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Conservation laws for voter-like models on random directed networks

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Abstract. We study the voter model, under node and link update, and the related invasion process on a single strongly connected component of a directed network. We implement an analytical treatment in the thermodynamic limit using the heterogeneous mean-field assumption. From the dynamical rules at the microscopic level, we find the equations for the evolution of the relative densities of nodes in a given state on heterogeneous networks with arbitrary degree distribution and degree–degree correlations. We prove that conserved quantities as weighted linear superpositions of spin states exist for all three processes and, for uncorrelated directed networks, we derive their specific expressions. We also discuss the time evolution of the relative densities that decay exponentially to a homogeneous stationary value given by the conserved quantity. The conservation laws obtained in the thermodynamic limit for a system that does not order in that limit determine the probabilities of reaching the absorbing state for a finite system. The contribution of each degree class to the conserved quantity is determined by a local property. Depending on the dynamics, the highest contribution is associated with influential nodes reaching a large number of outgoing neighbors, not too influenceable ones with a low number of incoming connections, or both at the same time.

Keywords: disordered systems (theory), network dynamics, stochastic processes

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Contents

1. Introduction

Conservation laws are intimately related to symmetries in the systems they hold for. They play an important role in the characterization and classification of different nonequilibrium processes of ordering dynamics. For example, in kinetic Ising models one distinguishes between Glauber (spin flip) and Kawasaki (spin exchange) dynamics. Kawasaki dynamics fulfills a microscopic conservation law, such that the total magnetization is conserved in each individual dynamical step of a stochastic realization. This conservation law does not hold for Glauber; as a consequence, the Glauber and Kawasaki dynamics give rise to different scaling laws for domain growth in coarsening processes [1] and they define different nonequilibrium universality classes.

In other types of nonequilibrium lattice models non-microscopic conservation laws are known to hold. They are statistical conservation laws in which the conserved quantity is an ensemble average defined over different realizations of the stochastic dynamics for the same distribution of initial conditions. Examples of such conservation laws occur for the voter model [2, 3] or the invasion process [4]. In particular, the role of the conservation law of the magnetization and of the Z_2 symmetry (± 1 states) in the voter dynamics universality class has been studied in detail in the critical dimension $d = 2$ of regular lattices [5]. The voter model is a paradigmatic model of consensus dynamics in the social context [6, 7] or, in the biological context, of competition of plant species in ecological communities [8]. In general, any Markov chain with at least two absorbing states reachable from all other configurations has a conserved quantity when averaged over the ensemble. Such a quantity determines the probability to eventually reach a particular absorbing configuration in a finite system. In some cases, this conservation law is of rather trivial nature as in the zero-temperature Ising Glauber dynamics where the magnetization sign is conserved. The voter model, the zero-temperature Ising Glauber dynamics, and other related models of

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J. Stat. Mech. (2009) P10024

language evolution [9] or population dynamics [10], belong to the class of models with two absorbing states while epidemic spreading dynamics, like the contact process [11] or the Susceptible–Infected–Susceptible model [12], usually have a single absorbing state with no conservation law.

While some of these questions have been studied for spin lattice models for a long time, conservation laws for dynamical processes on complex networks [13]–[16] still remain a challenge. This issue has been considered for the voter model [2, 3] or the invasion process [4] on undirected uncorrelated networks [17]–[21]. The link update dynamics for the voter model has been found to conserve the global magnetization [22], while the node update dynamics [22] and the invasion process [20] preserve a weighted global magnetization where the contribution of each spin is calibrated by some function of the degree of the corresponding node in the undirected network. Such ensemble average conservation laws characterize processes with two absorbing states accessible to the dynamics, which compete to maintain an active state in the thermodynamic limit. In finite networks, the conserved quantities give the probabilities of reaching the uniform states and so act as a bridge that enables some probabilistic predictive power of the final dynamical state based on information about the initial conditions. In addition, different finite size dynamical scaling properties can be related to different conservation laws [22].

Much less has been done exploring dynamical processes on directed networks, with the exception of the Ising model [23] and Boolean dynamics mainly applied to biological problems [24]. However, interactions between pairs of elements are asymmetric in different systems including some social networks [25], where social ties are perceived or implemented differently by the two individuals forming the connected pairs. Directed network representations rather than undirected ones become more informative and adjusted to reality. In general, directed networks present characteristic large-scale connectivity structures, the so-called bow-tie architecture formed by a strongly connected component as a core structure and peripheral in- and out-components [26]. This organization, coupled to the initial condition of the dynamics running on top, have an impact both on the evolution of the processes and the final possible states of the systems [27]–[29]. In the voter model, leaf nodes in the in-component never change their state, thus sending an invariable signal that can potentially propagate to the rest of the components of the system. This is closely related to phenomena such as the presence of zealots [30, 31] in undirected networks. Both input or output directional large-scale components and zealotry imply at the end an external forcing on the dynamical processes that prevents reaching one of the absorbing states even for a finite network. This is clearly illustrated by the evolution of dynamical processes running on networks at the transition from a pure strongly connected component to a complete bow-tie structure. In an isolated and strongly connected component, the voter dynamics keeps an active dynamical state in the thermodynamic limit, but it leads to a consensus (absorbing state) in a finite network as it happens on undirected networks. Thus, the appearance of an input component in the large-scale structure of the network prevents the system from reaching an absorbing state for random initial conditions [28].

In this paper, we focus on dynamics of coupled two-state spin variables and consider conserved quantities that are weighted sums of the spin values. Specifically, we investigate the form of the conservation law for the voter model—under node and link update—and the invasion process in directed networks with arbitrary degree distribution and

J. Stat. Mech. (2009) P10024

degree–degree correlations, and otherwise maximally random. The directionality of the interactions is therefore encoded in the topology. We restrict to a single strongly connected component so that the absorbing state can be reached in a finite system, which seems realistic for a number of densely connected real networks like the world trade web [32]. In section 2, we present a detailed study of the node update version of the voter model and implement an analytical treatment using the heterogeneous mean-field assumption in the thermodynamic limit. From the dynamical rules at the microscopic level, we find the equations for the evolution of the relative densities of nodes in one of the two possible states on heterogeneous networks with arbitrary degree distribution and degree–degree correlations. In this case, we prove that a conserved quantity as a weighted linear superposition of spin states exists. In section 3, we discuss the node update voter model in uncorrelated directed networks to derive an analytical expression for the conservation law and we also discuss the exponential decay of the relative densities to their homogeneous stationary value, which is basically a function of the conserved quantity. We show how the conserved quantity determines the probability of reaching one of the two states in a finite network. In sections 4 and 5, we present the results of applying the same methodology to the voter model with link update and the invasion process, respectively. We conclude in section 6 with a summary of results and open questions for future research.

2. The voter model on strongly connected components

In the voter model under node update (VM), each node of a network can exist in one of two possible states, 1 or 0[4]. In a single dynamical event, a randomly selected node copies the state of one of its neighbors, also selected at random. The link update dynamics of the voter model selects instead a link [22]. Time is increased by $1/N$, so that the physical time is incremented by 1 after N of such events. On undirected networks, the node update voter model conserves the ensemble average of a weighted magnetization, where the contribution of each spin is multiplied by the degree of the corresponding node.

As defined above, the interactions in the voter dynamics are instantaneously asymmetric since the updates always go in the same direction once the original node is chosen independently of the undirectionality of the substrate. Hence, the discussion of the voter model on directed networks comes out as a natural one, where the directionality of the interaction is decoupled from the dynamics and encoded in the structure of the substrate. The straightforward generalization of the voter model on directed networks under node update consists of selecting a node at random, and then assigning to it the state of one of its incoming neighbors, also chosen at random. We will discuss this dynamics next in this section and section 3, and the voter model with link update will be discussed later in section 4.

2.1. Directed networks

The topological structure of directed networks is more complex than the one of undirected graphs. In purely directed networks, without bidirectional links, the edges

[4] We use these values $s = 1, 0$ in order to simplify computations instead of the usual spin notation $\sigma = \pm 1$. There is a direct mapping between both schemes $\sigma = 2s - 1$, and therefore for all the properties defined as a function of the states. For instance, the total magnetization m in the $\{\pm 1\}$ scheme is related to the total magnetization m' in the $\{0, 1\}$ scheme through $m = 2m' - 1$.

J. Stat. Mech. (2009) P10024

are differentiated into incoming and outgoing, so that each vertex has two coexisting degrees $k_{\rm in}$ and $k_{\rm out}$, with total degree $k = k_{\rm in} + k_{\rm out}$. Hence, the degree distribution for a directed network is a joint degree distribution $P(k_{\rm in}, k_{\rm out}) \equiv P(\mathbf{k})$ of in- and out-degrees that, in general, may be correlated. We consider degree correlations $P_{\rm in}(\mathbf{k}'|\mathbf{k})$ and $P_{\rm out}(\mathbf{k}'|\mathbf{k})$, which respectively measure the probability to reach a vertex of degree $\mathbf{k}'$ leaving from a vertex of degree $\mathbf{k}$ using an incoming or outgoing edge of the source vertex, and are related through the following degree detailed balance condition [33]:

$$k_{\rm out} P(\mathbf{k}) P_{\rm out}(\mathbf{k}'|\mathbf{k}) = k'_{\rm in} P(\mathbf{k}') P_{\rm in}(\mathbf{k}|\mathbf{k}'). \tag{1}$$

This ensures that the network is closed and $\langle k_{\rm in}\rangle = \langle k_{\rm out}\rangle$. Apart from the prescribed degrees and two-point correlations, networks are maximally random. At the macroscopic scale, the giant weakly connected component, i.e. the set of nodes that can communicate to each other when considering the links as undirected [34]–[38], becomes internally structured in three giant connected components, as well as other secondary structures such as tubes or tendrils, forming a bow-tie architecture [26]. The main component is the strongly connected component (SCC), a central core formed by the set of vertices that can be reached from each other following a directed path. The other two main components are peripheral components, the in-component (IN) formed by all vertices from which the SCC is reachable by a directed path but that cannot be reached from there, and the out-component (OUT) formed by all vertices that are reachable from the SCC by a directed path but cannot reach the SCC themselves. Percolation theory for purely directed networks was first developed for uncorrelated networks [37]–[41] and directed random networks with arbitrary two-point degree correlations and bidirectional edges [33].

We restrict to networks forming a strongly connected component without peripheral components that would act on the SCC as sources of external forcing. We will see that, within the strongly connected component, conservation laws preserve weighted magnetizations, where the weights are dictated by the directed degrees.

2.2. From microscopic dynamics to the drift equation under the heterogeneous mean-field assumption

In this section, we first describe the voter model as a stochastic process on the microscopic level of single nodes. Then we derive a macroscopic description by coupled Langevin equations capturing the stochasticity in the drift terms. Each equation is for the expected state averaged over the subset of nodes having the same in- and out-degrees. This so-called heterogeneous mean-field approach allows us to deal with dynamical processes on complex networks. It is based on two assumptions: (i) all nodes in the same degree class are statistically equivalent, that is, nodes with different degrees have to be treated as intrinsically different (the 'heterogeneous' part of the assumption) and (ii) any stochastic variable associated with the dynamics of a node belonging to a certain degree class is described by the average over the class (the 'mean-field' part of the assumption). Notice that the 'heterogeneous' approximation may not be completely true for single network realizations as two nodes with the same degree can have different environments. However, when random network ensembles are considered instead of single realizations, the averaging over network realizations ensures that all nodes within the same degree class are statistically equivalent. To our knowledge, the heterogeneous mean field was

J. Stat. Mech. (2009) P10024

first presented in [42, 43] and recently used to study the contact process [11]. In [21, 44], a homogeneous mean-field pair approximation was developed instead.

We focus on the microscopic state of nodes at some time t. Let $s_u(t), u = 1, \ldots, N$, be a stochastic binary variable defined for each of the N nodes in the network which describes its state, 0 or 1. The vector $\mathbf{s}(t) \equiv \{s_u(t)\}, u = 1, \ldots, N$, completely defines the dynamical state of the system at time t. Two more independent binary stochastic variables $\mu_u(\mathrm{d}t)$ and ξ_u are defined in order to model the transitions between states of single nodes in an iteration. After a time interval $\mathrm{d}t$, the variable $\mu_u(\mathrm{d}t)$ for a given node u takes the value 1 or 0 if u was chosen or not, respectively. In case node u was selected, then ξ_u assumes the value 1 [0] if u copies a neighbor with state 1 [0]. We assume that the occurrence of events in the voter dynamics follows an independent Poisson process for each node, with constant rate λ for all of them, which corresponds to a Monte Carlo step. In the remainder we will be set to $\lambda = 1$ without loss of generality. Thus, $\mu_u(\mathrm{d}t)$ and ξ_u have probability distributions:

$$P(\mu_u(\mathrm{d}t)) = \mathrm{d}t\, \delta_{\mu_u(\mathrm{d}t),1} + (1 - \mathrm{d}t)\delta_{\mu_u(\mathrm{d}t),0}, \tag{2}$$

$$P(\xi_u) = \Phi_u/k_{u,\mathrm{in}}\delta_{\xi_u,1} + (1 - \Phi_u/k_{u,\mathrm{in}})\delta_{\xi_u,0}, \tag{3}$$

where $k_{u,\mathrm{in}}$ is the incoming degree of node u, and we have defined $\Phi_u(t) = \sum_v a_{vu}s_v(t)$. The adjacency matrix $\{a_{vu}\}$ encodes the topological properties of the directed network. Element a_{vu} has value one if there is a directed link from v to u and zero otherwise, so that $\Phi_u(t)$ stands for the number of state-one incoming neighbors of node u at time t. The matrix $\{a_{vu}\}$ is symmetric for undirected networks but for directed ones it is, in general, asymmetric.

In terms of the above variables, the dynamical state $s_u(t)$ of node u after an increment of time $\mathrm{d}t$ is

$$s_u(t + \mathrm{d}t) = \mu_u(\mathrm{d}t)\xi_u + (1 - \mu_u(\mathrm{d}t))s_u(t). \tag{4}$$

This equation, together with equations (2) and (3), gives the complete description of the evolution of the system, making the formalism general and applicable to any network structure.

Although exact, this microscopic description is unmanageable. In order to reduce the degrees of freedom, we apply a heterogeneous mean-field *approach* [12] so that nodes with the same degree $\mathbf{k}$ are assumed to be statistically independent and equivalent and can be aggregated in the same degree class $\Upsilon(\mathbf{k}) \equiv \Upsilon(k_{\mathrm{in}}, k_{\mathrm{out}})$. At this point, we have restricted to directed networks organized at the large scale into an SCC without IN and OUT. Properties are then defined for each degree class, which will be characterized by the relative density $m_{\mathbf{k}}(t)$, the ratio between the number of state-one nodes within class $\Upsilon(\mathbf{k})$ and its number of nodes $N_{\mathbf{k}}$:

$$m_{\mathbf{k}}(t) = \frac{\sum_{u\epsilon\Upsilon(\mathbf{k})} s_u(t)}{N_{\mathbf{k}}}. \tag{5}$$

In the thermodynamic limit, the relative densities $m_{\mathbf{k}}(t)$ can be considered as continuous variables. Their time evolution can be described by a Langevin equation [45] with drift and diffusion coefficients that are respectively given by the first and second infinitesimal moments of the stochastic variables $m_{\mathbf{k}}(t)$. Those moments can be derived from the

J. Stat. Mech. (2009) P10024

microscopic equation (4) along with the definition in equation (5). In the thermodynamic limit, it is possible to prove that the diffusion term has a dependence $1/\sqrt{N_k}$ on the system size as for undirected networks [21], so that the drift term A_k will dominate. It is given by the average value over all possible configurations of $m_{\mathbf{k}}(t+\mathrm{d}t)$ conditioned to the state of the system at time t:

$$\langle m_{\mathbf{k}}(t+\mathrm{d}t)\rangle_{m_{\mathbf{k}}(t)} = m_{\mathbf{k}}(t) + A_k(t)\,\mathrm{d}t. \tag{6}$$

From the microscopic dynamics

$$\langle s_u(t+\mathrm{d}t)\rangle_{\mathbf{s}(t)} = s_u(t) - \mathrm{d}t[s_u(t) - \Phi_u(t)/k_{u,\mathrm{in}}], \tag{7}$$

and summing this equation for all nodes in the degree class $\mathbf{k}$ and dividing by the number of nodes $N_{\mathbf{k}}$, we arrive at

$$\langle m_{\mathbf{k}}(t+\mathrm{d}t)\rangle_{m_{\mathbf{k}}(t)} = m_{\mathbf{k}}(t) - \mathrm{d}t\left[m_{\mathbf{k}}(t) - \frac{1}{N_{\mathbf{k}}}\frac{1}{k_{\mathrm{in}}}\sum_{u\epsilon\Upsilon(\mathbf{k})}\Phi_u(t)\right], \tag{8}$$

and from here to

$$A_k(t) = -m_{\mathbf{k}}(t) + \frac{1}{N_{\mathbf{k}}}\frac{1}{k_{\mathrm{in}}}\sum_{u\epsilon\Upsilon(\mathbf{k})}\Phi_u(t). \tag{9}$$

The adjacency matrix contained in $\Phi_u(t)$ can be coarse-grained as well, so that a differential equation for the relative densities can eventually be written. This coarse-graining restricts the validity of the equations to random complex networks (and not lattices), since we assume all nodes in the same degree class to be statistically independent and equivalent. With these assumptions

$$\begin{aligned}\sum_{u\epsilon\Upsilon(\mathbf{k})}\Phi_u(t) &= \sum_{\mathbf{k}'}\sum_{v\epsilon\Upsilon(\mathbf{k}')}\sum_{u\epsilon\Upsilon(\mathbf{k})} a_{vu}s_v(t)\\ &= \sum_{\mathbf{k}'} E_{\mathbf{k}'\mathbf{k}}m_{\mathbf{k}'}(t),\end{aligned} \tag{10}$$

where $E_{\mathbf{k}'\mathbf{k}}$ is the number of connections from the class of vertices of degree $\mathbf{k}'$ to the class of vertices of degree $\mathbf{k}$. The generally asymmetric matrix E is the coarse-grained adjacency matrix, giving weighted connections between degree classes rather than between single nodes. With the detailed balance condition of equation (1), we obtain

$$E_{\mathbf{k}'\mathbf{k}} = k'_{\mathrm{out}}P_{\mathrm{out}}(\mathbf{k}|\mathbf{k}')N_{\mathbf{k}'} = k_{\mathrm{in}}P_{\mathrm{in}}(\mathbf{k}'|\mathbf{k})N_{\mathbf{k}}. \tag{11}$$

Inserting these results into equation (9), we arrive at the equation for the evolution of the relative density in the degree class $\mathbf{k}$ of a purely directed correlated network (disregarding diffusion terms):

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = -m_{\mathbf{k}}(t) + \sum_{\mathbf{k}'}P_{\mathrm{in}}(\mathbf{k}'|\mathbf{k})m_{\mathbf{k}'}(t). \tag{12}$$

Let us recall that this result is valid for the ensemble of networks defined by the degree distribution $P(\mathbf{k})$ and the degree correlations $P_{\mathrm{in}}(\mathbf{k}'|\mathbf{k})$ and $P_{\mathrm{out}}(\mathbf{k}'|\mathbf{k})$, but otherwise maximally random. Notice that large enough networks present good statistical quality

J. Stat. Mech. (2009) P10024

at the level of degree classes and are also well described by this equation. Finally, in the thermodynamic limit, the Langevin equation loses its noise term because of the dependence on the system size and reduces to equation (12), so that $m_{\mathbf{k}}(t)$ becomes a deterministic variable. Nevertheless, since the process is linear, equation (12) is always valid even for finite systems understanding that, in this case, the variables are averages over realizations of the process with the same distribution of initial conditions.

2.3. Conserved quantity on directed networks with degree–degree correlations

For correlated networks, $m_{\mathbf{k}}(t) = \sum_{\mathbf{k}'} P_{\rm in}(\mathbf{k}'|\mathbf{k}) m_{\mathbf{k}'}(t)$ in the stationary state. This equation corresponds indeed to an eigenvector problem, since $\{m_{\mathbf{k}}(t)\}$ can be thought of as the eigenvector of the matrix $\{P_{\rm in}(\mathbf{k}'|\mathbf{k})\}$ with eigenvalue one. By normalization of the conditional probability, a solution is the uniform vector $m_{\mathbf{k}}(t) = m\ \forall \mathbf{k}$.

We prove next that, within the heterogeneous mean-field approach and for the correlated directed networks we are considering, there is a conserved quantity given as a linear superposition of the form $\omega = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} m_{\mathbf{k}}(t)$. From equation (12), its evolution is given by

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\omega + \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \varphi_{\mathbf{k}} P_{\rm in}(\mathbf{k}'|\mathbf{k}) m_{\mathbf{k}'}(t), \tag{13}$$

and imposing that $\mathrm{d}\omega/\mathrm{d}t = 0$, we obtain

$$\sum_{\mathbf{k}} \varphi_{\mathbf{k}} m_{\mathbf{k}}(t) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} P_{\rm in}(\mathbf{k}|\mathbf{k}') \varphi_{\mathbf{k}'} m_{\mathbf{k}}(t). \tag{14}$$

For each density

$$\varphi_{\mathbf{k}} = \sum_{\mathbf{k}'} P_{\rm in}(\mathbf{k}|\mathbf{k}') \varphi_{\mathbf{k}'}. \tag{15}$$

This is an eigenvector equation that has a solution if the matrix $\{P_{\rm in}(\mathbf{k}|\mathbf{k}')\}$ has an eigenvalue equal to one with $\{\varphi_{\mathbf{k}}\}$ the corresponding eigenvector. One can prove that this eigenvector with eigenvalue one exists by summing both sides of the previous equation over $\mathbf{k}$. Using the normalization of the conditional probability $\sum_{\mathbf{k}} P_{\rm in}(\mathbf{k}|\mathbf{k}') = 1$, one eventually arrives at a trivial identity. The fact that the coefficients $\varphi_{\mathbf{k}}$ that modulate the contributions of the different $m_{\mathbf{k}}$ to the conserved weighted magnetization correspond to the entries of the eigenvector of a certain characteristic matrix with eigenvalue one also applies to other similar dynamical processes, such as the link dynamics and the invasion process, as we will show.

This proves that a conserved quantity of the form of a linear functional exists but, in general, it is not possible to derive its value without further specifying the form of the degree–degree correlations in the network.

In fact, a conserved quantity as a linear functional $\omega = \sum_u c_u s_u$ exists for any directed network. To see this, we note that the stationarity of ω is equivalent to

$$0 = \sum_u c_u \dot{s}_u = \sum_u c_u \sum_v a_{vu}(s_v - s_u), \tag{16}$$

J. Stat. Mech. (2009) P10024

where the expression for the time derivative of the states $\dot{s}_u$ is obtained from equation (4). For the quantity ω to be stationary under all choices of state vectors, the vector (c_u) must be a left eigenvector for eigenvalue zero of the Laplacian matrix L defined by

$$L_{uv} = \delta_{vu} k_{u,\mathrm{in}} - a_{vu}. \tag{17}$$

Since each row of L sums to zero, the columns of L are linearly dependent and zero is an eigenvalue of L. Thus a non-trivial coefficient vector (c_u) for the conserved quantity ω can be found for any network by solving the eigenvector problem for L.

In general, each entry of an eigenvector depends on the whole matrix, i.e. on all details of the network structure. Thus the coefficients (c_u) cannot be computed from local information only. The restriction to uncorrelated networks in the following sections enables us to identify the major contribution resulting from local structure and express the coefficients in terms of node degrees.

3. Voter model on uncorrelated SCCs

When two-point correlations are absent, the transition probabilities become independent of the degree of the source vertex. In this situation

$$P_{\mathrm{out}}(\mathbf{k}'|\mathbf{k}) = \frac{k'_{\mathrm{in}} P(\mathbf{k}')}{\langle k_{\mathrm{in}} \rangle}, \qquad P_{\mathrm{in}}(\mathbf{k}'|\mathbf{k}) = \frac{k'_{\mathrm{out}} P(\mathbf{k}')}{\langle k_{\mathrm{in}} \rangle}, \tag{18}$$

and using these expressions, equation (12) becomes

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = -m_{\mathbf{k}}(t) + \omega_{\mathrm{out}}, \tag{19}$$

where we have defined

$$\omega_{\mathrm{out}} = \frac{1}{\langle k_{\mathrm{in}} \rangle} \sum_{\mathbf{k}} k_{\mathrm{out}} P(\mathbf{k}) m_{\mathbf{k}}(t). \tag{20}$$

Therefore, in the stationary state $m_{\mathbf{k}} = \omega_{\mathrm{out}} \, \forall \mathbf{k}$. Indeed, as mentioned above, in the stationary state of correlated networks $m_{\mathbf{k}} = m \, \forall \mathbf{k}$ and then $m_{\mathbf{k}} = \omega_{\mathrm{out}} = m \, \forall \mathbf{k}$ also holds in the presence of correlations.

Also from equation (19), it is easy to see that ω_{out} is a conserved quantity in uncorrelated networks. However, in general it is not preserved when degree–degree correlations are present. This is in contrast to undirected networks, where the conserved quantity $\omega = (\sum_k kP(k)m_k(t))/\langle k \rangle$ is preserved even in the correlated case and indeed for any structure [22]. Going back to the uncorrelated case, notice that the out-degree is the quantity that weights the contribution of the nodes to the conserved quantity. From a local perspective, what seems therefore important in the VM is to be able to influence a large number of partners.

In uncorrelated networks, the convergence of the state-one relative densities to their stationary value can be easily computed. From equation (19), taking into account that ω_{out} is a conserved quantity and for a given initial condition $m_{\mathbf{k}}(0)$, it is straightforward

J. Stat. Mech. (2009) P10024

to arrive at the solution

$$m_{\mathbf{k}}(t) = \omega_{\text{out}} + (m_{\mathbf{k}}(0) - \omega_{\text{out}})\, \mathrm{e}^{-t}, \tag{21}$$

where we have substituted $\langle k_{\text{in}} \rangle$ by $\langle k_{\text{out}} \rangle$. Thus, all the densities decay exponentially fast to the stationary value $m_{\mathbf{k}}^{\text{st}} = \omega_{\text{out}}$ and the relaxation time is, for all of them, equal and independent of the degrees.

In the thermodynamic limit, the partially ordered stationary state is stable, while finite size fluctuations eventually bring the system to one of the two possible unanimity states. The probability P_1 that the system ends up with all nodes in state one ($m_{\mathbf{k}} = 1, \forall \mathbf{k}$) is given by the initial condition, which fixes the value of the conserved quantity at the beginning of the process. To see this, one takes into account that ω_{out} is an ensemble average conserved quantity of the form in equation (20), from which

$$P_1 = \omega_{\text{out}}. \tag{22}$$

This is in agreement with the fact that, in general, the Markov property of a stochastic process, if present, trivially ensures that the exit probability is a conserved quantity corresponding to a time-translation invariance. If the process has one absorbing state, the exit probability has a constant value one but, if the process has two or more absorbing barriers, the probability of reaching one of those is not trivial any more.

It is also interesting to investigate what happens to the quantity $v_i(t) = (\sum_{\mathbf{k}} k_{\text{in}} P(\mathbf{k}) m_{\mathbf{k}}(t))/\langle k_{\text{in}} \rangle$, which involves in-degree instead of out-degree. In the uncorrelated case, and disregarding fluctuations, $v_i(t) = (v_i(0) - \omega_{\text{out}})\mathrm{e}^{-t} + \omega_{\text{out}}$, that is, in general, v_i decays exponentially fast to ω_{out}. The quantity $v_i(0)$ depends on the initial condition. If this is homogeneous over degree classes, then $v_i(0) = \omega_{\text{out}}$ and $v_i(t)$ remains constant.

In order to check the convergence of the state-one relative densities to the conserved quantity, we have run numerical simulations of the voter model dynamics on a random uncorrelated network of size $N = 10^5$, scale-free in-degree distribution with exponent 2.5 and exponential out-degree distribution. To obtain an initial state that is inhomogeneous in the densities $m_{\mathbf{k}}$, we have chosen an initial configuration in which half of the nodes with the lowest out-degree have state zero, and the other half have state one. In this way, initial densities $m_{\mathbf{k}}(0)$ in classes with k_{out} lower than 4 were small or zero, while densities in classes with k_{out} larger than 4 were one.

In figure 1, we plot the average of the conserved quantity ω_{out} and the densities for classes $\mathbf{k} = (k_{\text{in}}, k_{\text{out}}) = (2, 1)$, $(4, 3)$ and $(3, 9)$ versus time, over 100 independent realizations starting from the same initial condition as mentioned above. As predicted by the theory, we observe that $\langle \omega_{\text{out}} \rangle$ stays constant over time, whereas the three densities converge to the average of the stationary value $m_{\mathbf{k}}^{\text{st}}$, in a time of order 10. We note that, apart from finite size fluctuations, the convergence of the densities to $m_{\mathbf{k}}^{\text{st}}$ happens for every realization. This can be seen in figure 2, where we show the evolution of $m_{(2,1)}$ and $m_{(3,9)}$ versus ω_{out} in a single run. After a short transient, the densities and the conserved quantity start to evolve in a coupled manner (except from small deviations around the $m_{\mathbf{k}} = \omega_{\text{out}}$ line) and they fluctuate from 0 to 1 until they reach the homogeneous zero state. We also observe that fluctuations in $m_{(3,9)}$ are larger than in $m_{(2,1)}$, given that degree distribution make the number of nodes in class $(2, 1)$ larger than in class $(3, 9)$.

J. Stat. Mech. (2009) P10024

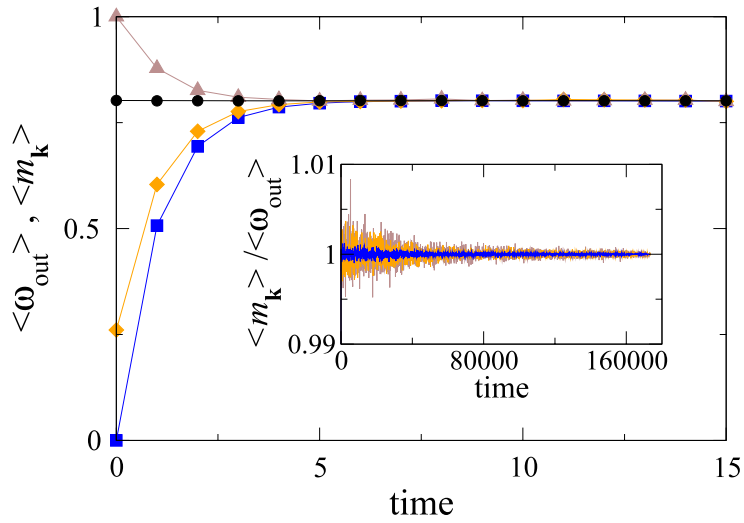

Figure 1. Time evolution of the conserved quantity ω_{out} (circles) and the densities of state-one nodes $m_{\mathbf{k}}$ in degree classes $\mathbf{k} = (k_{\text{in}}, k_{\text{out}}) = (2, 1)$ (squares), $(4, 3)$ (diamonds) and $(3, 9)$ (triangles), for the voter model dynamics. Curves correspond to averages over 100 realizations on a single random uncorrelated network with $N = 10^5$ nodes, scale-free in-degree distribution with exponent 2.5 and exponential out-degree distribution. While $\langle\omega_{\text{out}}\rangle$ remains roughly constant over time, the densities quickly decay to the stationary value $\langle\omega_{\text{out}}\rangle$. The inset shows that the ratio between the densities of state-one nodes (same degree classes as in the main graph) and the conserved quantity is close to one during the entire evolution.

4. Voter model with link update

The same assumptions and procedures apply to the link update voter model and the invasion process. The link update (LU) dynamics selects first a directed connection, so that the node at the tail will always transmit its state to the neighbor at the head.

The microscopic dynamics of the link update voter model is described by

$$s_u(t + \mathrm{d}t) = \mu_u(\mathrm{d}t)\xi_u + (1 - \mu_u(\mathrm{d}t))s_u(t), \tag{23}$$

whereas for the voter dynamics ξ_u is given by equation (3) and the binary variable $\mu_u(\mathrm{d}t)$ for the selection of a link has a probability distribution

$$P(\mu_u(\mathrm{d}t)) = k_{u,\text{in}}\,\mathrm{d}t\,\delta_{\mu_u(\mathrm{d}t),1} + (1 - k_{u,\text{in}}\,\mathrm{d}t)\delta_{\mu_u(\mathrm{d}t),0}. \tag{24}$$

A factor $\lambda/(N\langle k_{\text{in}}\rangle)$ has been reabsorbed in the definition of $\mathrm{d}t$. Proceeding as for the voter model (we skip the details), we arrive at the equation for the evolution of the relative densities $m_{\mathbf{k}}$ for the different degree classes:

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = -k_{\text{in}}m_{\mathbf{k}}(t) + k_{\text{in}}\sum_{\mathbf{k}'} P_{\text{in}}(\mathbf{k}'|\mathbf{k})m_{\mathbf{k}'}(t). \tag{25}$$

Regarding the stationary state, the same result as for the voter model is found. The state-one relative densities behave again as $m_{\mathbf{k}}(t) = \sum_{\mathbf{k}'} P_{\text{in}}(\mathbf{k}'|\mathbf{k})m_{\mathbf{k}'}(t)$, and by normalization of the conditional probability a solution is the uniform vector $m_{\mathbf{k}}(t) = m\,\forall\mathbf{k}$. We can once

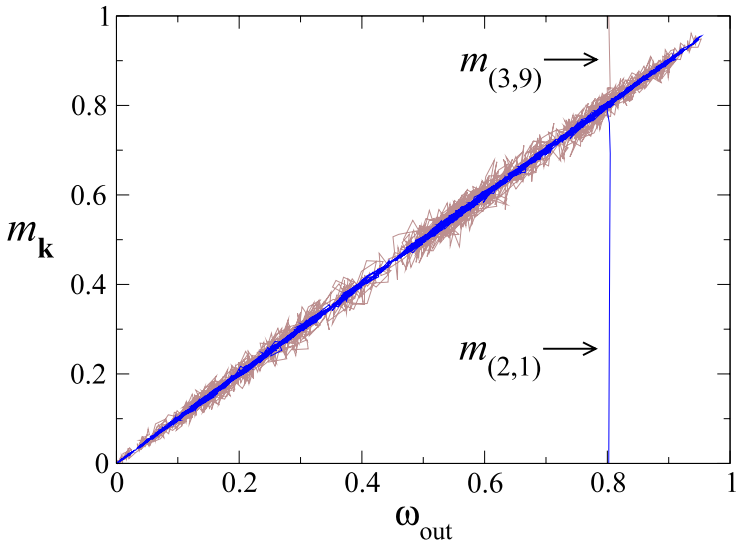

Figure 2. Densities of state-one nodes $m_{(2,1)}$ and $m_{(3,9)}$ versus ω_{out} in a single realization of the voter model dynamics on the same network of figure 1. The trajectories of classes $(2,1)$ and $(3,9)$ start at the positions $(0.8,0)$ and $(0.8,1.0)$, respectively. Then they quickly hit and move along the diagonal $m_{\mathbf{k}} = \omega_{\text{out}}$, until they reach the zero-state consensus point $m_{(2,1)} = m_{(3,9)} = 0$.

again prove, within the heterogeneous mean-field approach and for correlated strongly connected components, that a conserved quantity of the form $\omega = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} m_{\mathbf{k}}(t)$ exists and is defined by the eigenvector problem

$$\tilde{\varphi}_{\mathbf{k}} = \sum_{\mathbf{k}'} P_{\text{in}}(\mathbf{k}|\mathbf{k}')\tilde{\varphi}_{\mathbf{k}'}, \tag{26}$$

where now $\tilde{\varphi}_{\mathbf{k}} = k_{\text{in}}\varphi_{\mathbf{k}}$. In general, it is not possible to derive these coefficients without further specifying the form of degree–degree correlations in the network.

When two-point correlations are absent

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = -k_{\text{in}} m_{\mathbf{k}}(t) + k_{\text{in}}\omega_{\text{out}}(t). \tag{27}$$

In the stationary state, $m_{\mathbf{k}} = \omega_{\text{out}}(t)\,\forall \mathbf{k}$, but $\omega_{\text{out}}(t)$ is not a conserved quantity for the link update process as it was for the voter model. Instead, the conserved quantity is

$$\begin{aligned}\omega_{\text{oi}} &= \left\langle \frac{k_{\text{out}}}{k_{\text{in}}} m_{\mathbf{k}}(t) \right\rangle \Bigg/ \left\langle \frac{k_{\text{out}}}{k_{\text{in}}} \right\rangle \\ &= \sum_{\mathbf{k}} \frac{k_{\text{out}}}{k_{\text{in}}} P(\mathbf{k}) m_{\mathbf{k}}(t) \Bigg/ \left\langle \frac{k_{\text{out}}}{k_{\text{in}}} \right\rangle,\end{aligned} \tag{28}$$

which follows from equation (27). Compare this expression with that for the total magnetization in uncorrelated undirected networks $w = \omega = (\sum_k P(k) m_k(t))/\langle k \rangle$ which corresponds to the conserved quantity for those structures [22]. The dependence of the conserved weighted magnetization on the ratio between out- and in-degree for directed networks highlights the fact that in LU it is important to have both a high out-degree to be influential and at the same time to have a low in-degree in order not to be too

J. Stat. Mech. (2009) P10024

influenceable. Notice that the ratio of the directed degrees is well defined since we are assuming that the network is organized at the macroscopic scale into an SCC without peripheral components all nodes having at least one incoming and one outgoing link. Finally, in finite systems the probability of the state-one absorbing state is given by the conserved quantity, $P_1 = \omega_{\rm oi}$, and so fixed by the initial condition.

The derivation of how the state-one relative densities converge to their stationary values in uncorrelated networks is more intricate than for the voter model, but we can make use of a quasi-stationary approximation [45] in order to solve equation (27), exploiting the fact that $\omega_{\rm oi}$ is the conserved quantity. In the stationary state $\omega_{\rm out} = \omega_{\rm oi}$, and we approximate the equation by

$$\frac{{\rm d}m_{\mathbf{k}}(t)}{{\rm d}t} = -k_{\rm in} m_{\mathbf{k}}(t) + k_{\rm in}\omega_{\rm oi}. \tag{29}$$

For a given initial condition $m_{\mathbf{k}}(0)$, the solution is

$$m_{\mathbf{k}}(t) = \omega_{\rm oi}(m_{\mathbf{k}}(0) - \omega_{\rm oi}){\rm e}^{-k_{\rm in}t}. \tag{30}$$

As in the voter model, all the densities decay exponentially fast to the stationary value $\omega_{\rm oi}$, but in contrast not all the densities decay with the same velocity, which depends on the in-degree. Higher in-degree classes have smaller relaxation times and decay faster than lower ones, but the transient is always faster as compared to the VM.

5. Invasion process

The invasion process (IP) picks nodes at random that export their state to a randomly chosen outgoing neighbor. A certain node u will update its state in a passive form only when one of its incoming neighbors v is selected as the first node in one iteration of the dynamics and then v chooses u among all its outgoing neighbors to transmit its state. In this situation, it is more convenient to work with the probability of node u undergoing a state update with final state 1, $\xi_u^{(1)}$, and the probability of node u undergoing a state update with final state 0, $\xi_u^{(0)}$. The probability distributions of these dichotomic stochastic variables are

$$P(\xi_u^{(1)}) = \Phi_u^1\,{\rm d}t\,\delta_{\xi_u^{(1)},1} + (1 - \Phi_u^1\,{\rm d}t)\delta_{\xi_u^{(1)},0}, \tag{31}$$

$$P(\xi_u^{(0)}) = \Phi_u^0\,{\rm d}t\,\delta_{\xi_u^{(0)},1} + (1 - \Phi_u^0\,{\rm d}t)\delta_{\xi_u^{(0)},0}, \tag{32}$$

with

$$\Phi_u^1(t) = \sum_v a_{vu} s_v(t)/k_{v,{\rm out}}, \tag{33}$$

$$\Phi_u^0(t) = \sum_v a_{vu}(1 - s_v(t))/k_{v,{\rm out}} \tag{34}$$

and the parameter λ of the Poisson process for the happening of events reabsorbed in ${\rm d}t$. Using these expressions, the dynamics is described at the microscopic scale by

$$s_u(t + {\rm d}t) = \xi_u^{(1)}({\rm d}t)(1 - \xi_u^{(0)}({\rm d}t)) + (1 - \xi_u^{(1)}({\rm d}t))(1 - \xi_u^{(0)}({\rm d}t))s_u(t). \tag{35}$$

Following the same methodology as for the voter model, the drift equations for the relative densities in the different degree classes are

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = k_{\mathrm{in}} \sum_{\mathbf{k}'} \frac{1}{k'_{\mathrm{out}}} P_{\mathrm{in}}(\mathbf{k}'|\mathbf{k})(m_{\mathbf{k}'}(t) - m_{\mathbf{k}}(t)). \tag{36}$$

The existence of a conserved quantity $\omega = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} m_{\mathbf{k}}(t)$ in the correlated case is governed by the eigenvalue problem

$$\tilde{\varphi}_{\mathbf{k}} = \sum_{\mathbf{k}'} \frac{P_{\mathrm{in}}(\mathbf{k}|\mathbf{k}')/k_{\mathrm{out}}}{\sum_{\mathbf{k}''} P_{\mathrm{in}}(\mathbf{k}''|\mathbf{k}')/k''_{\mathrm{out}}} \tilde{\varphi}_{\mathbf{k}'}, \tag{37}$$

where $\tilde{\varphi}_{\mathbf{k}} = \varphi_{\mathbf{k}} k_{\mathrm{in}} \sum_{\mathbf{k}''} P_{\mathrm{in}}(\mathbf{k}''|\mathbf{k})/k''_{\mathrm{out}}$. Summing both sides of this equation over $\mathbf{k}$, one arrives once more at a trivial identity and so a conserved quantity exists in general on networks with degree–degree correlations. As we see next, we can be more specific on uncorrelated networks, for which equation (36) reduces to

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = \frac{k_{\mathrm{in}}}{\langle k_{\mathrm{in}} \rangle}(m(t) - m_{\mathbf{k}}(t)), \tag{38}$$

where $m(t) = \sum_{\mathbf{k}} P(\mathbf{k}) m_{\mathbf{k}}(t)$ is the total density of state-one nodes in the network.

In the stationary state, $m_{\mathbf{k}}(t) = m(t)\,\forall \mathbf{k}$, but here $m(t)$ is not a conserved quantity for the IP in uncorrelated directed networks. Instead, the conserved quantity is

$$\begin{aligned} \omega_{\mathrm{in}}(t) &= \left\langle \frac{m_{\mathbf{k}}(t)}{k_{\mathrm{in}}} \right\rangle \Bigg/ \left\langle \frac{1}{k_{\mathrm{in}}} \right\rangle \\ &= \sum_{\mathbf{k}} \frac{1}{k_{\mathrm{in}}} P(\mathbf{k}) m_{\mathbf{k}}(t) \Bigg/ \left\langle \frac{1}{k_{\mathrm{in}}} \right\rangle. \end{aligned} \tag{39}$$

In finite systems, the probability of the state-one absorbing state is given by this conserved quantity, $P_1 = \omega_{\mathrm{in}}$, and is therefore fixed by the initial condition. The dependence of the weights on the inverse of the in-degree implies that those nodes with low in-degree, so less influenceable, have the highest contribution and control the process. This dependence on the in-degree is analogous to the dependence on the degree of the conserved quantity $w = \omega = (\sum_k 1/kP(k)m_k(t))/\langle k \rangle$ in uncorrelated undirected networks [20].

After a transient, $m(t)$ reaches the value ω_{in}, so that the stationary values of the relative densities are $m_{\mathbf{k}}(t) = \omega_{\mathrm{in}}\,\forall \mathbf{k}$. This result tells us that all the densities become independent of $\mathbf{k}$ and reach the same stationary value, as in the previous processes.

The derivation of how the state-one relative densities converge to their stationary value in uncorrelated networks is more intricate than for the voter model, but like for the link update we can make use of a quasi-stationary approximation [45] in order to solve equation (38). Substituting into equation (38) that in the stationary state $m(t) = \omega_{\mathrm{in}}$:

$$\frac{\mathrm{d}m_{\mathbf{k}}(t)}{\mathrm{d}t} = \frac{k_{\mathrm{in}}}{\langle k_{\mathrm{in}} \rangle}(\omega_{\mathrm{in}} - m_{\mathbf{k}}(t)). \tag{40}$$

For a given initial condition $m_{\mathbf{k}}(0)$, the solution is

$$m_{\mathbf{k}}(t) = \omega_{\mathrm{in}} + (m_{\mathbf{k}}(0) - \omega_{\mathrm{in}})\, \mathrm{e}^{-(k_{\mathrm{in}}/\langle k_{\mathrm{in}} \rangle)t}. \tag{41}$$

All the densities decay exponentially to the stationary value ω_{in}. Higher in-degree classes decay faster than lower ones with a relaxation time that is proportional to the inverse

J. Stat. Mech. (2009) P10024

Table 1. Conserved quantities for voter-like models in strongly connected components of directed networks. First column, existence of conserved quantity for correlated networks; second column, conserved quantity for uncorrelated networks; third column, stationary values for the relative densities; fourth column, density decay.

	$\omega_{\rm corr}$	$\omega_{\rm unc}$	$m_{\mathbf{k}}^{\rm st}$	$m_{\mathbf{k}}$
VM	$\exists$	$\omega_{\rm out} = \frac{1}{\langle k_{\rm out}\rangle}\sum_{\mathbf{k}} k_{\rm out} P(\mathbf{k}) m_{\mathbf{k}}(t)$	$\omega_{\rm out}$	$m_{\mathbf{k}}^{\rm st} + (m_{\mathbf{k}}(0) - m_{\mathbf{k}}^{\rm st})\mathrm{e}^{-t}$
LU	$\exists$	$\omega_{\rm oi} = \frac{1}{\langle k_{\rm out}/k_{\rm in}\rangle}\sum_{\mathbf{k}} \frac{k_{\rm out}}{k_{\rm in}} P(\mathbf{k}) m_{\mathbf{k}}(t)$	$\omega_{\rm oi}$	$m_{\mathbf{k}}^{\rm st} + (m_{\mathbf{k}}(0) - m_{\mathbf{k}}^{\rm st})\mathrm{e}^{-k_{\rm in}t}$
IP	$\exists$	$\omega_{\rm in} = \frac{1}{\langle 1/k_{\rm in}\rangle}\sum_{\mathbf{k}} \frac{1}{k_{\rm in}} P(\mathbf{k}) m_{\mathbf{k}}(t)$	$\omega_{\rm in}$	$m_{\mathbf{k}}^{\rm st} + (m_{\mathbf{k}}(0) - m_{\mathbf{k}}^{\rm st})\mathrm{e}^{-(k_{\rm in}/\langle k_{\rm in}\rangle)t}$

of the in-degree, as is the case for LU. Due to the average degree in the relaxation time, however, transients are generally slower in the IP than in the LU. When compared with the VM, the IP dynamics exhibits a slower transient for degree classes with in-degree below average while those with in-degree above the average converge faster to the stationary state.

6. Conclusions

We have introduced an analytical formalism from microscopic dynamics to show that three different nonequilibrium dynamical models with two absorbing states running on strongly connected components of directed networks with heterogeneous degrees and degree–degree correlations have associated ensemble average conservation laws. These conservation laws, summarized in table 1, have been fully determined when degree–degree correlations are absent. The existence of ensemble average conservation laws is a general characteristic of Markov processes with two or more absorbing states.

Let us briefly discuss the validity and underlying assumptions of the conservation laws. For a stochastic process, existence of a quantity that is conserved in every *single realization* would require the state transition graph to be disconnected into components that are not mutually accessible. Then a non-trivial conserved quantity would be constant on a given component and vary across components. The voter-like models studied here, however, have weakly connected state transition graphs. For any pair of configurations (a, b), there is a trajectory of positive probability either from a to b or from b to a. Thus any quantity can be conserved only in an ensemble, i.e. as an average over the statistical distribution of trajectories starting from the same initial condition.

The constraints imposed on the dynamics by the conservation laws lead to interesting and non-trivial behavior. From a practical point of view, they are related to the stationary values and the characteristic relaxation times of the relative densities of nodes in state one in each degree class and, in finite systems, gives the probabilities of reaching the two possible absorbing states. In this sense, the conservation laws obtained in the thermodynamic limit for a system that does not order in that limit (i.e. does not reach the absorbing state) determine the probabilities of reaching each absorbing state for a finite system. The contribution of each node to the conserved global weighted magnetization is always a specific function of the directed degrees. In the case of the VM, the out-degree

J. Stat. Mech. (2009) P10024

is the weight that controls the importance of the node as a measure of its influence, while in the IP it is the inverse of the in-degree, and in the LU it is the ratio between out and in-degree. In all cases, the conserved quantities are determined by local properties that encode the importance of each node in the network. Depending on the dynamics, what seems important from a local perspective is to be influential reaching a large number of neighbors, or not to be too influenceable, with a low number of incoming connections, or both at the same time.

From a broad perspective, these studies help in the understanding of how the rich structure of real systems affects the dynamical processes that run on top. However, many questions still remain unsolved. In which specific way do degree correlations alter the results for uncorrelated networks? How is the diffusive fluctuations regime in SCCs of finite directed networks? Is the finite size scaling of consensus times the same as in undirected networks? On the other hand, it seems realistic to restrict to SCCs for a number of densely connected systems, like for instance the world trade web [32], but in sparse directed networks the whole structure of core and peripheral components should be taken into account. Numerical simulations in some specific model networks [28] show that the appearance of an input component seems to prevent the system, even if finite, from reaching an absorbing state for specific initial conditions. How does the complete structure of a directed network couple to the initial conditions of the dynamics to induce the presence of zealots and how do they affect in quantitative terms the behavior of the whole system still needs further research.

During the final completion of this work, we became aware of recent work [46] discussing the fixation probabilities of mutants for voter-like dynamics on directed networks. Since there exists a direct relation between fixation probabilities of mutants and exit probabilities, and so conserved quantities, some of the results derived in that paper—without reference to conservation laws—concerning the dependence on the directed degrees are in correspondence to some of our results on uncorrelated strongly connected components.

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