promotion policy. The analysis shows that the policy to apply for maintaining or restoring the diversity of languages depends heavily on the volatility. When the volatility is high, the desired set does not contain any attractor, and we are in a similar case as the last one presented in chapter 2. It is necessary to act regularly to maintain the balance between the languages, otherwise one language finally gets extinct. In addition, the set of resilient states is smaller. The model of this example is relatively simple, but nevertheless we get a situation where the usual definition of resilience based on attractors would not be applicable. In the next chapters, we study other examples of this approach, applied to other fields.

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Appendix: Derivation of macroscopic equations

1. The case of neutral volatility $a = 1$: the biased voter model

In order to illustrate how one can derive and interpret equations for the macroscopic evolution of the system, we consider in this Appendix some mathematical details on different network topologies, with special reference to the simple case $a = 1$ which is also known as the biased voter model.

1.1 Fully-connected networks

In general, and given switching probabilities $P(X \to Y)$ and $P(Y \to X)$, one has that in the case that the switch occurs, the density x is reduced by $1/N$, thus the average change in the density of X-speakers can be described by the following rate equation

$$\frac{dx}{dt} = \frac{1}{1/N}\left[(1-x)P(Y \to X)\frac{1}{N} - xP(X \to Y)\frac{1}{N}\right]. \tag{3.A.1}$$

Using the transition probabilities (3.4) in Eq. (3.A.1) one arrives to equation (3.5). For $a = 1$ and $S = 1/2$, the transition rates in Eqs. (3.4) become linear in the densities x and y

$$\begin{aligned} P(X \to Y) &= \frac{1}{2}y, \\ P(Y \to X) &= \frac{1}{2}x. \end{aligned} \tag{3.A.2}$$

Thus apart from the constant prefactor $1/2$, the dynamics is equivalent to adopting the state of a randomly chosen neighbor, that is reminiscent of the voter model (Liggett, 1985). If $S \neq 1/2$, the preference for one of the languages makes the voter model to be biased in one direction. For $a = 1$, equation (3.5) becomes the well known logistic or Verhulst equation

$$\frac{dx}{dt} = (2S-1)x(1-x), \tag{3.A.3}$$

whose solution is

$$x(t) = \frac{x_0}{x_0 + (1-x_0)e^{-(2S-1)t}}, \tag{3.A.4}$$

with $x_0 = x(t=0)$. For a uniform initial condition, $x_0 = 1/2$. Thus

$$x(t) = \frac{1}{2}\Big\{1 + \tanh\left[(S-1/2)t\right]\Big\}, \tag{3.A.5}$$

and

$$\rho(t) = \frac{1}{2}\Big\{1 - \tanh^2\left[(S-1/2)t\right]\Big\}. \tag{3.A.6}$$

The analytical solutions from Eqs. (3.A.5) and (3.A.6) agree very well with the results from numerical simulations of the model with $S \neq 1/2$, for large enough systems. This is so, because finite-size fluctuations effects are negligible compared to bias effects, even for a small bias (Stauffer et al., 2007).

For $S = 1/2$, the bias is exactly zero, and one obtains that in an infinite large network $dx/dt = 0$, thus x and ρ are conserved. However, in a finite network, fluctuations lead the system to one of the dominance states. To find how the system relaxes to the final state, one needs to calculate the evolution of the second moment $\langle x^2 \rangle$, related to the fluctuations in x, where the symbol $\langle\ \rangle$ represents an average over many realizations. This leads to a decay of the average density of opposite-state links of the form (see Vazquez and Eguiluz, 2008)

$$\langle \rho \rangle = \langle 2x(1-x) \rangle = \langle \rho(0) \rangle \, e^{-2t/N}. \tag{3.A.7}$$

1.2 Complex networks

Given the switching probabilities of Eqs. (3.7), if the switch occurs, the density x is reduced by $1/N$, while the density ρ changes by $2(k-2n_y)/\mu N$. Therefore, in analogy to the case of fully connected networks, but now plugging the transition probabilities from Eqs. (3.7) into Eq. (3.A.1), we write the average change in the density of X speakers as

$$\begin{aligned} \frac{dx}{dt} &= \sum_k \frac{P_k(1-x)}{1/N} \sum_{n_x=0}^{k} B(n_x,k) S \left(\frac{n_x}{k}\right)^a \frac{1}{N} \\ &- \sum_k \frac{P_k x}{1/N} \sum_{n_y=0}^{k} B(n_y,k)(1-S) \left(\frac{n_y}{k}\right)^a \frac{1}{N}, \end{aligned} \tag{3.A.8}$$

and similarly, the change in the density of opposite-state links as

$$\begin{aligned}\frac{d\rho}{dt} &= \sum_k \frac{P_k\,(1-x)}{1/N} \sum_{n_x=0}^{k} B(n_x,k) S \left(\frac{n_x}{k}\right)^a \, \frac{2(k-2n_x)}{\mu N} \\ &+ \sum_k \frac{P_k\, x}{1/N} \sum_{n_y=0}^{k} B(n_y,k)(1-S) \left(\frac{n_y}{k}\right)^a \, \frac{2(k-2n_y)}{\mu N}, \end{aligned} \tag{3.A.9}$$

where we denoted by $B(n,k)$, the probability that a node of degree k and state $X(Y)$ has n neighbors in the opposite state $Y(X)$.

Defining the a-th moment of $B(n_x,k)$ as

$$\langle n_x^a \rangle_k \equiv \sum_{n_x=0}^{k} B(n_x,k) n_x^a,$$

and similarly for $B(n_y,k)$, we arrive to the equations

$$\frac{dx}{dt} = \sum_k \frac{P_k}{k^a} \left[S(1-x)\langle n_x^a \rangle_k - (1-S)x \langle n_y^a \rangle_k \right], \tag{3.A.10}$$

$$\begin{aligned}\frac{d\rho(t)}{dt} = \sum_k \frac{P_k}{\mu\, k^a} \Big\{ & S(1-x) \left[k\langle n_x^a \rangle_k - 2\langle n_x^{(1+a)} \rangle_k \right] + \\ & (1-S)x \left[k\langle n_y^a \rangle_k - 2\langle n_y^{(1+a)} \rangle_k \right] \Big\}. \end{aligned} \tag{3.A.11}$$

In order to develop an intuition about the temporal behavior of x and ρ from Eqs. (3.A.10) and (3.A.11), we analyze the simplest case $a = 1$. A rather complete analysis of the time evolution and consensus times of this model on uncorrelated networks, for the symmetric case $S = 1/2$, can be found in (Vazquez and Eguiluz, 2008). Following a similar approach, here we study the general situation in which the prestige S takes any value. To obtain close expressions for x and ρ, we use the fact that in uncorrelated networks dynamical correlations between the states of second nearest neighbors vanish, and also the system is "well mixed", in the sense that the different types of links are uniformly distributed over the network. Therefore, we assume that the probability that a link picked at random is of type xy is equal to the global density of xy links, ρ. Then, $B(n_x,k)$ becomes the binomial distribution with

$$P(x|y) = \rho/2y \tag{3.A.12}$$

as the single event probability that a neighbor of a node with state y has state x. $P(x|y)$ is calculated as the ratio between the total number of links $\rho\mu N/2$ from nodes in state y to nodes in state x, and the total number of links $Ny\mu$ coming out from nodes in state y. Taking $a = 1$ in Eqs. (3.A.10) and (3.A.11), and replacing the first and second moments of $B(n_x,k)$ by

$$\begin{aligned}\langle n_x \rangle &= P(x|y)k, \\ \langle n_x^2 \rangle &= P(x|y)k + P(x|y)^2 k(k-1),\end{aligned}$$

leads to the two coupled closed Eqs. (3.8)-(3.9) for x and ρ.

Given that $\rho \geq 0$, equation (3.8) shows that if $S > 1/2$ ($S < 1/2$), x approaches to 1 (0), and ρ goes to zero, as expected. Even though an exact analytical solution of Eqs. (3.8) and (3.9) is hard to obtain, we can still find a solution in the long time limit, assuming that ρ decays to zero as

$$\rho(t) = A\, e^{-t/2\tau(S)}, \quad \text{for} \;\; t \gg 1, \tag{3.A.13}$$

where A is a constant given by the initial state and $\tau(S)$ is another constant that depends on S. To calculate the value of τ, that quantifies the rate of decay towards the solutions $x = 1$ or $x = 0$, we first replace the ansatz from Eq. (3.A.13) into Eq. (3.8), and solve for x. We obtain

$$x = \begin{cases} 1 + (1-2S)\tau(S)\rho & \text{if } S > 1/2; \\ (1-2S)\tau(S)\rho & \text{if } S < 1/2. \end{cases}$$

Then, to first order in ρ

$$x(1-x) = \begin{cases} (2S-1)\tau(S)\rho & \text{if } S > 1/2; \\ -(2S-1)\tau(S)\rho & \text{if } S < 1/2. \end{cases} \tag{3.A.14}$$

Replacing the above expressions for $x(1-x)$ into Eq. (3.9), we arrive to the following value of τ

$$\tau(S) = \begin{cases} \frac{\mu-2S}{2(2S-1)(\mu-2)} & \text{if } S > 1/2; \\ \frac{2(1-S)-\mu}{2(2S-1)(\mu-2)} & \text{if } S < 1/2. \end{cases} \tag{3.A.15}$$

Finally, the fraction of X speakers for long times behave as

$$x = \begin{cases} 1 - \frac{(\mu-2S)A}{2(\mu-2)} \exp\left[-\frac{(2S-1)(\mu-2)}{\mu-2S}t\right] & \text{if } S > 1/2; \\ \frac{[\mu-2(1-S)]A}{2(\mu-2)} \exp\left[-\frac{(1-2S)(\mu-2)}{\mu-2(1-S)}t\right] & \text{if } S < 1/2. \end{cases} \tag{3.A.16}$$

Using the expression for $\tau(S)$ from Eq. (3.A.15) in Eq. (3.A.14) we find that for $S = 1/2$ is $\rho = \frac{2(\mu-2)}{(\mu-1)}x(1-x)$, in agreement with previous results of the voter model on networks (Vazquez and Eguiluz, 2008). Eqs. (3.A.13) and (3.A.16) have the same form as Eqs. (3.A.6) and (3.A.5) in the long time limit, for fully connected networks. We can check that we recover that expressions by taking $\mu = N - 1 \gg 1$ in Eqs. (3.A.13) and (3.A.16). This result means that the evolution of x and ρ in the biased voter model on uncorrelated networks is very similar to the mean-field case, with the time rescaled by the constant τ that depends on the topology of the network, expressed by the mean connectivity μ. From the above equations we observe that the system reaches the dominance state $\rho = 0$ in a time of order τ. For the special case $S = 1/2$, τ diverges, thus Eqs. (3.A.13) and (3.A.16) predict that both x and ρ stay constant over time. However, as mentioned in the previous section, the absorbing state is reached by finite-size fluctuations. Taking fluctuations into account, one finds that the approach to the final state is described by the expression (Vazquez and Eguiluz, 2008):

$$\langle\rho\rangle = \frac{(\mu-2)}{2(\mu-1)}\left(1 - m_0^2\right)e^{-2t/\tau}, \tag{3.A.17}$$

where m_0 is the initial magnetization and τ is the relaxation time that depends on the system size and the first and second moments of the degree distribution.

2. Equation for the field $\phi_{\mathbf{r}}$

In order to obtain an equation for the time evolution of the field $\phi_{\mathbf{r}}(t)$ we use a standard method (Vazquez and López, 2008): We can interpret X and Y speakers as particles with spins $s = -1$ (down arrow) and $s = 1$ (up arrow) respectively. In other words, we map the language model into a spin-1/2 model, like the Ising model for ferromagnetism. Then, we define by $\phi_{\mathbf{r}}(t)$ the spin field at site $\mathbf{r}$ at time t, which is a continuous representation of the spin at that site ($-1 < \phi < 1$). This is done by placing Ω spin particles at each site of the lattice, each representing a different realization of the dynamics, and replacing $\phi_{\mathbf{r}}(t)$ by the average spin value $\phi_{\mathbf{r}}(t) \to \frac{1}{\Omega}\sum_{j=1}^{\Omega} S_{\mathbf{r}}^{j}$, where $S_{\mathbf{r}}^{j}$ is the spin of the j-th particle inside site $\mathbf{r}$. Within this formulation, the dynamics is the following. In a time step of length $\delta t = 1/\Omega$, a site $\mathbf{r}$ and a particle from that site are chosen at random. The probability that the chosen particle has spin $s = \pm 1$ is equal to the fraction of $\pm$ spins in that site $(1 \pm \phi_{\mathbf{r}})/2$. Then the spin flips with probability

$$\begin{aligned} P(- \to +) &= (1-S)\left(\frac{1+\psi_{\mathbf{r}}}{2}\right)^{a}, \\ P(+ \to -) &= S\left(\frac{1-\psi_{\mathbf{r}}}{2}\right)^{a}, \end{aligned} \tag{3.A.18}$$

where $\psi_{\mathbf{r}} \to \frac{1}{4}\sum_{\mathbf{r}'/\mathbf{r}} \phi_{\mathbf{r}'}(t)$ is the average neighboring field of site $\mathbf{r}$, and the sum is over the 4 first nearest-neighbors sites $\mathbf{r}'$ of site $\mathbf{r}$. If the flip happens, $\phi_{\mathbf{r}}$ changes by $-2s/\Omega$, thus its average change in time is given by the rate equation

$$\frac{\partial \phi_{\mathbf{r}}(t)}{\partial t} = [1-\phi_{\mathbf{r}}(t)]\,P(- \to +) - [1+\phi_{\mathbf{r}}(t)]\,P(+ \to -), \tag{3.A.19}$$

where the first (second) term corresponds to a $- \to +$ $(+ \to -)$ flip event. We have also rescaled the time by $1/\Omega$. To obtain a closed equation for ϕ, we substitute the expression for the transition probabilities Eq. (3.A.18) into Eq. (3.A.19), and write it in the more convenient form

$$\frac{\partial \phi}{\partial t} = \frac{(1-S)}{2^{a}}(1-\phi)(1+\psi)(1+\psi)^{a-1} - \frac{S}{2^{a}}(1+\phi)(1-\psi)(1-\psi)^{a-1}, \tag{3.A.20}$$

where ϕ and ψ are abbreviated forms of $\phi_{\mathbf{r}}$ and $\psi_{\mathbf{r}}$ respectively. We now replace the neighboring field ψ in the terms $(1+\psi)$ and $(1-\psi)$ of Eq. (3.A.20) by $\psi \equiv \phi + \Delta\phi$, where Δ is defined as the standard Laplacian operator $\Delta\phi_{\mathbf{r}} \equiv \frac{1}{4}\sum_{\mathbf{r}'/\mathbf{r}}(\phi_{\mathbf{r}'} - \phi_{\mathbf{r}}) = \psi_{\mathbf{r}} - \phi_{\mathbf{r}}$, and obtain

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= 2^{-a}(1-\phi^{2})\left[(1-S)(1+\psi)^{a-1} - S(1-\psi)^{a-1}\right] \\ &+ 2^{-a}\left[(1-S)(1-\phi)(1+\psi)^{a-1} + S(1+\phi)(1-\psi)^{a-1}\right]\Delta\phi. \end{aligned} \tag{3.A.21}$$

Using a Taylor series expansions around $\psi = 0$ in the right hand side of Eq. (3.A.21),

$$\begin{aligned} (1\pm\psi)^{a-1} &= 1+(a-1)\psi+\frac{1}{2}(a-1)(a-2)\psi^{2}+\frac{1}{6}(a-1)(a-2)(a-3)\psi^{3} \text{ and} \\ (1-\psi)^{a-1} &= 1-(a-1)\psi+\frac{1}{2}(a-1)(a-2)\psi^{2}-\frac{1}{6}(a-1)(a-2)(a-3)\psi^{3} \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= 2^{-a}(1-\phi^{2})\Big\{(1-2S)+(a-1)\psi+\frac{(1-2S)}{2}(a-1)(a-2)\psi^{2} \\ &+ \frac{1}{6}(a-1)(a-2)(a-3)\psi^{3}\Big\} + 2^{-a}\Big\{[1-(1-2S)\phi]\left[1+\frac{1}{2}(a-1)(a-2)\psi^{2}\right] \\ &+ (1-2S-\phi)\left[(a-1)\psi+\frac{1}{6}(a-1)(a-2)(a-3)\psi^{3}\right]\Big\}\Delta\phi \end{aligned} \tag{3.A.22}$$

We finally replace ψ by $\phi + \Delta\phi$ in Eq. (3.A.22) and expand to first order in $\Delta\phi$, assuming that the field ϕ is smooth, so that $\Delta\phi \ll \phi$. Neglecting ϕ^{3} and higher order terms in the diffusion coefficient that multiplies the laplacian, we arrive to an equation for the spin field