Local transfer of optical angular momentum to matter

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september 15, 2004

Abstract

We resolve a paradoxical difference between the local density of optical angular momentum and the associated torque exerted on a trapped particle.

Mechanical properties of light like linear and angular momentum allow for the trapping, rotation and manipulation of microscopic objects [1]. They underpin important devices such as optical tweezers [2] and spanners [3, 4], with both technological and biological applications. Light beams elliptically polarized or with helical phase profiles have been shown to impart rotations to absorbing objects [3, 4]. A recent experiment shows that both the spin and the orbital angular momentum transferred to a small particle held off-axis are proportional to the *local intensity* of the beam [5]. An earlier experiment has shown the mechanical equivalence of spin and orbital angular momentum on small absorbing particles held on the beam axis [4].

These experimental observations appear to be in conflict with theoretical considerations [6], which predict that the density of angular momentum carried by a monocromatic paraxial beam of frequency ω in the propagation direction z is

$$j_z = \epsilon_0 \omega l |u|^2 - \frac{\epsilon_0}{2} \omega r \sigma \frac{\partial}{\partial r} |u|^2. \tag{1}$$

Here $u(r, \phi, z) = u_0(r, z) e^{il\phi}$ is the characteristic helical phase profile of the field leading to an orbital angular momentum proportional to l, and σ is the degree of polarization, taking the extremes values ± 1 for right and left circular polarizations. Eq. (1) shows that the orbital component is indeed proportional to the local intensity, consistent with experimental observations [5]. The spin angular momentum density, however, depends on the radial gradient of the intensity

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[6]. This suggests that the local sign of the spin density can be different from the global polarization state of the beam, σ, and hints at the possibility that a particle taken away from the beam axis would change its spinning direction, depending on the local sign of the gradient in Eq. (1). Moreover, the local spin density vanishes for a circularly polarized plane wave [7], as well as along the circle of maximum intensity in a donut mode, exactly where small particles are typically trapped and observed to rotate [5].

The conflict between the local (density) and global (polarization) features in the transfer of spin angular momentum is a long standing problem [6, 7, 8, 9, 10]. We might argue that each photon carries angular momentum $\hbar(l+\sigma)$ per photon and that each absorbed photon, therefore, must transfer this to the absorbing body. However a consistent analysis within the Maxwell formalism of the relation between the density of angular momentum and the effective local torque on small objects is still missing and remains a problem.

The aim of this letter is to resolve the problem of the local and global features of the angular momentum of light in a consistent treatment of the interaction with dielectric objects. In spite of the functional dependence of the spin, we obtain results consistent with experiments through the identification of the angular momentum flux [11] as the relevant quantity connected to the effective torque. The dynamics induced by the spin and orbital torques is then shown to be in agreement with experimental observation. We also identify the origin of the puzzling functional dependence of the spin density in the constraint of transversality for the electromagnetic fields.

Following Ref. [6] we consider the field described by the Lorenz-gauge vector potential

$$\vec{A}(\vec{x},z,t) = [\alpha\hat{x} + \beta\hat{y}]u(\vec{x},z)e^{ikz-i\omega t} \tag{2}$$

where $\vec{x} = (x,y)$, u is a solution of the paraxial wave equation, and $|\alpha|^2 + |\beta|^2 = 1$. From the electric and magnetic fields associated with $\vec{A}$ we can calculate the total angular momentum density $\vec{j} = \vec{r} \times \frac{\epsilon_0}{2}(\vec{E} \times \vec{B}^* + c.c.)$. Its component in the propagation direction is given in by Eq. (1), where $\sigma = i(\alpha\beta^* - \alpha^*\beta)$. To describe interaction of the beam with an object of area S we consider the integral of the density j_z over S. The orbital angular momentum is then proportional to the light intensity in the object area while the integrated spin is

$$\epsilon_0\omega\sigma \int_{\phi_a}^{\phi_b} d\phi \left[\int_{r_a}^{r_b} drr|u|^2 - \frac{1}{2}r^2|u|^2 \Bigg|_{r_a}^{r_b} \right]. \tag{3}$$

We identify a first contribution proportional to the integrated intensity - the *polarization intensity* term-, and a second *boundary* term. For objects that are large with respect to the radius of the illuminating beam [14, 4] the boundary term vanishes. In general, however, the local value of the spin in a small region depends on the relative size of the intensity and boundary terms in Eq. (3). Note that these two contributions have opposite signs and can lead to a total

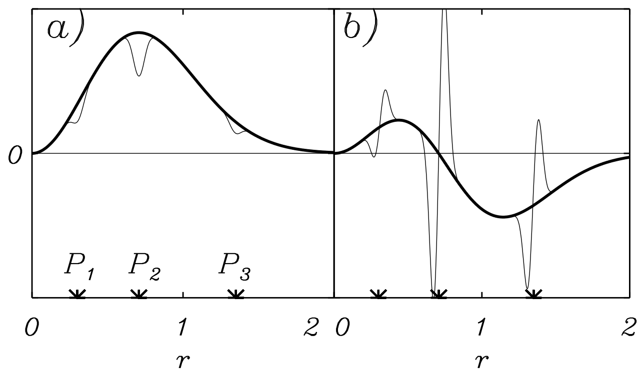

Figure 1: a) Intensity $|u(r, \phi, z = 0)|^2 \propto r^2 \exp -2r^2$ (thick line) of a circularly polarized beam ($\sigma = 1$). Intensity $(M_i u)^2$ of the beam interacting with small absorbers placed in P_i ($i = 1, 2, 3$) (thin lines), with mask functions $M_i(r, \phi = 0) = 1 - c \exp(-(\frac{r-r_i}{d})^2)$ ($c, d \ll 1$). b) Radial profile of the spin angular momentum density, for the unperturbed beam $j_z \propto r\partial_r |u|^2$ (thick line) and for the beam interacting with absorbers $j_z \propto r\partial_r |M_i u|^2$ (thin line).

that is opposite in sign to the global polarization σ. Hence not only the density of angular momentum but also the integral of this quantity over the object area suggests a discrepancy with experimental observations.

For a proper description of the effects of the beam on the illuminated object we have to evaluate the extent to which the beam is modified by the interaction. We consider a dielectric object weakly interacting with the beam, that is a material with refractive index differing only slightly from unity. In this weak dielectric limit we describe the effect of the object through a mask function $M(\vec{x})$ acting on the vector potential so that the change in the potential after interacting with the medium is then $\vec{A} \to M\vec{A}$ (AMA ansatz) with $\vec{A}$ given in Eq. (2) [12]. In the weak dielectric limit the transmission is almost unity and we can neglect the wave reflected by the object and other scattering effects. Different dielectrics allow for different mechanisms of transfer of angular momentum with real and complex masks M describing absorption and phase shift phenomena respectively. An illustration of the effect of an absorbing object on the intensity of the beam is plotted in Fig. 1a. Possible anisotropy of the object can be included through two different functions $M(\vec{x})$ and $N(\vec{x})$ acting on the x and y components of the potential (2). For an object placed in the region $0 < z < \Delta z$ the output potential (for $z > \Delta z$) is

$$\vec{A}(\vec{x}, z, t) = [M(\vec{x})\alpha\hat{x} + N(\vec{x})\beta\hat{y}]u(\vec{x}, 0)e^{ikz - i\omega t}. \tag{4}$$

We assume the object is thin so that we can neglect the z variation of the slowly varying function u in Eq. (4). It is important to note that our vector potential description has the calculational advantage that we do not need to worry about transversality, which is taken account of automatically in the form of the scalar potential. The x and y components of the $\vec{E}$ and $\vec{B}$ fields after the object are obtained from the input ones by replacing $\alpha u \to \alpha M u$ and $\beta u \to \beta N u$ [13].

The change of angular momentum density carried by the light beam is

$$
\begin{aligned}
j_z - j_z' &= \omega \frac{\epsilon_0}{2} \{ -r \frac{\partial}{\partial r} \left[|u|^2 (\sigma - \sigma') \right] \\
&+ \left[|u|^2 2l \left(1 - |\alpha M|^2 - |\beta N|^2 \right) \right] \\
&- i|u|^2 (|\alpha|^2 M \frac{\partial}{\partial \phi} M^* + |\beta|^2 N \frac{\partial}{\partial \phi} N^* - c.c.) \}
\end{aligned}
\tag{5}
$$

where the prime distinguishes the output angular momentum j_z', and $\sigma' = i(\alpha\beta^* M N^* - c.c)$. The first term in Eq. (5) is the spin angular momentum density variation, arising for both absorption (for real $M = N$) and polarization rotation ($M \neq N$). The contribution of the second term describes the change of orbital angular momentum due to absorption, for which $|\alpha M|^2 + |\beta N|^2 \neq 1$. Finally, the last term in Eq. (5) is non-vanishing only for M, N *not* being real functions and accounts for objects introducing a variation on the field phase profile with respect to the input $e^{il\phi}$. As an example, in Fig. 1b we show the change in the local spin angular momentum introduced by an absorber on a circularly polarized beam. The effect of the mask function is clearly visible, introducing a strong local oscillation in the spin density, depending on the gradient of the mask function $r\partial_r |Mu|^2$.

The transfer of angular momentum to an object is obtained integrating the density Eq. (5). For a paraxial field $c \int (j_z - j_z') d\vec{x}$ is simply the angular momentum $flux$ over a plane perpendicular to the propagation direction before and after the object [11]. The spin flux is then

$$
\epsilon_0 \omega \int_{\phi_a}^{\phi_b} d\phi \left[\int_{r_a}^{r_b} dr r (\sigma - \sigma') |u|^2 - (\sigma - \sigma') \frac{r^2 |u|^2}{2} \Bigg|_{r_a}^{r_b} \right].
$$

The (first) intensity term gives a spin transfer with the same sign of $(\sigma - \sigma')$. As we have already seen (Eq. (3)) the strange local effects arising from the radial gradient are due to the boundary (second) term. However the field is actually unchanged around the object ($\sigma(r_a) = \sigma'(r_a)$, $\sigma(r_b) = \sigma'(r_b)$) so that this boundary term variation does not contribute to the flux. We conclude that the peculiar behaviour of the local density of spin does not give rise to any mechanical effect. The relevant quantity in the interaction with an object is the change in the flux of angular momentum given only by the intensity term.

When the flux of angular momentum is considered the predicted mechanical effects on a small object are in agreement with experimental evidence, as illustrated in Fig. 1. Indeed the spin detected by a small absorber, that is the difference of the integrals around P_i of the functions plotted in Fig. 1b, is *equivalent* to the intensity variation, given by the area between the thick and thin lines around P_i in Fig. 1a. Therefore the transferred angular momentum, i.e. the nett angular momentum flux, has the same sign at all points P_i, in spite of the local change of the sign of the spin density in Eq. (1). Moreover, we note that at P_2, where objects are typically trapped by gradient forces, the most pronounced transfer of angular momentum is observed. If the beam carries both spin and orbital angular momentum the amount transferred to the absorber is

$$\omega\epsilon_0 \int_{\phi_a}^{\phi_b} d\phi \int_{r_a}^{r_b} drr|u|^2(1-M^2)(l+\sigma), \tag{6}$$

confirming that spin and optical angular momentum have equivalent effects on absorbing objects [4].

In order to clarify the meaning of the final term in Eq. (5), we consider a dielectric with $M(\vec{x}) = N(\vec{x}) = e^{i\theta(\vec{x})}$. The fact that M is complex means that the term $(M\partial_\phi M^* - c.c.)$ is non zero. However for a small regular object, $\theta(\phi = 0) = \theta(\phi = 2\pi)$, the integral will vanish. Hence a transparent *regular* object is able to locally modify the phase profile of the beam but not to change its orbital angular momentum, which is a global property of the front. A transparent object can change the total orbital angular momentum only if it described by a discontinuous function θ. An example is the spiral phase-plate [15], that “adds a phase” increasing with the angular position over all the profile of the beam ($\theta(r,\phi) \propto \phi$).

To complete our analysis we derive the mechanical effect, that is the resulting *torque*, arising by the transfer of angular momentum to an illuminated object. In the Maxwell formalism the torque and the temporal variation of the angular momentum are related to the angular momentum flux by a continuity equation [17], that in vacuum reduces to Eq. (14) of Ref. [11]. For paraxial and monochromatic fields the integrated continuity equation has the simple form

$$c\int d\vec{x}\ (j_z - j_z') = \int d\vec{x} \int_0^{\Delta z} dz\ g_z(\vec{x}, z). \tag{7}$$

Hence the nett flux, on the left hand side, is naturally connected to the torque g_z in the propagation direction on the illuminated object.

The torque density is [8]

$$g_z = rf_\phi + \frac{1}{2}[(\vec{P} \times \vec{E}^*)_z + c.c], \tag{8}$$

with f_ϕ being the azimuthal component of the Lorentz force $\vec{f} = 1/2[(\vec{P}\cdot\vec{\nabla})\vec{E}^* + (\partial_t\vec{P}) \times \vec{B}^* + c.c]$ [16], and $\vec{E}, \vec{P}, \vec{B}$ being the electric, polarization and magnetic fields inside the dielectric region. These fields are obtained by propagating our external fields into the weak dielectric and lead to the relationship between the refractive index and the mask functions

$$M = \exp[ik(n_x - 1)\Delta z], \quad N = \exp[ik(n_y - 1)\Delta z]. \tag{9}$$

Direct calculation of the torque with the fields in the dielectric region confirms that the torque balances the angular momentum change (5) we obtained within the AMA ansatz. The fact that Eq. (7) is satisfied by our solution is a confirmation that the AMA ansatz gives a solution of the Maxwell equations, able to capture the fundamental features of the transfer of angular momentum process.

Given the Lorentz force and the torque we can describe the dynamics of a small isotropic absorber (with mask M as in Fig 1). An object held off axis, centred in $\vec{x}_0 \neq 0$, is trapped in a circle of maximum intensity in the donut beam. The damped dynamics of the centre of mass $\vec{x}_0 = (r_0, \phi_0)$, after initial transients have died away, is governed by

$$\gamma r_0 \dot{\phi}_0 = (\vec{F} \cdot \hat{\phi}_0) \tag{10}$$

where γ is the damping coefficient, associated with the viscosity of the medium in which the particle is suspended, and $\vec{F} = \Delta z \int \vec{f} d\vec{x}$ is the total Lorentz force. For a small object the azimuthal total Lorentz force in Eq. (10) is proportional to $f_\phi(\vec{x}_0)$. By the relation between the force and the orbital torque $g_z^{orb} = r f_\phi$ (see Eq. 8) we then obtain the angular velocity

$$\dot{\phi}_0 \propto l \frac{|u(\vec{x}_0)|^2}{r_0^2}. \tag{11}$$

Hence beams carrying orbital angular momentum induce a motion of a small object around the beam axis with angular velocity proportional to the local intensity of the beam divided by the squared distance to the beam axis. This functional dependence has been observed in the recent experiment in Ref. [5].

It remains to examine the dynamics of the object held off-axis about its centre of mass $\vec{x}_0$. We consider a new frame of reference with origin at $\vec{x}_0$, so that the total orbital torque vanishes while the spin torque g_z^{spin} is unchanged, with density given by the second term in Eq. (8). In the new coordinates (ρ, ψ) we then obtain

$$\dot{\psi} \propto \frac{g_z^{spin}(\vec{x}_0)}{\gamma'} \propto \sigma |u(\vec{x}_0)|^2. \tag{12}$$

Therefore an elliptically polarized beam ($\sigma \neq 0$) induces a spinning motion of a small absorber around its own axis with an angular velocity proportional to the local intensity, in agreement to what observed in Ref. [5].

If the object is held on the beam axis then the total Lorentz force vanishes so that the centre of mass does not move. Now both the spin and the orbital torque contribute to the spinning of the object around its axis

$$\dot{\phi} = \frac{\int d\vec{x} g_z(\vec{x})}{\gamma'} \propto (\sigma + l) \int d\vec{x} |u(\vec{x})|^2 [1 - M^2(\vec{x})], \tag{13}$$

and we obtain the mechanical equivalence observed in [4].

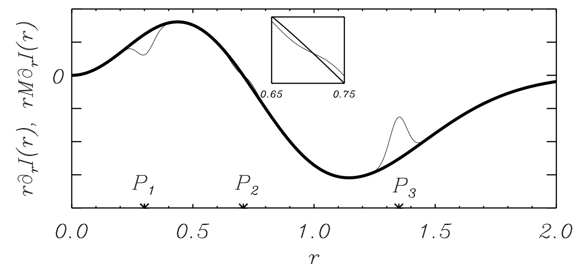

Figure 2: Radial profile of the spin angular momentum density with the EME ansatz. $j_z \propto r\partial_r |u|^2$ (thick line) and $j'_z \propto rM_i\partial_r |u|^2$ (thin line). The insert is a zoom in the region of vanishing momentum.

We have presented a consistent description of the transfer of angular momentum to small objects, identifying the flux of angular momentum as the relevant quantity leading to torque effects. The peculiar functional form of the spin density does not have any mechanical effect on small objects but we should ask the reason for its dependence on the radial gradient of the intensity. The fact that this functional dependence appears in the spin but not in the orbital angular momentum suggests a possible explanation related to the vectorial character of the field. We can test this idea by considering an absorbing object ($M \in R, M < 1$) and deliberately ignoring the constraints of transversality by obtaining the output fields as $\vec{E} \rightarrow M\vec{E}$, $\vec{B} \rightarrow M\vec{B}$. The output fields given by this EME ansatz are not solutions of the Maxwell equations, as they are *not* transverse. With this ansatz the angular momentum after the object would be

$$j'_z = M^2 j_z. \tag{14}$$

This is fundamentally different to the result obtained using the AMA ansatz in that the density does not depend on the gradient of M. Eq. (14) leads to a sign of the transferred angular momentum determined by the *local* sign of the density of spin angular momentum of the incoming field. As shown in Fig. 2 the integral of the difference of the thick and thin curves has a different sign at P_1 and P_3, leading to rotations in opposite directions. In such a picture an object interacting with the beam would be influenced by the local functional form of the spin density rather than the polarization state of the beam. In particular the amount of rotation predicted in P_2 is now nearly negligible (see insert in Fig. 2). The key physical principle neglected in this description is transversality. This suggests that the form of the spin angular momentum density is strongly determined by the transversality constraint for the fields. The correct spin transfer is predicted within the AMA ansatz, in which the electromagnetic field is a (transverse) solution of the Maxwell equations.

We have shown within the Maxwell formalism that, in spite of the gradient form of the spin angular momentum density, the transfer of both spin and orbital angular momenta depends on the local light intensity and thereby resolved a long-standing controversy. This has been possible through the identification of the angular momentum flux as the proper quantity in the description of mechanical effects of the optical angular momentum. The flux before and after the object have been related to the effective torque in the medium showing the consistency of our treatment. Given the torque we have found that our calculations of the object dynamics are in full agreement with experimental observations. Finally we have demonstrated the relevance of the transversality constraint for the electromagnetic field in the peculiar functional form of the spin angular momentum density.

We gratefully acknowledge useful discussions with E. Anderrson, R. Loudon, M. Padgett and L. Allen and financial support from the UK Engineering and Physical Sciences Research Council (GR/S03898/01) and the European Commission through the project QUANTIM (IST-2000-26019).

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