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Anomalous lifetime distributions and topological traps in ordering dynamics

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Abstract – We address the role of community structure of an interaction network in ordering dynamics, as well as associated forms of metastability. We consider the voter and AB model dynamics in a network model which mimics social interactions. The AB model includes an intermediate state between the two excluding options of the voter model. For the voter model we find dynamical metastable disordered states with a characteristic mean lifetime. However, for the AB dynamics we find a power law distribution of the lifetime of metastable states, so that the mean lifetime is not representative of the dynamics. These trapped metastable states, which can order at all time scales, originate in the mesoscopic network structure.

Introduction. – Statistical mechanics and complex network theory have been applied to different disciplines, ranging from biology to sociology. From this perspective, social systems are modelled as a collection of agents, located at the nodes of a network, interacting through simple rules. Social networks of human interaction are structured into cohesive groups [1], and increased knowledge of this structure [2–4] has sparked the creation of new network models [3,5–9]. These models allow us to study the effect of the structure of social interactions on the dynamics taking place on the networks, and on the associated collective phenomena emerging from the interactions among the agents.

The mesoscopic structure of a social network, and in particular its community structure, has been found to influence dynamics taking place on it in ways that cannot be explained by global level statistics in several cases [4,10–12]. In this paper we address the role of such mesoscopic structure on ordering dynamics or consensus processes: the question is when the interaction of agents with several options leads to an ordered state with a single option (consensus) or when disordered states (possibly metastable) with coexisting options prevail. We consider two dynamical models. The first one is the prototype *voter model* [13] whose dynamics in complex networks is known to be generally determined by global properties such as the effective network dimensionality [14]. Secondly, we consider the *AB model* [15] introduced to describe language competition, which gives a natural context for the community concept. These two dynamical models are studied in a class of networks [9] incorporating nontrivial community structure which introduces structural correlations.

Two dynamical models of competing options. – The *voter model* [13] concerns the competition of two equivalent but excluding options A and B. The state of a node is updated by imitation of a randomly chosen neighbor. The *AB model* [15] includes a third non-excluding mixed AB state, with the additional rule that a node cannot change state from A to B or vice versa without going through the AB state. In studies of dynamics of language competition, the voter model gives a microscopic version [16] of the Abrams-Strogatz [17] model for the competition of two socially equivalent languages. In this context the third state of non-excluding options of the AB model is naturally associated with bilingualism [18]. More generally the AB model describes competition of two equivalent social norms which can coexist at the individual level.

In both models, an agent changes its state with a probability which depends on the states of its neighbors. The

(a) These authors contributed equally to this work.

fraction of first neighbors in state A [B, AB] of an agent is called the *local density* σ_A [σ_B, σ_{AB}]. For the voter model, the state AB is not allowed and the probabilities of a node changing state are defined as follows:

$$p_{A\to B} = \sigma_B, \qquad p_{B\to A} = \sigma_A. \tag{1}$$

The AB model is defined by the following update rules:

$$p_{A\to AB} = \frac{1}{2}\sigma_B, \qquad p_{B\to AB} = \frac{1}{2}\sigma_A \tag{2}$$

$$p_{AB\to A} = \frac{1}{2}(1-\sigma_B), \qquad p_{AB\to B} = \frac{1}{2}(1-\sigma_A). \tag{3}$$

In our simulations we start from random initial conditions for the state of the agents in a network with N nodes (see below) and we use random asynchronous node update: at each time step a single node is randomly chosen and updated according to the transition probabilities eq. (1) or eqs. (2), (3). We normalize time so that every unit of time includes N time steps.

A question of interest is under which conditions consensus is reached (all nodes hold the same option), and which is the process of emergence and growth of spatial domains where the nodes are in the same state (*coarsening*). Both models are symmetric by interchange of A and B, so that reaching consensus in either of these two states is a symmetry-breaking process. To describe the dynamics of the system we use as order parameter the *interface density* ρ, which is defined as the fraction of links which connect nodes in different states. The ensemble average interface density $\langle\rho\rangle$ is considered, where the ensemble average, indicated as $\langle\cdot\rangle$, denotes average over realizations of the stochastic dynamics starting from different random initial conditions. Interface density decreases as domains grow in size. If one of the states becomes dominant, the interface density decreases along with the disappearing state. Zero interface density indicates that an absorbing state, consensus, has been reached. Coarsening in the voter model is driven by interfacial noise, while for the AB model earlier results indicate that coarsening is curvature driven: boundaries tend to straighten out, reducing curvature and leading to the growth of spatial domains [15]. It turns out that domains of AB agents are never formed. Instead, AB agents place themselves in the interface between A and B domains.

The dependence of the voter model dynamics on network dimensionality, disorder and degree distribution has been carefully studied [14,19–21]. A main result is that $d=2$ is the critical dimensionality for this model. This means that for $d \leqslant 2$ there is coarsening, *i.e.* unbounded growth (in the thermodynamic limit) of domains in which all nodes are in the same state. However, for $d>2$ there is no coarsening beyond an initial transient. In finite networks of $d>2$ there exist long-lived metastable states in which ρ takes a plateau value. The inverse of this plateau value is the characteristic size of coexisting A and B domains. Eventually a finite-size fluctuation takes the system to one of the two consensus-absorbing states. We note that complex networks are typically high-dimensional structures for which these metastable states naturally occur [14].

Coarsening processes leading to consensus often come to a halt due to the appearance of metastable states that can be of different nature. Coarsening and metastable properties depend on the dynamical model as well as on network characteristics. The type of metastability encountered for the voter model is characterized by the fact that all realizations of the process are of the same class (qualitatively similar) and that the metastable states have a finite lifetime for a finite system. For the voter model the mean lifetime of these states scales as $\tau \sim N$ [14]. We call this type of metastable states *dynamical metastable states*. A different type of metastability, which we call *trapped metastable states*, occurs in situations in which different realizations of the process are of different type. While some of them follow a coarsening process until finite-size effects come into play, others get stuck in topological traps. The latter correspond to trapped metastable states that can be of two types: they might have a finite lifetime in finite systems, as it occurs for the AB model with stripe-like configurations in regular two-dimensional lattices [15], or they might be infinitely long lived as it occurs in zero-temperature kinetic Ising models [22–25]. In summary, different forms of metastability can appear for the voter and AB models considered here, but every realization is expected to have a finite lifetime in a finite system.

A class of social type networks. – Several models have been designed to capture some of the characteristics of social networks, based on mechanisms such as geographical proximity [8], social similarity [3,7], and local search [5,6,9]. A combination of random attachment with local search for new contacts has proved fruitful in generating well-known features of social networks, such as assortativity, broad degree distributions, and community structure [9]. The term "community" is typically used in the context of groups of nodes with dense internal and sparse external connections; exact definitions differ [26–29]. The community structure leads naturally to high values of the clustering coefficient.

The algorithm to generate this class of networks [9] consists of two growth processes: 1) random attachment, and 2) implicit preferential attachment resulting from following edges from the randomly chosen initial contacts. The local nature of the second process gives rise to high clustering, assortativity and community structure. Starting from any small connected seed network of N_0 vertices, new nodes are added as follows (see fig. 1): i) Pick $n_{init} \geqslant 1$ random nodes as initial contacts. ii) Pick $n_{sec} \geqslant 0$ neighbors of each initial contact as secondary contacts. iii) Connect the new node to the initial and secondary contacts.

Throughout this paper, we will use the *standard parameters* [9]: the number of initial contacts is selected according to the probabilities $p(n_{init}=1)=0.95$, $p(n_{init}=2)=0.05$;

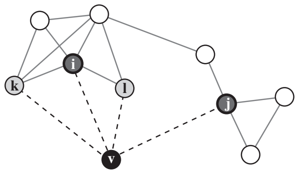

Fig. 1: Growth process of the network. The new vertex v links to one or more randomly chosen initial contacts (here i, j) and possibly to some of their neighbors (here k, l).

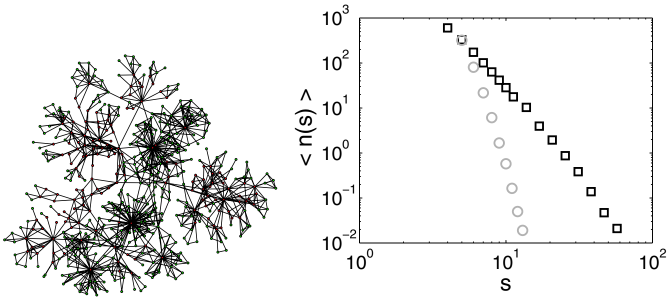

Fig. 2: Left: a partial view of the network centered on a randomized selected node. Right: average number $\langle n(s) \rangle$ of k-clique-communities of size s for $k = 4$ (□) and $k = 5$ (◯), in networks of size $N = 10000$, averaged over 400 realizations.

and the number of secondary contacts from each initial contact, n_{sec}, is chosen from a uniform probability distribution between 0 and 3; the initial seed contains $N_0 = 10$ nodes.

The degree distributions of the resulting networks are found to decay slower than exponential [9]. Using the k-clique algorithm [29] for detecting communities, a broad distribution of community sizes is found in the model (fig. 2).

For reference, we use randomized versions of the same networks, where the degree sequence is kept intact but edges are randomly rewired under the restriction that the network must stay connected [30]. This eliminates community structure, clustering, and degree correlations. The randomized networks are therefore locally treelike.

Results. – We have considered the update rules eqs. (1) for the voter model, or eqs. (2), (3) for the AB model in the class of networks described above. We followed the development over time of the interface density and of the fraction of runs that had not yet reached consensus at any particular time. When results for the original and randomized networks differ, we can conclude that structural characteristics other than the degree distribution are responsible for the differences.

Interface density. The average interface density $\langle \rho \rangle$ on the class of networks considered here, and on their randomized counterparts is shown in fig. 3. For the voter model (fig. 3a), we obtain that the structure of the network does not alter the qualitative behavior. In both classes of networks we observe plateau values of $\langle \rho \rangle$ associated with dynamical metastable states. Still, the plateau value

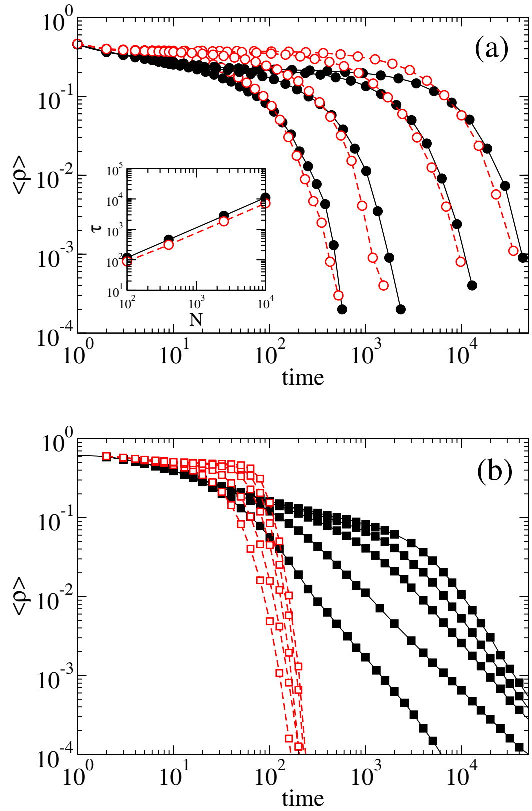

Fig. 3: Time evolution of the average interface density in networks with communities (solid symbols) and randomized networks (empty symbols) with the same degree sequences. (a) Voter model. Network sizes increase from left to right: $N = 100$, 400, 2500, 10000. Averages are taken over 100 different realizations of the model network, with 10 runs in each. Inset: time to reach consensus scales with network size as $\tau \sim N^{\gamma}$, $\gamma \approx 0.96$ for the randomized and $\gamma \approx 0.98$ for the original networks. (b) AB model. Network sizes increase from left to right: $N = 100$, 400, 2500, 10000, 40000. Averages taken over 400–5000 realizations (depending on system size) of the model network, and with 10 runs in each.

for networks with community structure is lower than for the randomized networks, indicating that the typical size of spatial domains where agents are in the same state is larger. We also observe in both cases that finite size fluctuations drive the system to an absorbing state. The characteristic time to reach consensus (mean lifetime of the metastable state) depends on network size but it does not depend sensitively on network structure. The inset in fig. 3a shows that the time to reach consensus depends linearly on network size for networks with communities and their randomized counterparts[1]. These results support the earlier finding made on networks without mesoscopic structure that effective dimensionality dominates voter model behavior [14].

[1]The slight deviation from linear scaling is due to violation of conservation laws when using node update dynamics on networks with nodes of very different degree (see [21]).

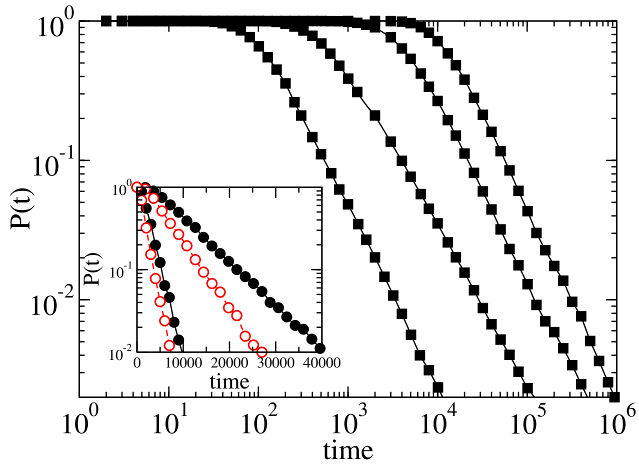

Fig. 4: Fraction of alive runs in time for networks with communities (solid symbols) and randomized networks (empty symbols). AB model (double-logarithmic plot); system sizes $N = 100$, 400, 2500, 10000 from left to right, with averages taken over different realizations of the network (400–5000 depending on system size), with 10 runs in each. Inset: voter model (semilogarithmic plot). System sizes $N = 2500$, 10000. Averages are taken over 100 different realizations of the networks, with 10 runs in each.

Figure 3b shows the average interface density for the AB dynamics. We observe significant differences between the original and the randomized version networks: a plateau value of $\langle\rho\rangle$ is observed for randomized networks, while a first dynamical stage of coarsening where spatial domains grow in size is found for large networks with communities. The plateau observed in randomized networks indicates that a dynamical metastable state of the class found in the voter model for both types of networks is rapidly reached. Moreover, in the randomized networks there is a fast decay towards an absorbing state with a characteristic time to reach consensus almost independent of system size. For the networks with a community structure we observe two dynamical stages in the evolution of $\langle\rho\rangle$. After an initial power law associated with coarsening there appears a second power law tail in the approach to the absorbing state. This last power law decay indicates that the mean lifetime to reach consensus for the AB model does not characterize the dynamics on these networks and that metastable states exist at all time scales, as we discuss below. Additionally, the difference with the randomized networks in several orders of magnitude for the extinction times, which increases with system size, shows that the network with communities slows down the dynamics significantly. All together these results manifest a sensitivity of the AB dynamics to the mesoscopic network structure which is not found for the voter dynamics.

Fraction of alive runs. Figure 4 shows the fraction $P(t)$ of realizations still alive at time t, *i.e.* the fraction of realizations which have not reached the absorbing state. For the voter model, the fraction of alive runs decreases exponentially in both the original and randomized networks (fig. 4, inset), in agreement with previous results for the voter model in high-dimensional complex networks [14]. A rather different result is found for the AB model (fig. 4). In our class of networks, we find a power law behavior $P(t) \sim t^{-\alpha}$, $\alpha \approx 1.3$, so that a mean lifetime of the realizations of the AB dynamics does not give a characteristic time scale. At any time there are live realizations which have not reached the absorbing state. Different parametrizations of the network model (not shown) produce the same qualitative phenomenon: we have modified the number of secondary contacts from each initial contact, n_{sec}, using uniform probability distributions between 0 and $1, 2, 4$, obtaining also a power law of the distribution of alive runs with an exponent smaller than 2, which indicates the robustness of this result. This behavior is different from the usual exponential decay of the tails of $P(t)$ observed for the voter, and AB dynamics either in regular, small world [15], random or Barabási-Albert scale-free networks (not shown), and reflects the existence of metastable states at all time scales. This fact indicates that the anomalous lifetime distribution is linked to the structure of the network at a mesoscopic level. Such structure seems to give rise to a number of traps that cause trapped metastable states at all time scales. To substantiate this claim we next look at some detailed dynamics.

Discussion. Further understanding of the dynamical process can be obtained by considering the measure called overlap, O [4]. This characteristic of a link between two nodes tells us essentially which fraction of their neighbors is shared by the nodes. Within a community, nodes tend to share many neighbors, and thus overlap is high, while edges between communities will have low or zero overlap. Considering dynamics of competing options on a network, the overlap can be used to identify spatially homogenous domains in the network: if the average overlap $\langle O\rangle$ of the links in the interface between domains is low, we may assume that the domain boundaries follow the community boundaries. On the other hand, if the overlap at the interfaces is high, it indicates that nodes within communities are in different states. For the voter model dynamics we have found that the average overlap of interface links drops to about 80 percent of the average value $\langle O\rangle = 0.27$ of the whole network, while in the AB model it drops to under 70 percent. This indicates that in both models the interfaces between domains lie preferably in low overlap links, so that domains of the same option follow the community structure, but in the AB model these domains are correlated with the communities closer.

The difference between the two dynamics is better understood by looking at snapshots of the dynamics (fig. 5) which show the characteristic behavior for each of the models, starting from random initial conditions ($t = 0$). In the voter model (left) the homogeneous domains of nodes with the same option appear to follow the community structure, but a particular community (topological

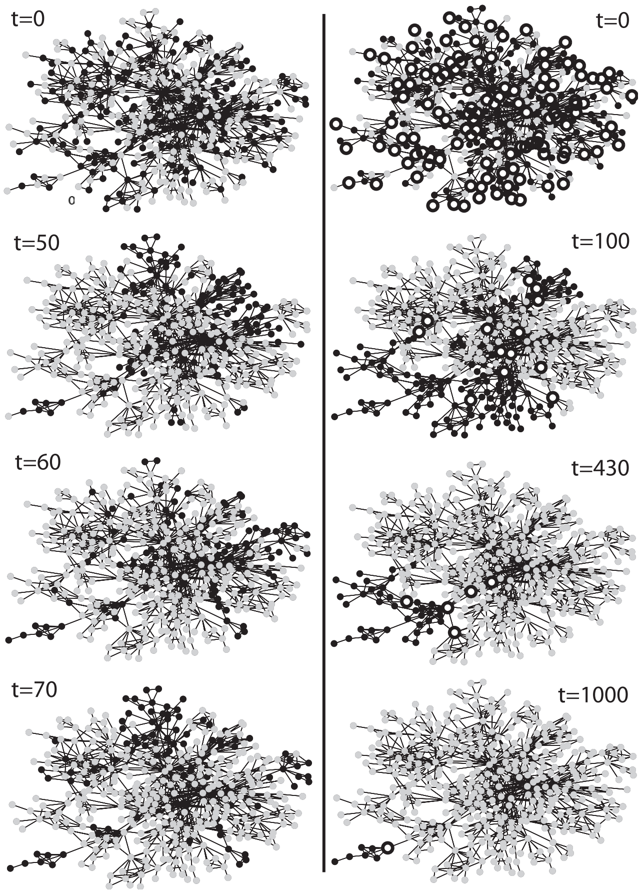

Fig. 5: Snapshots of the dynamics, with nodes in state A in black, B in grey, and AB in white circled in black. Simulations start from random initial conditions. Left: voter model. Right: AB model.

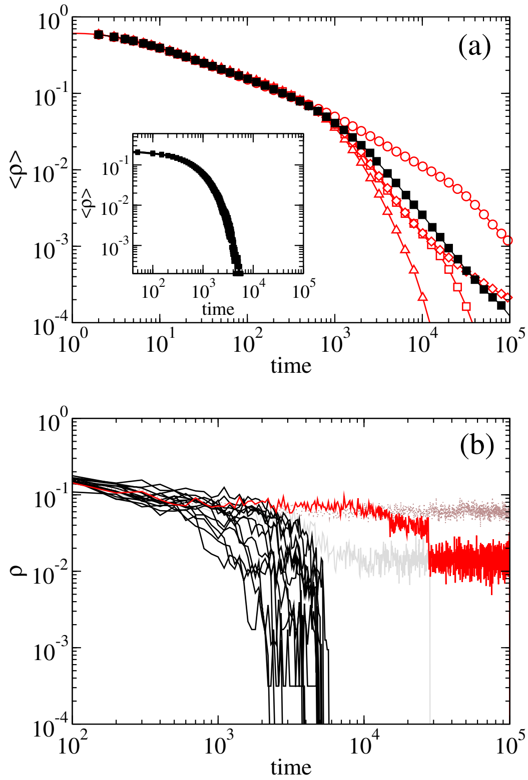

Fig. 6: (a) Time evolution for the AB model of the average interface density on different realizations of the network with 2500 agents; 20000 runs on each (empty symbols). The extreme cases were selected as examples of networks where *trapped metastable states* (see text) are found *often* (◯); and found *rarely* (△). For comparison, the average over 500 networks (10 runs on each) is also shown (■). Inset: time evolution for the voter model of the average interface density for four realizations of the networks of 2500 agents; 5000 runs on each network. (b) Time evolution of the interface density in single realizations of the AB dynamics on a network with 2500 agents. A class of realizations decay to the absorbing state after a coarsening stage (solid black lines), while others fall in long-lived trapped metastable states. The latter display several plateaus, indicating hierarchical levels of ordering before reaching the absorbing state, or cascading between several trapped metastable states.

region) may change the option adopted by the community rather quickly ($t = 50, 60, 70$). At variance with this behavior, in the AB model (right) spatial domains grow and homogenize steadily in a community without much fluctuation. For this dynamics, communities that have adopted a given option, and which are poorly linked to the rest of the network, take a long time to be invaded by a different option, acting therefore as topological traps. As an example of this we show two long-lived trapped metastable states at $t = 430$ and $t = 1000$, where the interface stayed relatively stable for a prolonged period (~ 100 and ~ 1000 time steps, respectively). These different behaviors reflect in the community structure two different interfacial dynamics: interfacial noise-driven dynamics for the voter model, and curvature-driven dynamics for the AB model with agents in the AB state at the interfaces.

Different realizations of the algorithm to construct the social-type network produce different detailed structures of the network. The power law for the fraction of alive runs in fig. 4 is a statistical effect of the average over such realizations. The time evolution of the average interface density on single realizations of the network, $\langle\rho\rangle$, is shown for the AB dynamics in fig. 6a. We observe different behaviors in the second stage of the decay of $\langle\rho\rangle$ depending on the specific realization of the network: from broad tails to exponential-like decays, with an intermediate behavior. On the other hand, and in agreement with our previous discussion, the voter model dynamics (fig. 6a, inset) is not sensitive to the details of the network structure. For the AB model some realizations of the network topology produce particularly long-lived metastable states, while in others, corresponding to exponential-like decay of $\langle\rho\rangle$, they are observed rarely. Plots of the interface density of individual runs on a given network show a class of realizations with different plateaus (ordering levels) where the system gets trapped for a long time (fig. 6b). These trapped metastable states, analogous to those displayed in fig. 5, right, correspond to the structure in the network. The variety of traps and associated different lifetimes seems to be the mechanism that causes an anomalous power law distribution for the lifetimes.

We note that although the details of each network realization matter for the occurrence of trapped metastable states, the community size distribution detected by the k-clique-percolation method [29] is the same for all the network realizations that we have considered. This and other available statistical methods seem not to be sufficient to discern between the network topologies producing many or few trapped metastable states.

Summary and conclusions. – We have considered two dynamical models, the voter and the AB model, in order to study metastable states and the role of community structure in the dynamics of consensus processes. The voter model dynamics, driven by interfacial noise, is not particularly sensitive to the mesoscopic structure of the network: we find that all realizations of the dynamics are of the same class, leading to a type of dynamical metastable states shared by other complex networks of high dimensionality without degree correlations. On the contrary, for the AB dynamics we find different classes of realizations leading to a power law distribution for the times to reach consensus. This is explained in terms of trapped metastable states associated with the structure of the network. Our result implies that a mean lifetime for these states does not give a characteristic time scale of the ordering dynamics. We note that a mean lifetime does not exist for the zero-temperature kinetic Ising model dynamics on regular or complex networks [24], due to realizations that lead to trapped metastable states of infinite lifetime in finite systems. The novelty of our finding is that we have realizations with any lifetime. For the AB model in a regular 2D lattice trapped metastable states with stripe-like configuration have been found [15], but in that case the distribution of lifetimes is exponential: $P(t) \sim e^{-\alpha t}$ and the mean lifetime is representative of the dynamics. The power law distribution for the lifetimes originates here in the multiplicity of different traps that reflects the mesoscopic structure of the networks. Simpler configurations of community structure should be considered in the future in order to gain a deeper understanding of the microscopic mechanisms underlying consensus dynamics.

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