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On co-evolution and the importance of initial conditions

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ABSTRACT

We present a generic threshold model for the co-evolution of the structure of a network and the binary state of its nodes. We focus on regular directed networks and derive equations for the evolution of the system toward its absorbing state. It is shown that the system displays a transition from a connected phase to a fragmented phase, and that this transition is driven by the initial configuration of the system, as different initial conditions may lead to drastically different final configurations. Computer simulations are performed and confirm the theoretical predictions.

1. Introduction

The relation between a cause and its effect is usually abrupt in complex systems, in the sense that a small change in the neighborhood of a subsystem may (or may not) trigger its reaction. This mechanism is at the heart of many models of self-organized criticality [1] where a cascade starts when the system has been frustrated beyond some threshold, e.g. the angle of a sand pile, but also in models for the diffusion of ideas in social networks [2–5] where the adoption of a new idea requires simultaneous exposure to multiple active acquaintances, and in integrate-and-fire neuron dynamics [6] where the voltage on a single neuron increases until a specified threshold is reached and it suddenly fires by emitting an action potential, thereby quickly returning to its reference. These types of model consist in cascading propagations on a fixed topology, i.e., a network of some sort, until a frozen configuration is reached, but they do not incorporate the well-known feedback existing between network topology and dynamics [7–16], namely that the topology itself may reorganize when it is not compatible with the state of the nodes. This reorganisation may originate from homophily and social balance in social networks or synaptic plasticity in neuron dynamics.

The purpose of this paper is to clarify the important role of initial conditions for such models where the nodes' states co-evolve with the network architecture. Let us describe this effect for the generic co-evolutionary threshold dynamics defined as follows (CTD). We present its ingredients in terms of the diffusion of opinions in social networks [17,18] while keeping in mind that the model is applicable to more general systems. The system is made of a social network of interaction, whose N nodes are endowed with a binary opinion s, $+$ or $-$. The dynamics is driven by the threshold ϕ, such that $0 \leq \phi \leq 1$ and, in most cases of interest, $\phi > 1/2$. At each step, a randomly selected node i evaluates the opinion of its k_i neighbours (see Fig. 1). Let ϕ_i be the fraction of neighbours disagreeing with i. If $\phi_i \leq \phi$, node i breaks the links toward those disagreeing neighbours and rewires them to randomly selected nodes. If $\phi_i > \phi$, i adopts the state of the majority. By construction, the dynamics perdures until consensus, i.e., all agents having the same opinion, has been attained in the whole system or in disconnected components. This absorbing state obviously depends on the threshold ϕ but also, as we will discuss in detail below, on its initial condition, as some classes of initial configurations will lead to fragmentation while others will lead to global (connected) consensus.

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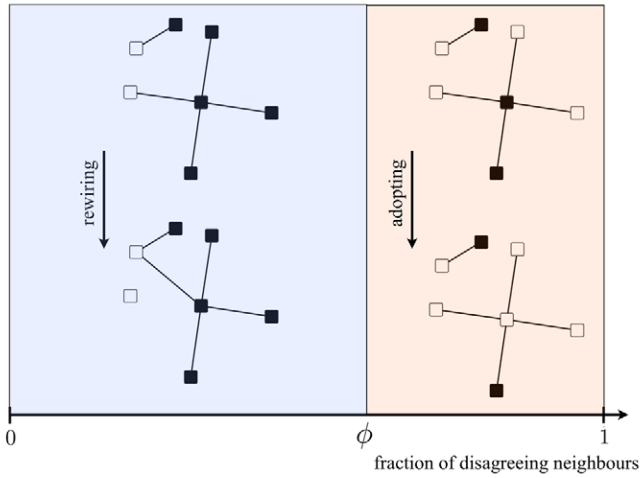

Fig. 1. Update process of CTD for two different configurations of neighbours of a randomly selected node (surrounded). When one out of four neighbours is in a different state, the central node breaks its links and creates a new link to a randomly chosen node. When three out of four neighbours are in a different state, the threshold ϕ is exceeded and the central node thus adopts the majority state.

A complete analysis of CTD requires extensive computer simulations, which is not the objective of this paper. We will instead focus on a simplified version of the model that can be studied analytically and pinpoint the key mechanisms responsible for its behaviour [19]. This model is introduced in Section 2 and studied analytically and numerically in Section 3, where we also describe in detail the transition between consensus and partial consensus. Finally, in Section 4, we conclude and discuss the implications of our work.

2. Simple model for co-evolution

Let us introduce a simplified version of CTD. In this version, the network is directed and all the nodes have two incoming links, i.e. each node is influenced by exactly two nodes, while their out-degree is initially Poisson distributed. Moreover, we will take $\phi = 1/2$, such that CTD now simplifies as follows. At each time step, a node i is selected at random. Let j and k be the two in-neighbours of i, namely the nodes at the extremities of its incoming links. If $s_i \neq s_j$ and $s_i \neq s_k$, node i switches its opinion, i.e., $s_i \to -s_i$. If the opinion of only one of its in-neighbours, say j, differs from s_i, i cuts its link from j and reconnects to a randomly chosen node, thereby maintaining its in-degree constant. If $s_i = s_j = s_k$, nothing happens. It is interesting to stress that this simplified version of CTD corresponds to the unanimity rule [20] when no rewiring is implemented. This model is well-known to exhibit a non-trivial relation between the initial and final densities of $+$ nodes, denoted by $n_{+;0}$ and $n_{+;\infty}$ respectively. We will show that the addition of the rewiring mechanism leads to a transition from a connected phase with consensus where all the nodes asymptotically belong to the same component, to a fragmented phase where two disconnected components of different opinions survive. The critical parameter of this transition is shown to be the initial density $n_{+;0}$ of $+$ nodes, i.e.,

$$n_{+;\infty} \begin{cases} = 0 \ (- \text{ consensus}) & \text{for } n_{+;0} < n_c, \\ \in]0, 1[\ (\text{fragmentation}) & \text{for } n_c < n_{+;0} < 1 - n_c, \\ = 1 \ (+ \text{ consensus}) & \text{for } n_{+;0} > 1 - n_c, \end{cases} \tag{1}$$

where n_c is the critical density.

3. Results

In order to analyze the system dynamics, let us follow the approach proposed in Ref. [20] and focus on the expected number $N_{s_0;s_1s_2}$ of configurations where a node in state s_0 receives its incoming links from a node in state s_1 and another node in state s_2. Let us denote by $\{s_0; s_1s_2\}$ such a triplet of nodes. By construction, s_i may be $+1$ or -1 and $\sum_{s_0s_1s_2} N_{s_0;s_1s_2} = N$. Moreover, the order of the links is not important and therefore $N_{s_0;s_1s_2} = N_{s_0;s_2s_1}$. By neglecting higher order correlations than those included in $N_{s_0;s_1s_2}$, it is possible to derive the set of equations

$$N_{+;++}(t+1) = N_{+;++} + \frac{1}{N}(N_{-;++} + n_+ N_{+;+-} + \pi_{-\to+} N_{+;+-} - 2\pi_{+\to-} N_{+;++})$$

$$N_{+;--}(t+1) = N_{+;--} + \frac{1}{N}(-N_{+;--} + \pi_{+\to-} N_{+;+-} - 2\pi_{-\to+} N_{+;--})$$

$$N_{+;+-}(t+1) = N_{+;+-} + \frac{1}{N}(-n_+N_{+;+-} + 2\pi_{-\to+}N_{+;--} + 2\pi_{+\to-}N_{+;++} - (\pi_{+\to-} + \pi_{-\to+})N_{+;+-})$$
$$N_{-;--}(t+1) = N_{-;--} + \frac{1}{N}(N_{+;--} + n_-N_{-;+-} + \pi_{+\to-}N_{-;+-} - 2\pi_{-\to+}N_{-;--}) \quad (2)$$
$$N_{-;++}(t+1) = N_{-;++} + \frac{1}{N}(-N_{-;++} + \pi_{-\to+}N_{-;+-} - 2\pi_{+\to-}N_{-;++})$$
$$N_{-;+-}(t+1) = N_{-;+-} + \frac{1}{N}(-n_-N_{-;+-} + 2\pi_{+\to-}N_{-;++} + 2\pi_{-\to+}N_{-;--} - (\pi_{+\to-} + \pi_{-\to+})N_{-;+-})$$

where n_+ and n_- are the density of $+$ and $-$ nodes respectively. $\pi_{+\to-}$ ($\pi_{-\to+}$) is the probability for a randomly selected $+$ $(-)$ node to switch its opinion to $-$ $(+)$. By construction, this quantity is the probability that a random $+$ $(-)$ node is connected to two $-$ $(+)$ nodes

$$\pi_{+\to-} = \frac{N_{+;--}}{N_+}, \qquad \pi_{-\to+} = \frac{N_{-;++}}{N_-}, \quad (3)$$

where $N_+ = \sum_{s_1,s_2} N_{+;s_1s_2}$ and $N_- = N - N_+$ are the expected numbers of $+$ and $-$ nodes respectively.

Let us describe in detail the first equation for $N_{+;++}$, the other ones being obtained in a similar way. Its evolution is made of several contributions. The first term is the probability that a $\{-;++\}$ triplet is selected and transforms into $\{+;++\}$ by unanimity rule. The second term is the probability that a $\{+;+-\}$ triplet is selected and the rewired link (originally from $+$ to $-$) arrives on a $+$ node (with probability n_+). The last two terms account for the possible change of the state of one of the two neighbours in the triplet, as they may also switch their opinion because of a unanimity rule in another triplet, and are evaluated by using the aforementioned $\pi_{+\to-}$ and $\pi_{-\to+}$.

As discussed in [20], several initial conditions may in principle be chosen for the system of equations (2), each of them leading to its own trajectory in the 6-dimensional dynamical space. Such initial conditions are subject to the normalisation $\sum_{s_0,s_1,s_2} N_{s_0;s_1s_2} = N$, and to the conservation laws

$$T_+ = 2N_+, \qquad T_- = 2N_-, \quad (4)$$

where the quantities

$$T_+ = 2N_{+;++} + 2N_{-;++} + N_{+;+-} + N_{-;+-}$$
$$T_- = 2N_{+;--} + 2N_{-;--} + N_{+;+-} + N_{-;+-} \quad (5)$$

are the total number of $+$ $(-)$ incoming neighbours in the triplets. Relations (4) simply mean that each node i that is a neighbor in a triplet $\{s_x; s_is_y\}$ is also at the summit of another triplet $\{s_i; s_{x'}s_{y'}\}$ (as it also receives two incoming links by construction). In order to select one of the several configurations $N_{s_0;s_1s_2}$ that still satisfy the above constraints, we will further assume that the initial configuration is uncorrelated and therefore that each node has the same probability $n_{+;0}$ to be $+$. Among all the possible configurations for which $N_+ = Nn_{+;0}$, we therefore select the initial condition

$$N_{+;++} = Nn_{+;0}^3, \qquad N_{+;--} = Nn_{+;0}(1-n_{+;0})^2$$
$$N_{+;+-} = 2Nn_{+;0}^2(1-n_{+;0})$$
$$N_{-;--} = N(1-n_{+;0})^3, \qquad N_{-;++} = N(1-n_{+;0})n_{+;0}^2$$
$$N_{-;+-} = 2N(1-n_{+;0})^2n_{+;0}. \quad (6)$$

Before going further, it is instructive to look at the total number $N_+ = \sum_{s_1,s_2} N_{+;s_1s_2}$ of $+$nodes whose time evolution is obtained by summing over the first three equations of (2)

$$N_+(t+1) = N_+ + \frac{1}{N}(N_{-;++} - N_{+;--}). \quad (7)$$

This relation shows that $N_{-;++} = N_{+;--}$ at stationarity. A careful look at the second equation of (2) shows, however, that $N_{+;--}$ has to decay until it reaches zero. The third equation of (2) also shows that the only stationary solution of $N_{+;+-}$ is also zero when $N_{-;++} = N_{+;--} = 0$, thereby confirming that the dynamics asymptotically reaches a frozen configuration where consensus is reached among the connected nodes. The dynamics is therefore driven by two types of triplets: the triplets $\{+;--\}$ and $\{-;++\}$ drive the system toward consensus, while the configurations $\{+;+-\}$ and $\{-;+-\}$ allow for a topological rearrangement of the network. This rearrangement implies that the only frozen states are those corresponding to consensus (in one or several clusters) [21] and drives the division of the system into disconnected clusters. The competition between these two types of mechanisms is crucial for the transition (1). One should also note that models for opinion dynamics are known to exhibit a coexistence of different opinions when applied to a static underlying network with modular structure [22]. In the case of CTD, in contrast, it is the rewiring of the links that reorganizes the system into modules and thereby allows for coexistence.

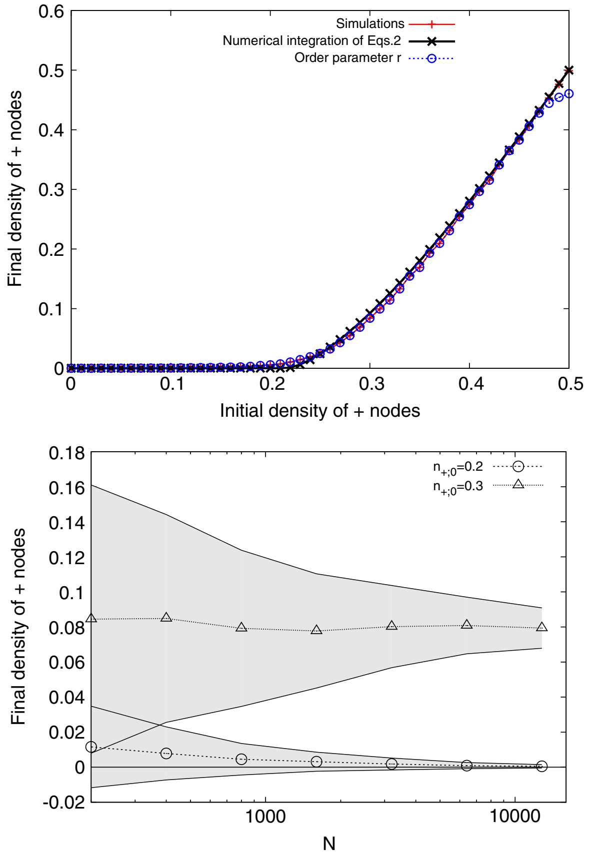

Fig. 2. In the upper figure, we plot $\langle n_{+,\infty}\rangle(n_{+,0})$, evaluated by integrating the set of Eqs. (2) and by performing numerical simulations of a network made of 1000 nodes, averaged over 1000 realisations of the random process. We also plot the order parameter $r = \langle 1/2 - |1/2 - n_{+,\infty}|\rangle$ which confirms that the system actually breaks into disconnected components when $n_c < n_{+,0} < 1 - n_c$. In the lower figure, we plot the size dependence of $\langle n_{+,\infty}\rangle(n_{+,0})$ for $n_{+,0} = 0.2$ and $n_{+,0} = 0.3$, averaged over 1000 simulations. While $\langle n_{+,\infty}\rangle(0.3)$ does not depend on the system size, $\langle n_{+,\infty}\rangle(0.2)$ is shown to tend to zero when N is increased. The grey areas cover the regions $\langle n_{+,\infty}\rangle \pm \sigma$, where $\sigma \equiv \sqrt{\langle n_{+,\infty}^2\rangle - \langle n_{+,\infty}\rangle^2}$.

By integrating recursively the system of equations (2) starting from the initial conditions (6), we obtain a non-trivial relation

$$\langle n_{+,\infty}\rangle(n_{+,0}) \tag{8}$$

between the initial density and the expected final density of +nodes, where we have emphasized that $\langle n_{+,\infty}\rangle \equiv N_{+;\infty}/N$ is an ensemble average by construction. This numerical integration confirms the above discussions, and clearly shows that a transition occurs at $n_c \approx 0.22$ (see Fig. 2). One should insist on the fact that this relation differs from the standard exit probability measured when the dynamics takes place on a static network [23–25]. In the latter models, $\langle n_{+,\infty}\rangle(n_{+,0})$ would measure the probability to end in a +consensus (in the whole system) starting from some initial density of +nodes. Relation (8) tells us instead whether or not the system has split into two disconnected clusters [26] and a different consensus has been reached in each cluster. $\langle n_{+,\infty}\rangle$ is thus the expected number of +nodes in the frozen state, but one should keep in mind that different realisations of the random process (with the same value for $n_{+,0}$) will typically lead to different values of $n_{+,\infty}$.

We have verified the accuracy of our calculations by performing numerical simulations of the model (see Fig. 2). To do so, we have considered systems made of 1000 nodes and have averaged the asymptotic density of +nodes (evaluated when the dynamics is frozen) over 1000 realisations for each value of $n_{+,0}$. In order to check that the system actually breaks into two clusters when $n_c < n_{+,0} < 1 - n_c$ (see Fig. 3), we have measured $r = \langle 1/2 - |1/2 - n_{+,\infty}|\rangle$. This order parameter would vanish if, for each realisation, $n_{+,\infty}$ is either zero or one, while $r = \langle n_{+,\infty}\rangle$ if the system breaks into two clusters. The simulation results show a very good agreement with the theoretical predictions. We have also verified that the discrepancies are due to finite-size effects by varying the system size and showing that $\langle n_{+,\infty}\rangle$ tends to zero when N is increased and $n_{+;0} < n_c$. Finally, our simulations indicate that the fluctuations around $\langle n_{+,\infty}\rangle$ decrease with the system size, i.e. $\sigma \equiv \sqrt{\langle n_{+,\infty}^2\rangle - \langle n_{+,\infty}\rangle^2} \sim N^{-1/2}$. This result implies that transition (1) is not only valid for an ensemble of realisations, but almost surely for any realisation of the stochastic process in the thermodynamic (large N) limit.

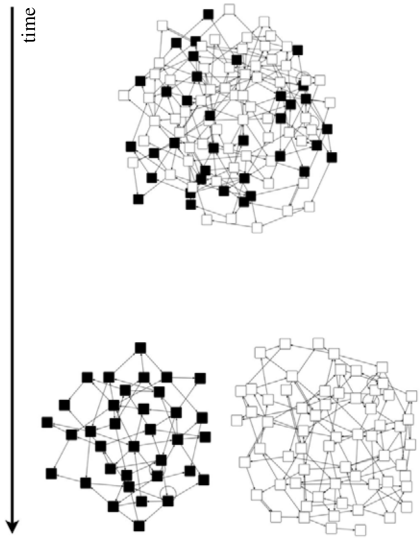

Fig. 3. Visualisation of the initial and final states of one realisation of the dynamics for a network made of $N = 100$ nodes. The initial density of +nodes $n_{+,0} = 0.4$. The asymptotic network is made of two clusters. The +cluster is now made of 35% of the nodes.

4. Conclusion

In this article, we have focused on a simple model where the binary state of a node and its links coevolve. We have shown that the system may undergo fragmentation, a generic feature that has been observed in other co-evolution network models, for instance based on the Axelrod [13] or the Voter model [16], but also in the case of coupled maps with a variable coupling strength [27]. In our model, the critical parameter is the initial condition, as a sufficient fraction, i.e. *critical mass*, of +nodes is necessary for such nodes to survive and to separate from the main cluster. This observation is particularly important because most numerical studies of models with co-evolution only focus on balanced initial configurations, i.e. $n_{+,0} = 1/2$ in our case, and thus neglect the possible dependence of fragmentation on the initial conditions.

The model studied in this article has been kept as simple as possible in order to allow for an analytical description. Compared to the more general CTD presented in the introduction, this simplified model introduces unrealistic ingredients such as the fact that the underlying network is directed and regular, and that the in-degree has the smallest non-trivial value, i.e., two. Additional computer simulations are therefore required in order to explore the role of the threshold ϕ on the asymptotic state in more complex directed or undirected networks. Finally, let us point to an interesting generalisation of CTD that would include two different thresholds ϕ_r and ϕ_a for either rewiring links from disagreeing neighbours or adopting the state of the majority. Such a model would unify two seminal threshold models, namely the Granovetter model for the diffusion of cultural traits [2] and the Schelling model for social segregation [28].

Added Note: After this manuscript was completed, we became aware of a related work by Mandra et al. [29].

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