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Stochastic resonance in bistable confining potentials

On the role of confinement

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Abstract. We study the effects of the confining conditions on the occurrence of stochastic resonance (SR) in continuous bistable systems. We model such systems by means of double-well potentials that diverge like $|x|^q$ for $|x| \to \infty$. For super-harmonic (hard) potentials with $q > 2$ the SR peak sharpens with increasing q, whereas for sub-harmonic (soft) potentials, $q < 2$, it gets suppressed.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.Ey Stochastic processes

1 Introduction

The simplest dynamical system displaying stochastic resonance (SR) is a Brownian particle bound into a one-dimensional double well under the action of a time oscillating tilt and subjected to fluctuating forces (noise) [1,2]. The SR mechanism can be revealed as a maximum in the amplitude of the periodic component of the average particle position as a function of the noise intensity (temperature). Due to fluctuations, the particle randomly jumps between the two potential wells with Kramers rate [3] that depends on the double well potential and temperature. When the average escape time of the particle out of the potential minima (i.e., the inverse of the Kramers rate) approximately equals the half time-period of the applied perturbation, the noise induced interwell jumps and the periodic force synchronize, thus leading to SR.

When studying the problem of a Brownian particle in a symmetric double well periodically tilted in time, the corresponding potential $U(x)$ is usually assumed to diverge like $U(x) \sim x^4$ at large x [1,3], so as to ensure a robust confining action. However, the divergence of the potential for $|x| \to \infty$ strongly affects the response of the system to an external time-periodic forcing. The goal of the present paper is to investigate how the Brownian motion in a double well changes with the confining strength of the one-dimensional potential $U(x)$. For simplicity we assume that $U(x) \sim |x|^q$ for $x \to \pm\infty$. By studying the dependence of a SR spectral quantifier on q, we conclude that bistability is a *necessary, but not sufficient* condition for a one-dimensional system to exhibit SR.

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2 Model

The model discussed in the following represents an overdamped Brownian particle with coordinate x. Its dynamics is described by the Langevin equation,

$$\eta \dot{x} = -U'(x) + A(t) + \xi(t), \tag{1}$$

where $(\dots)' \equiv d(\dots)/dx$. The confining potential,

$$U(x) = U_0 \exp\left(-x^2/L_0^2\right) + k|x|^q/q, \tag{2}$$

is obtained by superimposing a Gaussian repulsive barrier of height U_0 and width L_0, to a power-law potential well. To ensure confinement, our analysis is restricted to $q > 1$. The total potential is mirror symmetric at $x = 0$, i.e. $U(x) = U(-x)$. Depending on q a potential $U(x)$ is called hard (super-harmonic) for $q > 2$, or soft (sub-harmonic) for $q < 2$ [4]. The periodic drive $A(t)$ is chosen as

$$A(t) = A_0 \cos(\Omega t), \tag{3}$$

with amplitude, A_0, angular frequency, $\Omega \equiv 2\pi\nu$, and time origin arbitrarily set to zero. The fluctuating force $\xi(t)$ is modeled as a stationary zero-mean Gaussian noise with auto-correlation function $\langle \xi(t)\xi(t')\rangle = 2\eta k_{\mathrm{B}} T \delta(t-t')$. Here T is the temperature and η the friction coefficient.

For numerical purposes it is convenient to choose U_0, L_0, and $\tau \equiv \eta L_0^2/U_0$ as the new units respectively of energy, space and time. Correspondingly, the variables and the parameters appearing in equation (1) can be replaced by the dimensionless quantities $\tilde{x} = x/L_0$, $\tilde{t} = t/\tau$, $\tilde{k} = L_0^q k/U_0$, $\tilde{A}_0 = A_0 L_0/U_0$, $\tilde{\Omega} = \Omega\tau$ and $\tilde{T} = k_B T/U_0$.

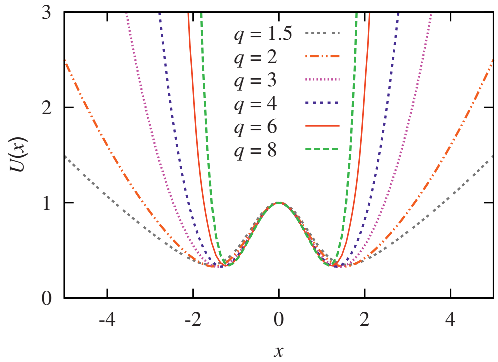

Fig. 1. (Color online) Rescaled potential (4) for $k = 0.2$ and q ranging between 1.5 and 8. The barrier height is approximately constant, $\Delta U \simeq 0.66$, and the minima $\pm x_m$ slowly shrink with q from $x_m \simeq 1.59$ down to $x_m \simeq 1.17$.

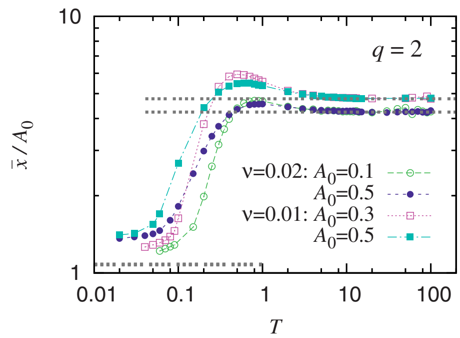

Fig. 2. (Color online) Rescaled amplitude $\bar{x}(T)/A_0$, defined in equation (6), versus T for the potential (4) with $k = 0.2$, $q = 2$. The horizontal dashed lines represent the intrawell oscillations, equation (9), with $\kappa = 1/|2k\ln(k/2)|$ for $T \to 0$, and $\kappa = k$ in the limit $T \to \infty$.

To avoid a cumbersome notation, in the following we omit all the tildes. In dimensionless notation the potential (2) reads,

$$U(x) = \exp(-x^2) + k|x|^q/q, \tag{4}$$

and the Langevin equation (1) can be rewritten as

$$\dot{x} = 2x\exp(-x^2) - k|x|^q/x + A_0\cos(\Omega t) + \sqrt{T}\xi(t), \tag{5}$$

after the Gaussian noise $\xi(t)$ has been further rescaled so that $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$. In the following we study how changing q influences the response of the particle to the periodic forcing signal $A(t)$. As a result of rescaling, the height, U_0, and the width, L_0, of the potential barrier, as well as the friction coefficient, η, have been set to one. The remaining tunable parameter k of the potential (4) will be kept fixed to $k = 0.2$ throughout the present paper. Due to the Gaussian nature of the potential barrier, the barrier height, ΔU, and the potential minima, $\pm x_m$, weakly depend on q (see Fig. 1); therefore, the observed residual SR dependence on q is mostly an effect of the varying confining strength of the potential.

We have simulated the behavior of the system by numerically integrating the rescaled Langevin equation (5) through a Milshtein algorithm [5,6]. Stochastic trajectories were simulated for different time lengths $t_{\max}$ and time steps Δt, so as to ensure appropriate numerical accuracy and transient effects subtraction. Average quantities have been obtained as ensemble averages over at least 10^4 trajectories.

3 Results

In the long time regime, after transient effects subsided, the response $\langle x(t)\rangle$ of a particle moving in a symmetric bistable potential $U(x)$ under the action of the signal (3) with small-amplitude, $A_0 x_m \ll \Delta U$, and low-frequency, $\Omega \ll U''(x_m)$, results from the interplay of inter- and the intrawell dynamics [1]. On ignoring for the time being the intrawell dynamics, the system response at low temperatures is dominated by its harmonic component [1,7–10]

$$\langle x(t\to\infty)\rangle = \bar{x}(T)\cos[\Omega t - \bar{\phi}(T)], \tag{6}$$

with amplitude, $\bar{x}(T)$, and phase, $\bar{\phi}(T)$, approximated by

$$\bar{x}(T) = \frac{A_0\langle x^2\rangle_0}{T}\frac{2r}{\sqrt{4r^2+\Omega^2}}, \tag{7}$$

$$\bar{\phi}(T) = \arctan(\Omega/2r). \tag{8}$$

Here $r \propto \exp(-\Delta U/T)$ is the Kramers rate and $\langle x^2\rangle_0$ the variance of the stationary unperturbed process $x(t)$ ($A_0 = 0$), both temperature dependent quantities. The amplitude $\bar{x}(T)$ can be manipulated by tuning the noise level. Note that equations (6)–(8) hold in the linear response theory limit, only, i.e., for $A_0 x_m \ll T$ and $\Omega > r$ [11,12].

According to equation (7), in the limit $T \to 0$ the amplitude $\bar{x}(T)$ vanishes due to the potential barrier. The rate r for the particle to overcome the potential barrier decreases to zero exponentially when lowering the temperature, that is $r \ll \Omega$. The interwell jumps are thus inhibited and the particle gets locked in either minima with probability 1/2; hence $\lim_{T\to 0}\langle x\rangle = 0$. In contrast, for high temperatures, $T \gg \Delta U$, r may grow much larger than Ω and, consequently, $\bar{x}(T) \simeq \langle x^2\rangle_0/T$. For a hard potential with $q > 2$ we show below that $\langle x^2\rangle_0 \sim T^{2/q}$, so that, again, $\lim_{T\to\infty}\bar{x}(T) = 0$. The occurrence of these limits for $T \to 0$ and $T \to \infty$ implies the existence of a maximum of $\bar{x}(T)$ for some optimal $T \sim \Delta U$. This is the so-called spectral characterization of SR [1].

3.1 Harmonic confining potentials

However, even if the approximate results (6)–(8) describe correctly the occurrence of SR in most bistable systems, Figures 2 and 3 ($q > 1$) clearly show that for $T \to 0$ the amplitude $\bar{x}(T)$ approaches a non-zero limit $\bar{x}(0) > 0$. This is a characteristic signature of the intrawell dynamics [11,12]. Moreover, for (and only for) $q = 2$ a similar behavior occurs also in the opposite limit $T \to \infty$: the curves $\bar{x}(T)$ attain an horizontal asymptote, see Figure 2. The coexistence of these two asymptotes, peculiar to $q = 2$, strongly suppresses the SR peak.

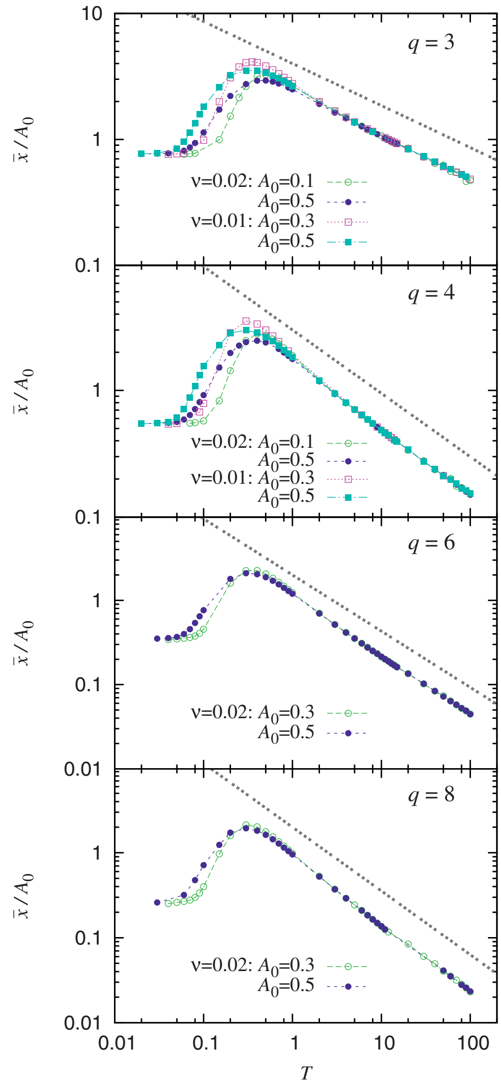

Fig. 3. (Color online) Rescaled amplitude $\bar{x}(T)/A_0$, defined by equation (6), versus T for the potential (4) with $k = 0.2$ and different $q > 2$ (hard potentials). The dashed lines are the decay power law $T^{2/q-1}$.

The nonzero $\bar{x}(T)$ limits for $T \to 0$ and $T \to \infty$ can be explained by noticing that an overdamped Brownian particle bound to a generic harmonic potential well, $U(x) = \kappa(x - x_0)^2/2$, responds to the signal (3) with amplitude

$$\bar{x} = A_0/\sqrt{\Omega^2 + \kappa^2}. \tag{9}$$

(Note also that its variance in the absence of forcing ($A_0 = 0$) is $\langle x^2 \rangle_0 = T/\kappa$).

In the low temperature limit, $T \to 0$, the particle described by the Langevin equation (5) is locked in either the right or left potential well, where it executes additional harmonic oscillations around the corresponding minima $x_0 = \pm x_m$ [1,11–13]. Such intrawell oscillations should not be mistaken for the interwell dynamics described by equation (6) [9]. Their amplitude is well reproduced by equation (9) with $\kappa \equiv U''(\pm x_m) = |2k \ln(k/2)|$.

In the high temperature limit, $T \to \infty$, the fluctuations $\xi(t)$ may grow so intense that the barrier of the bistable potential (4) becomes ineffective; the particle is thus effectively confined into a parabolic potential with $\kappa = k$ and centered at $x_0 = 0$. The amplitude of the periodic component of the particle response to the external force is then described again by equation (9) but with $\kappa = k$.

For small frequencies the rescaled amplitude $\bar{x}/A_0$ only depends on the curvature of the bistable potential at $x_0 = \pm x_m$ for $T \to 0$, $\bar{x}/A_0 = 1/|2k \ln(k/2)|$, and at $x_0 = 0$ for $T \to \infty$, $\bar{x}/A_0 = 1/k$.

The argument above can be easily generalized for any value of q at low temperatures, but it becomes untenable in the limit $T \to \infty$, where nonlinearity comes into play.

3.2 Hard confining potentials

As anticipated above, at high temperatures the presence of the central barrier can be ignored. This implies that for $T \to \infty$ equation (7) simplifies to

$$\frac{\bar{x}(T)}{A_0} = \frac{\langle x^2 \rangle_0}{T} = \frac{1}{T} \frac{\int_0^\infty dx\; x^2 \; \exp\left(-kx^q/qT\right)}{\int_0^\infty dx\; \exp\left(-kx^q/qT\right)}. \tag{10}$$

In equation (10) we made use of the inequality $r \gg \Omega$ and of the approximate expression $P_0(x) = \mathcal{N} \exp(-kx^q/qT)$ for the stationary probability density of the unperturbed process (5); $\mathcal{N}$ is an appropriate normalization constant. Note that for sufficiently low Ω, the condition $r \gg \Omega$ can be consistent with the approximations in equation (7), whereas suppressing the potential barrier makes the very definition of r meaningless.

An explicit calculation yields

$$\frac{\bar{x}(T)}{A_0} = \left(\frac{q}{k}\right)^{2/q} \frac{\Gamma(3/q)}{\Gamma(1/q)} \frac{1}{T^{1-2/q}}. \tag{11}$$

Ignoring the algebraic factors we conclude that

$$\lim_{T\to\infty} \bar{x}(T) \sim T^{2/q-1}. \tag{12}$$

From here one can see that $\bar{x}$ decreases with increasing T only for hard confining potentials with $q > 2$. In particular, for the prototypical case of a quartic potential, $q = 4$ [1], one finds $\bar{x}(T) \sim 1/\sqrt{T}$, as confirmed by the simulation results (see Fig. 3). For $q = 2$, one recovers the harmonic limit discussed in the foregoing subsection.

The decay law of $\bar{x}(T)$, equation (12), is clearly a consequence of the nonlinearity of the potential. Indeed, the same power law can be recovered by implementing the stochastic linearization scheme of reference [14]: in Gaussian approximation, for q an integer, $\lim_{|x|\to\infty} U(x) = \kappa x^2/2$ with $\kappa = (q-1)!!k\langle x^2 \rangle_0^{q/2-1}$; from the relation $\langle x^2 \rangle_0 = T/\kappa$, holding for harmonic potentials, equation (12) follows.

Moreover, $\bar{x}(T)$ cannot decrease faster than T^{-1}, which happens for $q \to \infty$. It should be noticed that $\bar{x}(T) \sim T^{-1}$ is the decay law predicted in two-state model approximation [7], where $\langle x^2 \rangle_0$ is replaced by x_m^2 (i.e., a constant).

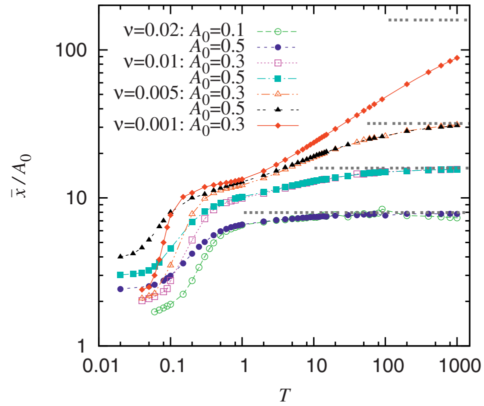

Fig. 4. (Color online) Rescaled amplitude $\bar{x}(T)/A_0$ versus T for the potential (4) with $k = 0.2$ and $q = 1.5$ (soft potential). The horizontal dashed lines represent the horizontal asymptotes $1/\Omega$ (see text). In place of the SR peak an inflexion point is detectable for low $\Omega = 2\pi\nu$.

3.3 Soft confining potentials

Equation (12) for $q < 2$ suggests that $\bar{x}(T)$ may diverge at high temperatures. However, when dealing with soft potentials, the linear theory approximations (6)–(8) must be used with caution. In the limit $T \to 0$ the interwell oscillation amplitude (7) is known to apply only for very small perturbation amplitudes [4]: this explains the residual A_0 dependence of the low T plateaus reported in Figure 4.

More importantly, in the high T limit, although the barrier of a soft potential is awash with noise, confinement gets so weak that the particle is driven up and down the potential walls primarily by the deterministic force $A(t)$, rather than by the noise. (For a comparison, we remind that a particle falls from $\pm\infty$ down to $\pm x_m$ in a finite time for $q > 2$ and in an infinite time for $q < 2$). In conclusion, on assuming that the Brownian particle oscillates as if it were (almost) free, its amplitude would read

$$\lim_{T\to\infty} \bar{x}(T) \sim A_0/\Omega. \qquad (13)$$

$\bar{x}(T)$ is then expected to develop high T plateaus also for $q < 2$, but, in contrast with the cases discussed in Section 3.1, such plateaus are inverse proportional to the drive frequency (also for low frequency drives, see Fig. 4).

In the case of sub-harmonic bistable potentials the hallmark of SR is thus the monotonic increase of the response amplitude with T, as opposed to the occurrence of a maximum often detected in the super-harmonic potentials. Such a behavior resembles the phenomenon of "SR without tuning" discussed in reference [15], with the important difference that here it has been observed in a *single* unit, rather than in a summing network of N excitable units.

4 Conclusions

We conclude this note with two important remarks:

(i) The coexistence of two locally stable minima separated by a potential barrier is commonly advocated to explain the occurrence of a SR peak in a continuous bistable dynamics. Here we have shown that this keeps being true as long as the confining action exerted by the potential is super-harmonic. Most notably, for harmonic and sub-harmonic potentials the periodic component of the system response may increase monotonically with the noise level.

(ii) In many experimental reports (see, for a review, Ref. [16]), the authors tried to characterize the SR peak by means of equation (6), without paying much attention to the T dependence of the quantity $\langle x^2\rangle_0$. In some cases they adopted an outright two-state model with $\langle x^2\rangle_0 = x_m^2$ [7]. This led to a poor fit of the decaying tail of $\bar{x}(T)$, whereas a more accurate fit could have given a valuable clue to better model the system at hand [17].

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