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Analytical solution of the voter model on uncorrelated networks

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Abstract. We present a mathematical description of the voter model dynamics on uncorrelated networks. When the average degree of the graph is $\mu \leqslant 2$ the system reaches complete order exponentially fast. For $\mu > 2$, a finite system falls, before it fully orders, in a quasi-stationary state in which the average density of active links (links between opposite-state nodes) in surviving runs is constant and equal to $\frac{(\mu-2)}{3(\mu-1)}$, while an infinitely large system stays ad infinitum in a partially ordered stationary active state. The mean lifetime of the quasi-stationary state is proportional to the mean time to reach the fully ordered state T, which scales as $T \sim \frac{(\mu-1)\mu^2 N}{(\mu-2)\mu_2}$, where N is the number of nodes of the network, and μ_2 is the second moment of the degree distribution. We find good agreement between these analytical results and numerical simulations on random networks with various degree distributions.

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1. Introduction

The *voter model* has become one of the most popular interacting particle systems [1, 2] with applications to the study of diverse processes like opinion formation [3, 4], kinetics of heterogeneous catalysis [5, 6] and species competition [7]. The general version of the model considers a network formed by nodes holding either spin 1 or -1. In a single event, a randomly chosen node adopts the spin of one of its neighbors, also chosen at random. Beyond this standard version, several variations of the model have been considered in the literature, to account for zealots or inhomogeneities (individuals that favor one of the states) [8], constrained interactions [9], non-equivalent states [10], asymmetric transitions or bias [11], noise [12], memory effects [13] and ecological diversity [14]. It is also known that several models presenting a coarsening process without surface tension belong to the voter model universality class [15].

In a regular lattice, the mean magnetization, i.e. the normalized difference in the number of 1 and -1 spins, is conserved at each time step. Thus, the magnetization is not a useful order parameter to study the ordering dynamics of the voter model. Instead, it is common in the physics literature to use as an order parameter the density of interfaces ρ, i.e, the fraction of links connecting neighbors with opposite spins. In a finite system, the only possible final state is the fully ordered state, in which all spins have the same value, either -1 or 1, and therefore all pairs of neighbors are aligned ($\rho = 0$). These are absorbing configurations given that the system cannot escape from them once they are reached [16]. Despite its non-trivial dynamics, an exact solution has been obtained for regular lattices of general dimension d [5, 6], becoming one of the few non-equilibrium models which are exactly solvable in any dimension. Indeed, the correspondence between the voter model and a system of coalescing random walkers helps to solve analytically many features of the dynamics [17, 18]. For $d \leqslant 2$, there is a coarsening process where the average size of ordered regions composed by sites holding the same spin

continuously grows. In the thermodynamic limit, the approach to the final frozen configuration is characterized by the monotonic decrease in ρ that decays as $\rho \sim t^{-1/2}$ in $1d$ and $\rho \sim (\ln t)^{-1}$ in $2d$ [5]. For $d > 2$, the density of active interfaces behaves as $\rho(t) \sim a - b\, t^{-d/2}$ [6], thus $\rho(t)$ reaches a constant value in the long time limit where the system reaches a stationary active state with nodes continuously flipping their spins. That is to say, full order is never reached. We need to clarify that the last is only true for infinite large systems, given that fluctuations in finite size lattices make the system ultimately reach complete order. The level of order in the stationary state is quantified by the two-spin correlation function $C_{ij} \equiv \langle S_i S_j \rangle$ between spins i and j that decays with their spatial separation $r = |i - j|$ as $C(r) \sim r^{(2-d)}$ [19], i.e. far apart spins become uncorrelated. Recent studies of the voter model on fractals with fractal dimension in the range (1,2), reveal that the system orders following $\rho(t) \sim t^{-\alpha}$, with the exponent α in the range (0,1) [20, 21].

The voter model has recently been investigated on complex networks [22]–[28], where its behavior seems to strongly depend on the topological characteristics of the network. A peculiar aspect is that the dynamics can be slightly modified giving different dynamical scaling laws. For instance with *node update*, i.e. selecting first a node and then one of its neighbors, the conservation of the magnetization is no longer fulfilled. Instead the degree-weighted magnetization, i.e. the sum over all nodes of its degree times its spin value, is in this case conserved at each time step. With *link update*, where a link is selected at random and then one of its ends is updated according to the neighbor's spin, the conservation of the magnetization is restored [24].

A striking feature of the voter model on several complex networks, including small-world, Barábasi–Albert (BA), Erdős–Rényi (ER), exponential and complete graph is the lack of complete order in the thermodynamic limit. In this paper, we provide an analytical insight into the incomplete ordering phenomenon in heterogeneous networks by studying the evolution and final state of the system using a simple mean-field (MF) approach. Despite the fact that this approach is meant to work well in networks with arbitrary degree distributions but without node degree correlations (uncorrelated networks), the qualitative results are rather general for many networks. We obtain analytical predictions for the density of active links (links connecting nodes with opposite spin) and the mean time to reach the ordered state as a function of the system size and the first and second moments of the degree distribution. These predictions explain numerical results reported in [24]–[27] and they agree with previous analytical results for ordering times [25].

The rest of the paper is organized as follows. In section 2, we define the model and its updating rule on a general network. We then develop in section 3 a MF approach for the time evolution of the density of active links and the link magnetization. This approximation reveals a transition at a critical value of the average connectivity $\mu = 2$. When μ is smaller than 2, complete order is reached exponentially fast, whereas for $\mu > 2$, the system quickly settles in a quasi-stationary disordered state characterized by a constant density of active links whose value only depends on μ, independent of the degree distribution. We find that ρ is proportional to the product of the spin densities with a proportionality constant that depends on μ. This relation allows us to derive an approximate Fokker–Planck equation for the magnetization in section 4. This equation is used in section 5 to study the relaxation of a finite system to the absorbing ordered state and in section 6 to obtain an expression for the survival probability of independent runs. The mean time to reach complete order, calculated in section 7, shows that the dependence of the results on the network topology enters through the first and the second moments of the

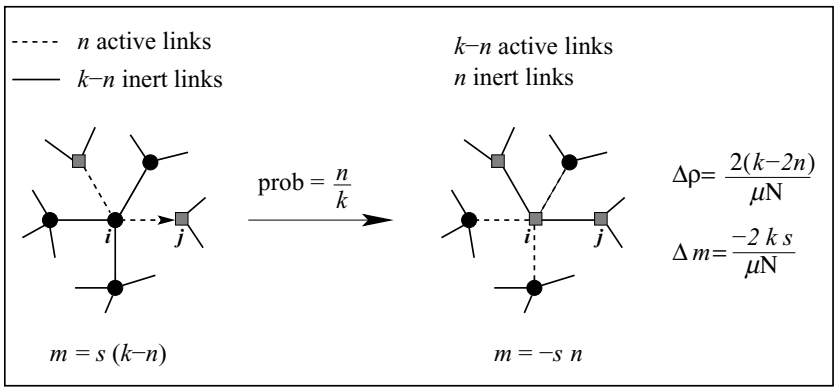

Figure 1. Update event in which a node i with spin $S_i = s$ (black circle) flips its spin to match its neighboring node spin $S_j = -s$ (gray square). The possible values of the spins are $s = \pm 1$. Changes in the density of active links ρ and the link magnetization $m = \rho_{++} - \rho_{--}$ are denoted by $\Delta\rho$ and Δm, respectively.

degree distribution only. Convergence to the ordered state slows down as μ approaches 2, where ordering times seem to diverge faster than N. The summary and conclusions are provided in section 8. In the appendix, we present some details of calculations.

2. The model

We consider a network composed of a set of N nodes and the links connecting pairs of nodes. We assume that the network has no degree correlations, i.e. the neighbors of each node are randomly selected from the entire set. We denote by P_k the degree distribution, which is the fraction of nodes with k links, subject to the normalization condition $\sum_k P_k = 1$. In the initial configuration, spins are assigned the values 1 or -1 with probabilities given by the initial densities σ_+ and σ_-, respectively. In a single time step, a node i with spin S_i and one of its neighbors j with spin S_j are chosen at random. Then i adopts j's spin ($S_i \to S_i = S_j$) (see figure 1). This step is repeated until the system reaches complete order and it cannot longer evolve.

3. MF theory

In order to obtain an insight about the time evolution of the system we develop a MF approach. There are two types of links in the system, links between nodes with different spin or *active links* and links between nodes with the same spin or *inert links*. Given that a single spin-flip update happens only when an active link is chosen, it seems natural to consider the *global density of active links* ρ as a parameter that measures the level of activity in the system.

In figure 1, we describe the possible changes in ρ and their probabilities in a time step, when a node i with spin $S_i = s$ ($s = 1$ or -1) and degree k is chosen. We denote by n the number of active links connected to node i before the update. With probability n/k an active link (in this example $i - j$) is randomly chosen. Node i flips its state changing the state of its links from active to inert and vice versa, and giving a local change of the number of active links $\Delta n = k - 2n$ and a global density change $\Delta\rho = \frac{2(k-2n)}{\mu N}$. Here, $\mu N/2$ is the total number of

links, $\mu \equiv \langle k \rangle = \sum_k k P_k$ is the number of links per node or average degree. Assembling these factors, the change in the average density of active links in a single time step of time interval $\mathrm{d}t = 1/N$ is described by the master equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \sum_k P_k \left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_k = \sum_k \frac{P_k}{1/N} \sum_{n=0}^{k} B(n,k) \frac{n}{k} \frac{2(k-2n)}{\mu N}, \tag{1}$$

where $B(n,k)$ is the probability that n active links are connected to a node of degree k, and $\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_k$ denotes the average change in ρ when a node of degree k is chosen. Given that, during the evolution, the densities of + and − spins are not the same, we expect that $B(n,k)$ will depend on the spin of node i. For instance, when the system is about to reach the + fully ordered state, we expect a configuration where most of the neighbors of a given node (independent of its spin) have + spin, thus the probability that a link connected to a node with spin + (−) is active will be close to zero (one). Therefore, we take $B(n,k)$ as the average probability over the two types of spins

$$B(n,k) = \sum_{s=\pm} \sigma_s \, B(n,k|s), \tag{2}$$

where $B(n,k|s)$ is the conditional probability that n of the k links connected to a node are active, given that the node has spin s. Substituting equation (2) into (1) we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{2}{\mu} \sum_k P_k \sum_{s=\pm} \sigma_s \sum_{n=0}^{k} B(n,k|s) \frac{n}{k}(k-2n) \tag{3}$$

$$= \frac{2}{\mu} \sum_k P_k \sum_{s=\pm} \sigma_s \left[\langle n \rangle_{k,s} - \frac{2}{k} \langle n^2 \rangle_{k,s} \right], \tag{4}$$

where $\langle n \rangle_{k,s}$ and $\langle n^2 \rangle_{k,s}$, are the first and the second moments of $B(n,k|s)$, respectively.

In order to calculate $B(n,k|s)$, we assume that only correlations between the states of first neighbors are relevant, neglecting second or higher neighbors correlations. Therefore, we consider the conditional probability $P(-s|s)$, that a neighbor of node i has spin $-s$ given that i has spin s, to be independent of the other neighbors of i. This is known in the lattice models literature with the name of *pair approximation*, and it is supposed to work only in networks without degree correlations. Thus, $B(n,k|s)$ becomes the binomial distribution with $P(-s|s)$ as the single event probability that a link connected to i is active. $P(-s|s)$ can be calculated as the average fraction of neighbors with spin $-s$ to a node with spin s, i.e. the ratio between the total number $\rho\mu N/2$ of $s \to -s$ links and the total number $\mu\sigma_s N$ of links connected to nodes with spin s. We have used the symmetry in the states of the voter model and assumed that the average degrees of nodes holding spins 1s and -1 are the same and equal to μ. We have numerically checked that the last is valid for the original voter model, but if the two states are not equivalent or a biased is introduced, the average degrees are different. Then, $P(-s|s) = \rho/2\,\sigma_s$, and the first and the second moments of $B(n,k|s)$ are

$$\langle n \rangle_{k,s} = \frac{k\rho}{2\sigma_s},$$

$$\langle n^2 \rangle_{k,s} = \frac{k\rho}{2\sigma_s} + \frac{k(k-1)\rho^2}{4\sigma_s^2}.$$

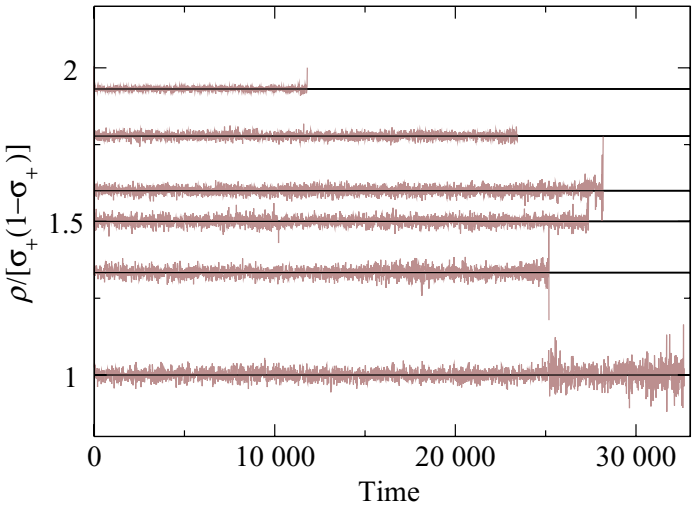

Figure 2. Ratio between the density of active links and the product of the spin densities versus time in one realization of the voter model dynamics on degree-regular (DR) random graphs with $N = 10\,000$ nodes and values of $\mu = 3$, 4, 5, 6, 10 and 30 (bottom to top). Solid horizontal lines are the constant values $4\xi = \frac{2(\mu-2)}{(\mu-1)}$.

Replacing these expressions for the moments in equation (4) and performing the sums we finally obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{2\rho}{\mu}\left[(\mu-1)\left(1 - \frac{\rho}{2\sigma_+(1-\sigma_+)}\right) - 1\right]. \tag{5}$$

Equation (5) is the master equation for the time evolution of ρ as a function of the spin density $\sigma_+(t)$. It has two stationary solutions, but depending on the value of μ, only one is stable. For $\mu \leqslant 2$, the stable solution $\rho = 0$ corresponds to a fully ordered frozen system. For $\mu > 2$, the stable solution is

$$\rho(t) = 4\xi(\mu)\sigma_+(t)\left[1 - \sigma_+(t)\right], \tag{6}$$

where we define

$$\xi(\mu) \equiv \frac{(\mu-2)}{2(\mu-1)}, \tag{7}$$

corresponding to a partially ordered system, composed by a fraction $\rho > 0$ of active links, as long as $\sigma_+ \neq 0, 1$.

In figure 2, we test equation (6) by plotting the time evolution of the ratio between ρ and $\sigma_+(1-\sigma_+)$ in a single realization, for various values of μ. We observe that, even though the ratio varies over time, it fluctuates around the constant value 4ξ predicted by equation (6). It is worth noting that the behavior of the ratio is the same from times of order one to the end of the realization, where fluctuations increase in amplitude before the system reaches complete order. We also notice that fluctuations decrease as μ increases, and they become zero in the complete graph case ($\mu = N - 1$), where we have $\rho(t) = 2\sigma_+(t)[1 - \sigma_+(t)]$, for $N \gg 1$.

In infinite large systems, fluctuations in $\sigma_+(t)$ vanish. Therefore, in a single realization we would see that $\sigma_+(t) = \sigma_+(0)$ for all $t > 0$ and that the system reaches an infinite long

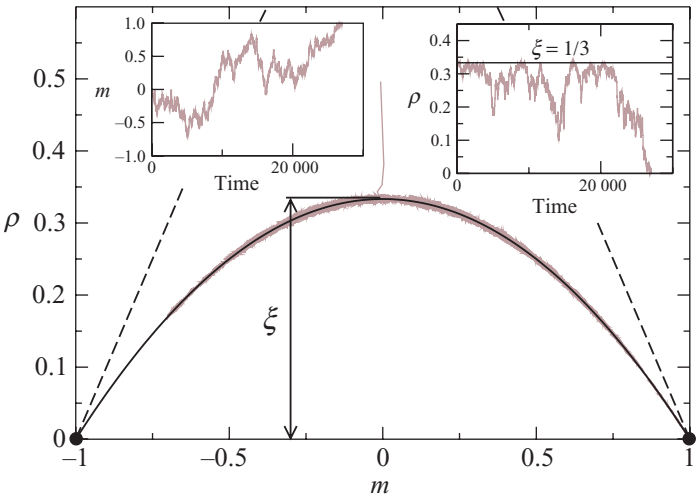

Figure 3. Trajectory of the system in a single realization plotted on the active links density-link magnetization ($\rho - m$) plane, for a DR random graph of size $N = 10^4$ and degree $\mu = 4$. Insets: time evolution of m (left) and ρ (right) for the same realization. We note that ρ and m are not independent but fluctuate in coupled manner, following a parabolic trajectory described by $\rho = \frac{1}{3}(1 - m^2)$ from equation (9) (solid line).

lived stationary state with $\rho = 4\xi\sigma_+(0)[1 - \sigma_+(0)] = \text{constant}$. Then, for networks with average degree $\mu > 2$, full order is never reached in the thermodynamic limit.

In finite size networks, fluctuations eventually drive the system to one of the two absorbing states, $\sigma_+ = 1$ or $\sigma_+ = 0$, characterized by the absence of active links ($\rho = 0$). Although the parameter ρ is useful for finding an absorbing state, it does not allow us to know which of the two states is reached. For this reason we introduce the *link magnetization* $m = \rho_{++} - \rho_{--}$, where ρ_{++} (ρ_{--}) are the density of links connecting two nodes with spins 1 (-1). It measures the level of order in the system, $m = 1$ ($m = -1$) corresponding to the $+$ ($-$) fully ordered absorbing state and $m = 0$ representing the totally mixed disordered state. Given that ρ becomes zero when m takes the values ± 1, we guess that ρ should be proportional to $1 - m^2$. To prove this, we first relate σ_s with ρ_{ss} ($s = \pm 1$) by calculating the total number of links coming out from nodes with spin s. This number of links is $\mu\sigma_s N$, from which $\rho\mu N/2$ are $s \to -s$ links and $\rho_{ss}\mu N$ are $s \to s$ links. We arrive at

$$\rho_{ss} = \sigma_s - \rho/2.$$

Then, the link magnetization is simply the spin magnetization

$$m = \rho_{++} - \rho_{--} = \sigma_+ - \sigma_- = 2\sigma_+ - 1. \tag{8}$$

Combining equations (6) and (8) we obtain that, neglecting fluctuations, ρ and m are related through the equation

$$\rho(t) = \xi[1 - m^2(t)]. \tag{9}$$

Figure 3 shows ρ versus m in one realization with $\mu = 4$ and $N = 10^4$. The system starts with equal density of $+$ and $-$ spins ($m = 0$ and $\rho = 1/2$), and after an initial transient of order one, in which m stays close to zero and ρ decays to a value similar to ξ, ρ fluctuates around

the parabola described by equation (9). This particular trajectory ends at the $(m = 1, \rho = 0)$ absorbing state.

4. Master equation for the link magnetization

In order to study the time evolution of the system we start by deriving a master equation for the probability $P(m, t)$ that the system has link magnetization m at time t. In a time step, a node with spin s and degree k flips its spin with probability $\sigma_s P(-s|s) = \rho/2$, after which the magnetization changes by $\Delta m = s\,\delta_k$, with $\delta_k = \frac{2k}{\mu N}$ (see figure 1), and with probability $\sigma_s[1 - P(-s|s)] = \sigma_s(1 - \rho/2\sigma_s)$ its spin remains unchanged. We have used that the density of s spins and the conditional probability $P(-s|s)$ in the subset of nodes with degree k is independent of k and equal to the global density σ_s (this was first noticed in [25, 27]). Using equation (9) we can write the probabilities of the possible changes in m due to the selection of a node of degree k as

$$\begin{aligned} W_{m\to m-\delta_k} &= \frac{\xi}{2}\left(1-m^2\right) P_k, \\ W_{m\to m+\delta_k} &= \frac{\xi}{2}\left(1-m^2\right) P_k, \\ W_{m\to m} &= \left[1-\xi\left(1-m^2\right)\right] P_k. \end{aligned} \tag{10}$$

Thus, the problem is reduced to the motion of a symmetric random walk in the $(-1, 1)$ interval, with absorbing boundaries at the ends and hopping distances and their probabilities that depend on the walker's position m and the degree distribution P_k. The time evolution of $P(m, t)$ is described by the master equation

$$\begin{aligned} P(m, t+\delta t) &= \sum_k P_k \left\{ W_{m+\delta_k\to m} P(m+\delta_k, t) + W_{m-\delta_k\to m} P(m-\delta_k, t) + W_{m\to m} P(m, t) \right\} \\ &= \sum_k P_k \left\{ \frac{\xi}{2}[1-(m+\delta_k)^2] P(m+\delta_k, t) + \frac{\xi}{2}[1-(m-\delta_k)^2] P(m-\delta_k, t) \right. \\ &\quad \left. + [1-\xi(1-m^2)] P(m, t) \right\}, \end{aligned} \tag{11}$$

where $\delta t = 1/N$ is the time step corresponding to a spin-flip attempt. In equation (11), the probability that the walker is at site m at time $t + \delta t$ is written as the sum of the probabilities for all possible events that take the walker from a site $m + \Delta$ to site m, with $\Delta = 0, \pm\delta_k$ and $k \geqslant 0$. The probability of a single event is the probability $P(m + \Delta, t)$ of being at site $m + \Delta$ at time t times the probability $W_{m+\Delta\to m}$ of hopping to site m. Expanding equation (11) to second order in m and first order in t we obtain

$$N\delta t \frac{\partial P}{\partial t} = \frac{2\,\xi}{\mu^2 N} \sum_k P_k k^2 \left\{ -2P - 4m\frac{\partial P}{\partial m} + (1-m^2)\frac{\partial^2 P}{\partial m^2} \right\}.$$

Thus, in the continuum limit ($\delta t = 1/N \to 0$ as $N \to \infty$), we arrive at the Fokker–Planck equation

$$\frac{\partial P(m, t')}{\partial t'} = \frac{\partial^2}{\partial m^2}[(1-m^2) P(m, t')], \tag{12}$$

where $t' \equiv t/\tau$ is a rescaled time,

$$\tau \equiv \frac{\mu^2 N}{2\xi(\mu)\mu_2} = \frac{(\mu-1)\mu^2 N}{(\mu-2)\mu_2} \tag{13}$$

is an intrinsic timescale of the system and $\mu_2 = \sum_k k^2 P_k$ is the second moment of the degree distribution. We shall see in section 7 that the time to reach the ordered state equals τ times a function of the initial magnetization. Note that, in complete graph, the corresponding Fokker–Planck equation derived for instance in [29], has the same form as equation (12) with $t' = t/N$, obtained as a particular case of a graph with distribution $P_k = \delta_{k,\mu}$, $\mu = N-1$ and $\mu_2 = \mu^2$. The general solution to equation (12) is given by the series expansion [29, 30]

$$P(m,t') = \sum_{l=0}^{\infty} A_l C_l^{3/2}(m)\, \mathrm{e}^{-(l+1)(l+2)\,t'}, \tag{14}$$

where A_l are coefficients determined by the initial condition and $C_l^{3/2}(x)$ are the Gegenbauer polynomials [31]. Equation (14) is of fundamental importance because it allows to find the two most relevant magnitudes in the voter model dynamics, namely, the average density of active links and the survival probability, as we shall see in sections 5 and 6, respectively.

5. Approach to the final frozen state

We are interested in how the average density of active links $\langle\rho\rangle$ decays to zero, where $\langle\cdot\rangle$ denotes an average over many independent realizations of the dynamics starting from the same initial spin densities. Using equation (9) we can write

$$\langle\rho(t')\rangle = \xi\langle 1-m^2(t')\rangle = \xi\int_{-1}^{1} \mathrm{d}m\,(1-m^2)P(m,t'), \tag{15}$$

with $P(m,t')$ given by equation (14). The solution to the above integral with an initial magnetization $m_0 = 2\sigma_+(0)-1$ is (see appendix A)

$$\langle\rho(t')\rangle = \xi(1-m_0^2)\,\mathrm{e}^{-2t'} \tag{16}$$

and replacing back t' and $\xi(\mu)$, we finally obtain

$$\langle\rho(t)\rangle = \frac{(\mu-2)}{2(\mu-1)}(1-m_0^2)\,\mathrm{e}^{-2t/\tau}. \tag{17}$$

We find that for $\mu > 2$, $\langle\rho(t)\rangle$ has an exponential decay with a time constant $\tau/2$, whose inverse gives the rate at which $\langle\rho\rangle$ decays. Given that τ is proportional to N (equation (13)), the decay becomes slower for increasing system sizes. Eventually, in the limit of an infinite large network $\langle\rho(t)\rangle$ remains at the constant value $\xi(1-m_0^2)$ as it was discussed in section 3, while in a finite network, $\langle\rho(t)\rangle$ reaches zero in a time of order τ.

We have simulated the voter model on various types of random networks: a DR random graph, an ER graph, an exponential network (EN) and a BA network. In figure 4, we observe that the analytical prediction (equation (17)) is in good agreement with numerical simulations on these four networks. For a fixed average degree μ and system size N, τ is determined by the second moment μ_2 of the network degree distribution P_k. For these particular networks, μ_2 can be written as a function of μ, because P_k only depends on μ and k. As a consequence of this, $\tau(\mu, N)$ is only a function of μ and N. The values of τ and μ_2 in the large N limit are

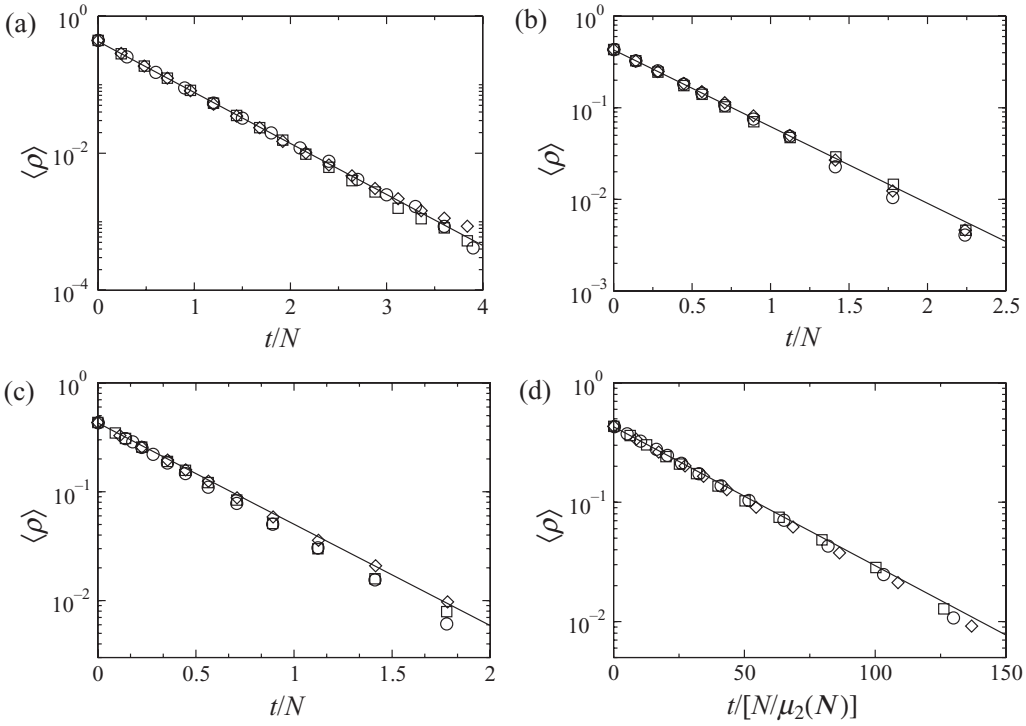

Figure 4. Time evolution of the average density of active links $\langle\rho(t)\rangle$ for (a) DR, (b) ER, (c) EN and (d) BA networks with average degree $\mu = 8$. The open symbols correspond to networks of different sizes: $N = 1000$ (circles), $N = 5000$ (squares) and $N = 10\,000$ (diamonds). Solid lines are the analytical predictions from equation (17). The average was taken over 1000 independent realizations, starting from a uniform distribution with magnetization $m_0 = 0$.

summarized in table 1. For the case of DR, ER and EN, $\langle\rho\rangle$ is a function of t/N as shown in figure 4 and μ_2 is finite and independent on N. We have checked that the scaling works very well for networks of size $N > 100$. For BA networks, μ_2 diverges with N (see calculation details in appendix B), thus we rescaled the x-axis by $N/\mu_2(N)$ in order to obtain an overlap for the curves of different system sizes.

6. Survival probability

In the last section, we found that the density of active links, when averaged over many runs, decays exponentially fast to zero. In estimating this average at a particular time t, we are considering all runs, even those that die before t and, therefore, contribute with $\rho = 0$ to the average. In order to gain an insight about the evolution of a single run [26], we consider the density of active links averaged only over surviving runs $\langle\rho^{\text{surv}}(t)\rangle$. If we define the survival probability $S(t)$ as the probability that the system has not reached the fully ordered state up to time t, then we can write $\langle\rho(t)\rangle = S(t)\langle\rho^{\text{surv}}(t)\rangle$.

In the $1d$ random walk mapping that we discussed in section 4, $S(t)$ corresponds to the probability that the walker is still alive at time t, that is to say, that it has not hit the absorbing boundaries $m = \pm 1$ up to time t. If at time $t = 0$, we launch many walkers from the same

Table 1. Node degree distribution P_k, its second moment μ_2 and the decay time constant of the average density of active links τ, for different networks.

Network	P_k	μ_2	$\tau(\mu, N)$
DR	$\delta_{k,\mu}$	μ^2	$\frac{(\mu-1)}{(\mu-2)}N$
ER	$e^{-\mu}\frac{\mu^k}{k!}$	$\mu(\mu+1)$	$\frac{\mu(\mu-1)}{(\mu+1)(\mu-2)}N$
EN	$\frac{2e}{\mu}\exp\left(-\frac{2k}{\mu}\right)$	$\frac{5}{4}\mu^2$	$\frac{4(\mu-1)}{5(\mu-2)}N$
BA	$\frac{\mu(\mu+2)}{2k(k+1)(k+2)}$	$\frac{\mu(\mu+2)}{4}\ln\left(\frac{\mu(\mu+2)^3 N}{(\mu+4)^4}\right)$	$\frac{4\mu(\mu-1)N/(\mu^2-4)}{\ln\left(\frac{\mu(\mu+2)^3}{(\mu+4)^4}N\right)}$
CG	$\delta_{k,N-1}$	$(N-1)^2$	N

position m_0, each of which represents an individual run, then $S(t)$ can be calculated as the fraction of surviving walkers at time t

$$S(t) = \int_{-1}^{1} \mathrm{d}m\, P(m, t). \tag{18}$$

The result of this integral for symmetric initial conditions ($m_0 = 0$) is given by the series (see appendix C)

$$S(t) = \sum_{l=0}^{\infty} \frac{(-1)^l (4l+3)(2l-1)!!}{(2l+2)!!} \exp\left(-\frac{2(2l+1)(l+1)\,t}{\tau(\mu, N)}\right). \tag{19}$$

As we observe in figure 5 there are two regimes. For $t \ll N$, is $S(t) \simeq 1$. For $t \gtrsim N/4$, only the first term corresponding to the lowest l ($l = 0$) gives a significant contribution to the series, thus neglecting the terms with $l > 0$ gives $S(t) \simeq \frac{3}{2}\exp\left(-\frac{t}{\tau(\mu,N)}\right)$. For a general initial condition m_0, we obtain that the survival probability decays as

$$S(t) \simeq \frac{3}{2}(1-m_0^2)\exp\left(-\frac{2(\mu-2)\,\mu_2}{(\mu-1)\mu^2}\frac{t}{N}\right), \quad \text{for } t > N. \tag{20}$$

Using equation (17) and (20), we finally obtain that the density of active links in surviving runs is

$$\langle \rho^{\mathrm{surv}}(t)\rangle = \frac{\langle \rho(t)\rangle}{S(t)} \simeq \begin{cases} \dfrac{(\mu-2)}{2(\mu-1)}(1-m_0^2)e^{-2t/\tau}, & \text{for } t \ll N; \\ \dfrac{(\mu-2)}{3(\mu-1)}, & \text{for } t \geqslant N. \end{cases} \tag{21}$$

We find that the average density of active links first decays and then reaches in a time of order N a plateau with value

$$\frac{2}{3}\xi(\mu) = \frac{(\mu-2)}{3(\mu-1)}. \tag{22}$$

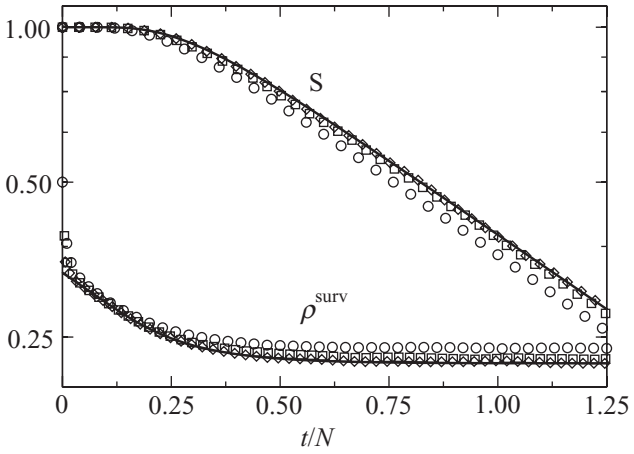

Figure 5. Survival probability S and average density of active links in surviving runs $\langle \rho^{surv} \rangle$ versus the rescaled time t/N for DR networks with degree $\mu = 4$ and sizes $N = 100$ (circles), $N = 400$ (squares) and $N = 1600$ (diamonds). Top and bottom solid lines are the analytical solutions $S(t)$ and $\langle \rho^{surv} \rangle = \langle \rho(t) \rangle / S(t)$, respectively, obtained using equations (19) and (17).

In figure 6, we plot the average height of the plateau as a function of μ obtained from numerical simulations on a BA network and a DR random graph. As equation (22) shows, the average plateau value $2\xi/3$ is only a function of the first moment of the distribution, as long as the network is random. The plateau is also independent of the initial condition m_0, and the system size N for N large.

A natural question is about the typical size of *spin domains* in the stationary state, where we use the term domain to identify a set of connected nodes with the same spin. Numerical simulations reveal that the system is always composed of two large domains with opposite spin until by fluctuations one of them takes over and the system freezes. This can be explained using percolation transition arguments on random graphs. Two connected nodes belong to the same domain if the link that connects them is inert, and this happens with probability $q = 1 - \rho$. Then, a domain that spans the system exists if $q > q_c = \frac{1}{\kappa - 1}$, with $\kappa = \frac{\mu_2}{\mu}$ [32]. This gives a critical density

$$\rho_c = \frac{\mu_2 - 2\mu}{\mu_2 - \mu}. \tag{23}$$

Given that $\mu_2 \geqslant \mu^2$, we have $\rho_c \geqslant \frac{\mu-2}{\mu-1} = 2\xi$, and because the density of active links in one realization is equal to or smaller than ξ (see figure 3), the system remains in the 'percolated phase', i.e. most of the nodes with the same spin are connected forming a giant domain of the order of the system size.

7. Ordering time in finite systems

A quantity of interest in the study of the voter model is the mean time to reach the fully ordered state when initially the system has magnetization m. In the random walk terminology of section 4, this is equivalent to the mean exit time $T(m)$, i.e. the time that the walker takes to

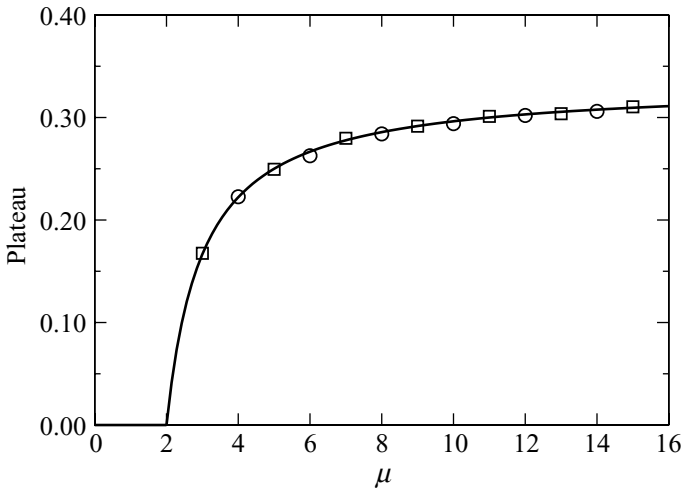

Figure 6. Average height of the plateau for BA (circles) and DR (squares) networks of size $N = 10\,000$. The solid line is the analytical prediction $\frac{(\mu-2)}{3(\mu-1)}$.

reach either absorbing boundary $m = \pm 1$ by the first time, starting from the position m. $T(m)$ obeys the following recursion formula:

$$T(m) = \sum_k P_k \left\{ \frac{\xi}{2}(1-m^2)\left[T(m+\delta_k)+\delta t\right] + \frac{\xi}{2}(1-m^2)\left[T(m-\delta_k)+\delta t\right] + \left[1-\xi(1-m^2)\right]\left[T(m)+\delta t\right] \right\},$$

with boundary conditions

$$T(-1) = T(1) = 0. \tag{24}$$

The mean exit time starting from site m equals the probability of taking a step to a site $m+\Delta$ times the exit time starting from this site. We then have to sum over all possible steps $\Delta = 0, \pm\delta_k$ and add the time interval δt of a single step. In the continuum limit ($\delta_k, \delta t \to 0$ as $N \to \infty$), this equation becomes

$$\frac{\mathrm{d}^2 T(m)}{\mathrm{d}m^2} = -\frac{\tau}{(1-m^2)}, \tag{25}$$

where τ is defined in equation (13). The solution to this equation is

$$T(m) = \tau\left[\frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right)\right],$$

or, in terms of the initial density of + spins $\sigma_+ = (1+m)/2$

$$T(\sigma_+) = -\frac{(\mu-1)\mu^2}{(\mu-2)\,\mu_2} N[\sigma_+ \ln\sigma_+ + (1-\sigma_+)\ln(1-\sigma_+)]. \tag{26}$$

This expression differs from the one obtained in [25] by a prefactor of $\frac{\mu-1}{\mu-2}$. However, this factor does not seem to change the scaling of $T(m)$ with the system size N that was found to be in good agreement with numerical simulations. In figure 7, we show the ordering time $t(\sigma_+)$ as

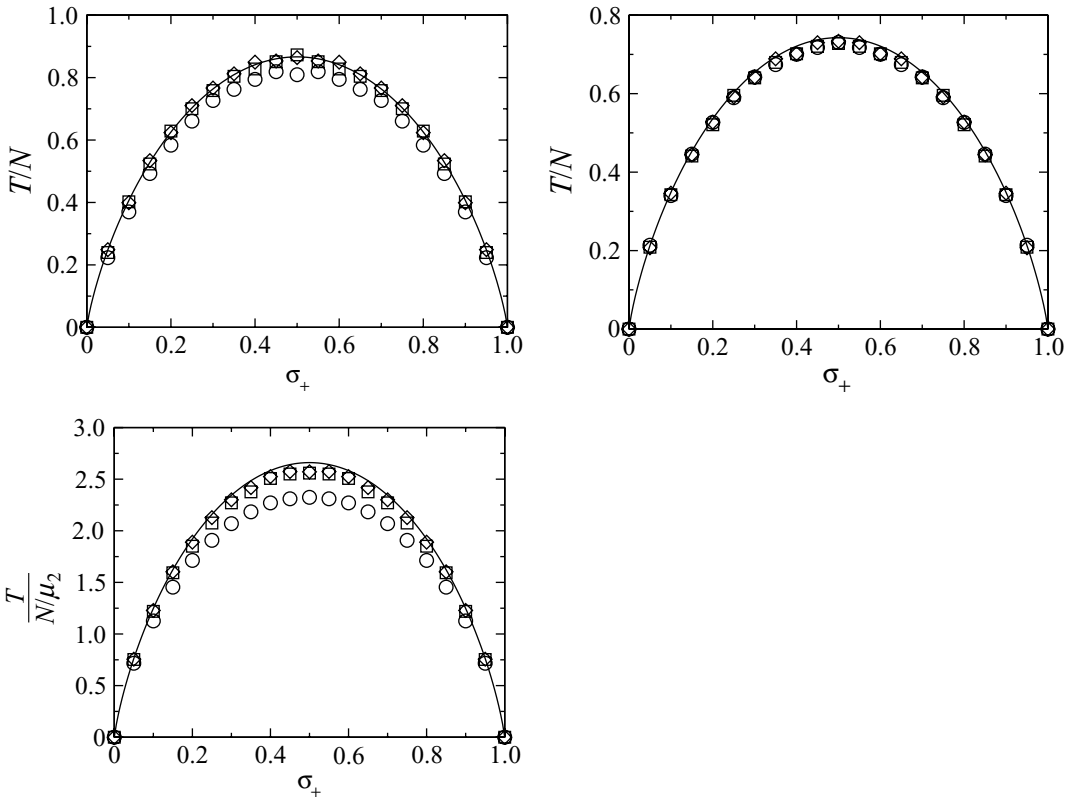

Figure 7. Scaled ordering times versus initial density of + spins σ_+ for networks of size $N = 10^2$ (circles), $N = 10^3$ (squares) and $N = 10^4$ (diamonds). Plots correspond to DR (top-left) and ER networks (top-right) with average degree $\mu = 4$ and BA networks (bottom-left) with $\mu = 20$. Solid lines are the analytical predictions from equation (26).

a function of the initial density of + spins, for a BA network with $\mu = 20$, ER and DR networks with $\mu = 6$.

For a fixed N, equation (26) predicts that $T(m)$ diverges at $\mu = 2$, but ordering times in the voter model are finite for finite sizes. To analyze this point, we numerically calculated T for an ER network as a function of μ for initial densities $\sigma_+ = \sigma_- = 1/2$ (see figure 8). For low values of μ, there is a fraction of nodes with zero degree that have no dynamics, thus we normalized T by the number of nodes N with degree larger than zero. As we observe in figure 8, when μ decreases the analytical solution given by equation (26) with $\mu_2 = \mu(\mu + 1)$ starts to diverge from the numerical solution. This disagreement might be due to the fact that our MF approach assumes that the system is homogeneous, and neglects every sort of fluctuations, which are important in networks with low connectivity. However, we still find that T reaches a maximum at $\mu \simeq 2$, where it seems to grow faster than N.

8. Summary and conclusions

In this paper, we have presented a MF approach over the density of active links that provides a description of the time evolution and final states of the voter model on heterogeneous networks in both infinite and finite systems. The theory gives analytical results that are in good agreement

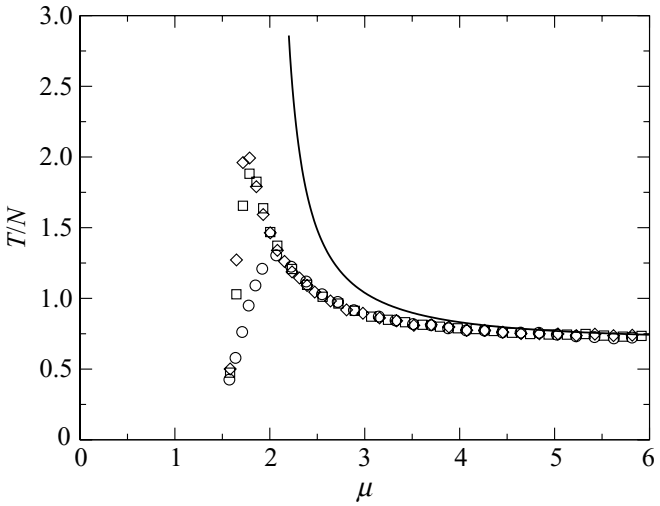

Figure 8. Scaled ordering times versus average degree μ for ER networks with $N = 100$ (circles), $N = 1000$ (squares) and $N = 2000$ (diamonds) nodes. The system size N was taken as the number of nodes in the network with degree larger than zero. The initial spin densities were $\sigma_+ = \sigma_- = 1/2$. The solid line is the solution given by equation (26).

with simulations of the model and also shows the connection between previous numerical and analytical results. The relation between the density of active links ρ and the density of + spins σ_+ expressed in equation (6) allows to treat random graphs as complete graphs, and to find expressions for ρ and the mean ordering time in finite systems. For large average degree values, equation (6) reduces to the expression for the density of active links in a complete graph. Therefore, this work confirms that uncorrelated networks with large enough connectivity are MF in character for the dynamics of the voter model. When the average degree μ is smaller than 2, the system orders, while for $\mu > 2$, the average density of active links in surviving runs reaches a plateau of height $\frac{(\mu-2)}{3(\mu-1)}$. Due to fluctuations, a finite system always falls into an absorbing, fully ordered state. The relaxation time T to the final absorbing state scales with the system size N and the first and second moments, μ and μ_2, respectively, of the degree distribution, as $T \sim \frac{(\mu-1)\mu^2 N}{(\mu-2)\,\mu_2}$.

The emergence of a transition between an active stationary state and a frozen ordered state at $\mu = 2$ is striking. Whether the transition is intrinsic to the voter model dynamics or it is connected to the topology of the network is an open question. It is worth noting that plateaus are also found on correlated networks with some level of node degree correlations, like for instance on small-world networks [22, 27], even though the plateau is lower than the one predicted by our theory. It might be interesting to modify the MF approach to account for degree correlations that correctly reproduce the behavior in very general networks.

Acknowledgments

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Appendix A. Average density of active links

To integrate equation (15), we use the series expansion equation (14) for $P(m, t')$ and write

$$\langle \rho(t') \rangle = \xi \sum_{l=0}^{\infty} A_l D_l \mathrm{e}^{-(l+1)(l+2)\, t'}, \tag{A.1}$$

where we define the coefficient

$$D_l \equiv \int_{-1}^{1} \mathrm{d}m (1-m^2) C_l^{3/2}(m).$$

To obtain the coefficients A_l, we assume that the initial magnetization is $m(t=0) = m_0$, i.e., $P(m, t=0) = \delta(m - m_0)$, from where the expansion for $P(m, t')$ becomes

$$\sum_{l=0}^{\infty} A_l C_l^{3/2}(m) = \delta(m - m_0).$$

Multiplying both sides of the above equation by $(1-m^2)\, C_{l'}^{3/2}(m)$ and integrating over m gives

$$\sum_{l=0}^{\infty} \frac{2(l+1)(l+2)}{(2l+3)} A_l \delta_{l,l'} = (1-m_0^2) C_{l'}^{3/2}(m_0), \tag{A.2}$$

where we used the orthogonality relation for the Gegenbauer polynomials equation MS 5.3.2 (8) in p 983 of [31] with $\lambda = 3/2$

$$\int_{-1}^{1} \mathrm{d}m C_l^{3/2}(m) C_{l'}^{3/2}(m)(1-m^2) = \frac{\pi \Gamma(l+3)}{4l!(l+3/2)[\Gamma(3/2)]^2} \delta_{l,l'} \tag{A.3}$$

and the identities $\Gamma(l) = (l-1)!$, $\Gamma(l+1) = l\Gamma(l)$ and $\Gamma(1/2) = \sqrt{\pi}$. Then, from equation (A.2) we obtain

$$A_l = \frac{(2l+3)(1-m_0^2)\, C_l^{3/2}(m_0)}{2(l+1)(l+2)}. \tag{A.4}$$

To find D_l, we use that the zeroth order polynomial is $C_0^{3/2}(m) = 1$, together with the orthogonality relation equation (A.3):

$$\begin{aligned} D_l &= \int_{-1}^{1} \mathrm{d}m C_l^{3/2}(m) C_0^{3/2}(m)(1-m^2) = \frac{\pi \Gamma(l+3)}{4l!(l+3/2)[\Gamma(3/2)]^2} \delta_{l,0} \\ &= \frac{2(l+1)(l+2)}{(2l+3)} \delta_{l,0}. \end{aligned} \tag{A.5}$$

Then, using equations (A.4) and (A.5), we find that the coefficients A_l and D_l are related by $A_l D_l = (1-m_0^2) C_l^{3/2}(m_0) \delta_{l,0}$. Replacing this relation in equation (A.1) and performing the summation, we finally obtain

$$\langle \rho(t') \rangle = \xi (1-m_0^2)\, \mathrm{e}^{-2t'},$$

as quoted in equation (16).

Appendix B. Calculation of μ_2 for BA networks

The BA network is generated by starting with a number of nodes m, and adding, at each time step, a new node with m links that connect to m different nodes in the network. When the number of nodes in the system is N, the total number of links is mN, and therefore the average degree is $\mu = 2m$. The expression for the resulting degree distribution, calculated for instance in [33], as a function of μ is

$$P(k) = \frac{\mu(\mu+2)}{2k(k+1)(k+2)} \tag{B.1}$$

and its second moment is

$$\begin{aligned}\mu_2 &= \int_{\mu/2}^{k_{\max}} k^2 P(k)\,\mathrm{d}k = \frac{\mu(\mu+2)}{2}\int_{\mu/2}^{k_{\max}} \frac{k\,\mathrm{d}k}{(k+1)(k+2)} \\ &= \frac{\mu(\mu+2)}{2}\ln\left[\frac{2(k_{\max}+2)^2(\mu+2)}{(k_{\max}+1)(\mu+4)^2}\right].\end{aligned} \tag{B.2}$$

The lower limit $\mu/2$ of the above integrals correspond to the lowest possible degree m, since nodes already have m links when they are added to the network. The reason for an upper limit $k_{\max}$ is that the contribution to μ_2 from large degree terms is important due to the slow asymptotic decay $P(k) \sim k^{-3}$, unlike for instance in ER or ENs, where $P(k)$ decays faster than k^{-3}, thus high degree terms become irrelevant. $k_{\max}$ is estimated as the degree for which the number of nodes with degree larger than $k_{\max}$ is less than one. Then

$$\frac{1}{N} = \frac{\mu(\mu+2)}{2}\int_{k_{\max}}^{\infty} \frac{\mathrm{d}k}{k(k+1)(k+2)} = \frac{\mu(\mu+2)}{4}\ln\left(\frac{(k_{\max}+1)^2}{k_{\max}(k_{\max}+2)}\right).$$

Assuming $k_{\max} \gg 1$, the expansion of the logarithm to first order in $1/k_{\max}$ is $1/k_{\max}^2$. Then, solving for $k_{\max}$, we obtain

$$k_{\max} \simeq \sqrt{u(u+2)/4}\,N^{1/2}, \tag{B.3}$$

i.e the maximum degree diverges with the system size.

Taking $k_{\max} \gg 1$ in equation (B.2) and replacing the value of $k_{\max}$ from equation (B.3) gives the expression quoted in table 1 for the second moment of a BA network

$$\mu_2 = \frac{\mu(\mu+2)}{4}\ln\left(\frac{\mu(\mu+2)^3 N}{(\mu+4)^4}\right). \tag{B.4}$$

Appendix C. Survival probability

By using the series representation equation (14), the survival probability quoted in equation (18) can be written as

$$S(t) = \sum_{l=0}^{\infty} A_l B_l\, \mathrm{e}^{-(l+1)(l+2)t'}, \tag{C.1}$$

where we define

$$B_l \equiv \int_{-1}^{1} \mathrm{d}m\, C_l^{3/2}(m). \tag{C.2}$$

To obtain the coefficients B_l, we use the derivative identity $C_l^{3/2}(m) = \frac{\mathrm{d}}{\mathrm{d}m} C_{l+1}^{1/2}(m)$ derived from equation MS 5.3.2 (1) in p 983 of [31] with $\lambda = 3/2$. Then

$$B_l = C_{l+1}^{1/2}(1) - C_{l+1}^{1/2}(-1) = 1 - (-1)^{l+1} = \begin{cases} 0, & l \text{ odd}, \\ 2, & l \text{ even}, \end{cases} \tag{C.3}$$

where we have used the relations $C_l^{1/2}(1) = 1 \ \forall l$ and $C_l^{1/2}(-1) = (-1)^l$ that follow from equation MO 98 (4) (p 983) and the parity of the polynomials (p 980) of [31], respectively.

An explicit function for the coefficients A_l of equation (A.4) can only be found for the $m_0 = 0$ case, given that for $m_0 \neq 0$ it seems that a closed expression for the polynomials $C_l^{3/2}(m_0)$ cannot be obtained. To obtain the coefficients $C_l^{3/2}(0)$ we use the recursion relation equation Mo 98 (4) (p 981) of [31] for $m \equiv x = 0$ and $\lambda = 3/2$, together with the values of the zeroth- and first-order polynomials $C_0^{3/2}(0) = 1$ and $C_1^{3/2}(0) = 0$. Then

$$C_l^{3/2}(0) = -\frac{(l+1)}{l} C_{l-2}^{3/2}(0) = \begin{cases} 0, & l \text{ odd}, \\ (-1)^{l/2} \dfrac{(l+1)!!}{l!!}, & l \text{ even}. \end{cases} \tag{C.4}$$

Plugging the above expression into equation (A.4) gives $A_l = 0$ for l odd and $A_l = \frac{(-1)^{l/2}(2l+3)(l-1)!!}{2(l+2)!!}$ for l even.

Then, using equation (C.3), the product $A_l\, B_l$ can be written as

$$A_l\, B_l = \begin{cases} 0, & l \text{ odd}, \\ \dfrac{(-1)^{l/2}(2l+3)(l-1)!!}{(l+2)!!}, & l \text{ even}. \end{cases} \tag{C.5}$$

Finally, making the variable change $l \to 2l$, equation (C.1) becomes

$$S(t') = \sum_{l=0}^{\infty} \frac{(-1)^l (4l+3)(2l-1)!!}{(2l+2)!!} \, \mathrm{e}^{-2(2l+1)(l+1)\, t'}. \tag{C.6}$$

Replacing t' by $t/\tau(\mu, N)$, we obtain the expression quoted in equation (19).

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