UNIVERSITAT DE LES ILLES BALEARS
DEPARTAMENT DE FISICA

# Paradoxical games: A Physics point of view.

*Tesi presentada per Pau Amengual Marí, en el Departament de Física de la Universitat de les Illes Balears, per optar al grau de Doctor en ciències Físiques.*

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*als meus pares*

# Resum

Des del descobriment per part del botànic R. Brown al segle XIX de la presència d'un moviment erràtic en sistemes de tamany microscòpic, més tard conegut com a moviment Brownià, grans avenços van ocórrer en el camp dels procesos estocàstics durant els anys següents. La inevitable presència de renou (o termes en les equacions del moviment que provenen de l'eliminació de graus de llibertat microscòpics, i que només es podien descriure de manera probabilística) en un sistema es pensava que jugava un paper *destructiu*. Recentment, però, han aparegut moltes situacions en les quals el renou pot jugar un paper constructiu. Així doncs, fenòmens com la *resonància estocàstica* [1] mostra que el renou pot millorar les propietats de transmissió d'un sistema [1]. També trobam les transicions de fase induïdes per renou, en les quals la presència de renou pot donar lloc a una transició de fase de no-equilibri cap a un estat de ruptura de simetria [2].

Un altre aplicació interessant té a veure amb fenòmens de transport: el renou pot ser emprat per a obtenir moviment unidireccional, açò és, es poden *rectificar* les fluctuacions causades pel renou tèrmic de l'ambient, obtenint així una corrent neta en el sistema. Aquest model es coneix com a *motor Brownià*. Bàsicament consisteix en un sistema de petita escala que es troba sotmès a fluctuacions tèrmiques les quals són rectificades mitjançant qualque tipus d'asimetria (ja sigui espaial o temporal) present en el sistema. Aquest fenòmen de transport es coneix com *efecte ratchet*. Depenent de la forma en la qual aquesta asimetria és introduïda podem distingir entre diferents tipus de motors Brownians. De totes maneres, el nostre interès es centrarà en un sol tipus de motor Brownià conegut com a *flashing ratchet*, i que es caracteritza per una partícula que es veu sotmesa a un potencial asimètric que s'encén i s'apaga periòdicament o aleatòriament.

El flashing ratchet serví com a *inspiració* al físic espanyol Juan M.R. Parrondo per a model.lar un exemple de caire pedagògic amb dos jocs A i B en els quals ocorria un efecte similar. Va crear aquests jocs l'any 1996, i els presentà de manera informal a Torino, Itàlia [3]. Aquests jocs, més tard coneguts com a *jocs de Parrondo*, eren dos jocs justos (o inclús jocs perdedors) quan s'hi jugava a un d'ells solament, mentre que si un els combinava de forma periòdica o inclús aleatòria, s'obtenia com a resultat un joc guanyador. Llavors aquest exemple mostrava d'una manera molt senzilla com quan

[1]Per a ser més precisos, existeix un valor òptim per a l'amplitud del renou, reflectit per la presència d'un màxim quan dibuixam la *relació entre la senyal i el renou* en funció del renou.

dues dinàmiques es combinaven no necesàriament donaven com a resultat una *suma* de dinàmiques. Tot el contrari, s'obtenia un resultat que era totalment inesperat. El resultat d'obtenir un joc guanyador a partir de dos jocs justos o perdedors es coneix com a la *paradoxa de Parrondo* [4–8].

Des de la seva aparició, aquests jocs varen atreure molt d'interès en altres camps, com per exemple teoria d'informació quàntica [9–12], teoria de control [13,14], sistemes d'Ising [15], formació d'estructures [16–18], resonància estocàstica [19], caminates aleatòries i difusions [20–24], sistemes dinàmics discrets [25–27], economia [28,29], motors moleculars en biologia [30, 31], biogènesis [32] i dinàmica de població [33, 34]. També han estat tractats com a processos de naixement i mort [35] i autòmates cel.lulars [36].

No obstant això, a pesar de que la connexió entre el flashing ratchet i els jocs de Parrondo era patent, no existia una relació precisa i quantitativa entre ambdós. És la finalitat doncs d'aquesta tesi poder aprofundir en aquesta connexió entre els jocs de Parrondo i el flashing ratchet. La tesi es divideix en deu capítols, dels quals el Capítol 1 constitueix una breu introducció als conceptes preliminars necessaris per a un millor enteniment dels capítols posteriors. Hi presentam els conceptes bàsics de la teoría de processos estocàstics, així com altres de teoria de cadenes de Markov i teoria de la informació.

El Capítol 2 està dedicat a una explicació detallada del ratchet Brownià. Concretament ens centrarem en el flashing ratchet, explicant el mecanisme físic que es troba darrera l'efecte ratchet. En aquest capítol també presentam detalladament els jocs de Parrondo tal com varen ser definits, juntament amb un anàlisi mitjançant cadenes de Markov a temps discret que ens conduirà a l'obtenció de la distribució de probabilitats estacionàries així com els ritmes de guany dels jocs. A més a més, presentarem de manera resumida altres versions dels jocs de Parrondo que podem trobar en altres treballs, i que es diferencien dels originals en les regles emprades per a escollir les probabilitats.

Els jocs A i B que apareixen a la paradoxa de Parrondo poden considerar-se com un procés de difusió sota l'acció d'un potencial extern. No obstant això, no tenen la forma general d'un procés natural de difusió, ja que el capital sempre canvia amb cada joc, mentre que en el cas més general de difusió la partícula pot moure's cap amunt o cap avall o romandre en la mateixa posició en un temps donat. En el Capítol 3 presentam una nova versió dels jocs de Parrondo, en els quals consideram una nova probabilitat de transició. Introduïm la probabilitat anomenada *self-transition*, amb la qual el capital del jugador pot romandre igual després d'haver jugat. Per tant aquesta nova versió pot considerar-se com una evolució natural dels jocs de Parrondo, dels quals els jocs originals en constitueixen un cas particular.

Després d'introduïr aquesta nova versió dels jocs, procedim a derivar una relació *quantitativa* entre els jocs de Parrondo i el model físic del ratchet Brownià. El treball original de Parrondo no feia aquesta comparació detallada. Aquesta relació funciona en ambdós sentits: emprant la nostra relació és possible obtenir nous jocs partint de potencials físics molt simples; de forma semblant, és possible generar nous models físics que presenten l'efecte ratchet a partir de la descripció teòrica d'un joc. El Capítol 4 està doncs dedicat a mostrar aquesta relació entre els jocs de Parrondo i el flashing ratchet, demostrant que pot ser establerta de forma rigurosa.

Per a ampliar encara més l'analogia establerta entre els jocs de Parrondo i el ratchet Brownià, analitzam al Capítol 5, des del punt de vista de la teoria de la informació, la relació entre el corrent (o guany) dels jocs i l'entropia de la informació (també coneguda com a *negentropy*). Aquesta relació, establerta anteriorment per al ratchet Brownià [37], presenta un efecte molt similar i per tant reforça l'equivalència entre els dos models.

Un altre punt d'interès fa referència als intercanvis energètics en els motors Brownians. Aquesta qüestió ha estat estudiada durant els darrers anys, i fins i tot podem trobar en la bibliografia existent diferents definicions per a l'eficiència. A més, aquesta qüestió ha subscitat interès per al cas dels jocs de Parrondo [8] ja que no existia una connexió clara entre l'energia que s'injecta al sistema, l'energia que se n'obté i en conseqüència l'eficiència dels jocs. Per tant el Capítol 6 està dedicat a un estudi de la relació entre l'eficiència d'un sistema físic i els jocs de Parrondo. Emprant el formalisme introduït prèviament en el Capítol 4, desenvolupam un mètode per avaluar l'eficiència dels jocs combinant resultats tant de models discrets com de continuus.

Tots els capítols anteriors determinen, des de diferents perspectives, la relació completa que existeix entre els jocs de Parrondo i el model del flashing ratchet. Cal destacar que en aquests jocs únicament hi intervé un sol jugador. Llavors, el següent pas a fer inclou un estudi de jocs amb més d'un jugador: és a dir, els *jocs de Parrondo col.lectius*. Ambdós Capítols 7 i 8 estàn dedicats als jocs col.lectius. Per una banda, estudiam al Capítol 7 diversos casos d'un joc col.lectiu introduït per Toral [38], en els quals s'estableix una redistribució de capital entre els jugadors. Obtenim, per a diferents combinacions dels jocs A i B, resultats analítics per al guany mitjà d'un sol jugador. D'altra banda, introduïm al Capítol 8 una nova versió de jocs col.lectius que presenten, a més de l'efecte de Parrondo, una inversió de corrent sota determinades circumstàncies. Aquesta nova propietat es caracteritza per obtenir un joc que pot ser guanyador o perdedor d'acord amb la freqüència de canvi entre els diferents jocs, un resultat que no s'observa per al cas d'un sol jugador. Analitzam en detall aquests nous jocs i explicam qualitativament el mecanisme que es troba darrera aquesta inversió de corrent.

El Capítol 9 fa referència a l'altre tema principal d'estudi d'aquest tesi. Consideram l'anàlisi d'un altre tipus de jocs en els quals tamb té lloc un resultat paradògic. Aquests jocs es coneixen com a *truels*, i poden considerar-se com a una extensió del duel a on hi participen tres individuus. De forma resumida, l'efecte paradògic que s'obté en aquests jocs és que el jugador més fort (o amb més aptituds) no necessàriament guanyarà el joc, sinó que en alguns casos el jugador més feble posseeix la major probabilitat de sobreviure.

Aquests jocs varen ser estudiats des del punt de vista de la teoria de jocs [39–42]. En aquest Capítol 9 reproduïm els resultats existents d'aquest camp en un llenguatge que, segurament, un físic troba més adequats per a un millor enteniment: el dels processos estocàstics. Obtenim les probabilitats de supervivència per a cada jugador, així com la distribució de guanyadors per a diferentes versions dels jocs analitzats. També empram les simulacions dels jocs (un procediment amb ordinador i amb llarga tradició en la física) per tal d'entendre els resultats en aquelles situacions en les quals molts jugadors competeixen entre ells emprant les regles dels *truels*. A més, estudiam l'efecte d'incloure una dependència espaial en el sistema i una generalització dels truels per a més de tres jugadors.

Finalment, en el Capítol 10 extraiem les conclusions sobre el treball presentat, així com les futures línies de treball que cal seguir.

Pau Amengual

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# Preface

Since the discovery by the botanist R. Brown in the nineteenth century of the presence of an erratic movement in small-scale systems, later known as Brownian motion, great advances occurred in the field of stochastic processes in subsequently years. The unavoidable presence of noise (or terms in the movement equations coming from the supression of microscopic degrees of freedom, which can only be described in a probabilistic manner) in a system was supposed to play a destructive role. Very recently, however, there have appeared many situations in which noise can lead to a constructive effect. For example, phenomena of stochastic resonance [1] shows that noise can enhance the transmission properties of a system [2]. We also find phase transitions induced by noise, where the presence of noise may induce a nonequilibrium phase transition to a symmetry breaking state [2].

Another interesting application deals with transport phenomena: noise can be used in order to obtain directed motion, i.e., one can rectify unbiased fluctuations caused by thermal environment obtaining a net current in the system. This model is known in the literature as Brownian motor. Basically it consists on a small-scale system subjected to thermal fluctuations which are *rectified* through some sort of asymmetry (either spatial or temporal) present in the system. This transport effect is known as *ratchet effect*. Depending on the way the asymmetry is introduced we may distinguish betwen different kinds of Brownian motors. However, our interest is focused on one class of Brownian motor known as *flashing ratchet*, characterized by a particle subjected to an asymmetric potential that is switched *on* and *off* either periodically or randomly.

The flashing ratchet served as *inspiration* to the Spanish physicist Juan M.R. Parrondo to design a pedagogical example with two coin tossing games A and B where a similar effect took place. He devised the games in 1996, presenting them in unpublished form in Torino, Italy [3]. These games, later known as *Parrondo games*, were both fair games (or even losing) when played alone, whereas if one combined them either in a periodic or even random fashion a winning game was obtained. Therefore this example showed in a simple manner that two dynamics, when combined, do not necessarily give a result being simply a *sum* of dynamics. All the contrary, it might turn to be a totally unexpected

[2]To be more precise, there exists an optimal value of the noise amplitude, reflected by the presence of a maximum when plotting the signal to noise ratio in terms of noise.

outcome. The result of a winning game out of two fair/losing games is known in the literature as *Parrondo's Paradox* [4–8].

Since their appearance, these games attracted much interest in other fields, for example quantum information theory [9–12], control theory [13, 14], Ising systems [15], pattern formation [16–18], stochastic resonance [19], random walks and diffusions [20–24], discrete dinamical systems [25–27], economics [28, 29], molecular motors in biology [30, 31], biogenesis [32] and population dynamics [33, 34]. They have also been considered as quasi-birth-death processes [35] and lattice gas automata [36].

However, even though the connection between the flashing ratchet and Parrondo's games was patent, there was no precise and quantitative relation between both. Therefore it is the aim of this thesis to deepen into this connection between Parrondo games and the flashing ratchet. The thesis is divided into ten Chapters, from which Chapter 1 constitutes a brief introduction to the necessary preliminary concepts needed for a better understanding of succeeding chapters. We present some basic concepts taken from theory of stochastic processes, as well as others from Markov chain theory and information theory.

Chapter 2 is devoted to a detailed explanation of the Brownian ratchet. Concretely we focus on the flashing ratchet, explaining the physical mechanism behind the ratchet effect. In this Chapter we also explain in detail the original Parrondo's games as they were defined, together with a detailed analysis by means of discrete-time Markov chains leading to the distribution of stationary probabilities as well as the rates of winning of the games. Besides, we briefly present other versions of Parrondo games present in the literature, differentiating from the originals on the rules used for selecting the probabilities.

Games A and B appearing in Parrondo's paradox can be thought of as diffusion processes under the action of a external potential. However, they do not have the more general form of a natural diffusion process, because the capital will always change with every game played, whereas in the most general diffusion process a particle can either move up or down or remain in the same position at a given time. In Chapter 3 we present a new version of Parrondo's games, where a new transition probability is taken into account. We introduce a *self-transition* probability, that is, the capital of the player now can remain the same after a game played. Thus the significance of this new version is a natural evolution of Parrondo's games, from which the original games consitute a particular case.

After introducing this new version of Parrondo games, we proceed to derive a *quantitative* relation between Parrondo's games and the physical model of the Brownian ratchet. Parrondo's original work did not make such a detailed comparison. The interest goes both ways: using our detailed comparison it is possible to derive new games starting from very simple physical potentials; similarly, it is possible to generate new physical models that undergo the ratchet effect starting from some game theoretical description. Chapter 4 is thus dedicated to present this relation between Parrondo's games and the flashing ratchet, showing that it can be established in a rigorous way.

To extend further the analogy between Parrondo games and the Brownian ratchet, we analyze in Chapter 5, from the point of view of information theory, the relation between the current (or gain) from the games and the information entropy (also known as negentropy). This relation, already established for the Brownian ratchet [37], presents a similar effect and hence reinforces the equivalence between both models.

Another point of interest concerns the energetics of Brownian motors. This question has been addressed in recent years, finding in the literature different definitions of efficiency. It has also raised interest in case of Parrondo games [8] as there is no clear connection between energy input, energy output and consequently the efficiency in the games. Thus, Chapter 6 is dedicated to a study of the relation between the energetics of a physical system and Parrondo's games. Making use of the formalism introduced previously in Chapter 4, we develop a method for evaluating the efficiency of the games combining results from both discrete and continuous models.

All previous Chapters determine, from different perspectives, a complete relation existing between Parrondo's games and the flashing ratchet. These games are played by one player only. Therefore, our next step involves a study of a game with more than one player: i.e. *collective Parrondo games*. Both Chapters 7 and 8 are dedicated to collective games. On one hand we study in Chapter 7 various cases from a collective game introduced by Toral [38], where a redistribution of capital takes place amongst players. We obtain, for different combinations of games A and B, analytical results for the average gain of a single player. On the other hand, we introduce in Chapter 8 a new version of collective games presenting, appart from the Parrondo effect, a current inversion under certain circumstances. This new feature is characterized by an outcome that can be winning or losing according to the frequency of change between the differen games, a result that is not observed in single player games. We analyze in detail these new games and explain qualitatively the mechanism behind this current inversion.

Chapter 9 is committed to the second main subject of study of the present thesis. It considers the analysis of another kind of games where, again, a counter-intuitive result takes place. These are the so–called truel games, and can be thought of as an extension of a duel played by three individuals. In brief, the paradoxical result appearing in these games is that the player with the highest performance does not necesarrily win the game, instead, even the weakest possesses a higher probability of winning in some cases.

These games were studied from the pont of view of game theory [39–42]. In this Chapter 9 we reproduce the existing results of this field in a language that, hopefully, a physicist finds more comfortable to understand: that of stochastic processes. We obtain the survival probabilities of each player, as well as the distribution of winners for the different versions of the games analyzed. We also use simulations of the games (a computer procedure with long tradition in physics) in order to understand the actual outcome in a situation in which many agents compete amongst themselves using the rules of truels. Furthermore, we study the effect of including spatial dependence in the system and a generalization of the truels to more than three players.

Finally, in Chapter 10 conclusions about the present work will be drawn, together with perspectives about future work.

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# Contents

Chapter 1

# Introduction

In the following sections of the present Chapter we introduce some basic concepts that will be used in foregoing Chapters. First we will present briefly some concepts on the theory of stochastic processes in Sec. 1.1, Markov processes in Sec. 1.2 and the Fokker-Planck equation in Sec.1.3. These will be followed by some introductory concepts about Markov chain theory in Sec.1.4 and finally some concepts on information theory in Sec. 1.5.

## 1.1 Stochastic processes

A stochastic process can be thought of as a system that evolves probabilistically in time, or more explicitly, a system where there exists at least one time–dependent random variable. Denoting this stochastic variable as $\mathbf{X}(t)$, we can measure its actual value $x_1, x_2, x_3, \ldots$ at different times $t_1, t_2, t_3, \ldots$ and so obtain the joint probability density of the variable $\mathbf{X}(t)$

$$P(x_1, t_1; x_2, t_2; x_3, t_3; \ldots) \tag{1.1}$$

which denotes the probability that we measured the value $x_1$ at time $t_1$, value $x_2$ at time $t_2$,. . ., etc.

Using these probability density functions we can also define *conditional probability densities* through

$$P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1; y_2, \tau_2; \ldots) = \frac{P(x_1, t_1; x_2, t_2; \ldots; y_1, \tau_1; y_2, \tau_2; \ldots)}{P(y_1, \tau_1; y_2, \tau_2; \ldots)}, \tag{1.2}$$

where it's been assumed that the times are ordered, i.e., $t_1 \geq t_2 \geq t_3 \geq \ldots \geq \tau_1 \geq \tau_2 \geq \ldots$.

The simplest stochastic process is that of complete independence

$$P(x_1, t_1; x_2, t_2; x_3, t_3; \ldots) = \prod_i P(x_i, t_i) \tag{1.3}$$

which means that the value of **X** at time $t$ is completely independent of its values at previous – or posterior – times.

The next step is to consider processes where the future state of the system depends on its actual state. This kind of processes are known in the literature as *Markov processes*.

## 1.2 Markov processes

This class of processes are characterized by the so called *Markov property*. A Markov process can be defined as a stochastic process with the property that for any set of *successive* times, i.e. $t_1 \geq t_2 \geq t_3 \geq \ldots \geq \tau_1 \geq \tau_2 \geq \ldots$, one has

$$P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1; y_2, \tau_2; \ldots) = P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1). \tag{1.4}$$

This previous statement means that we can define everything in terms of simple conditional probabilities $P(x_1, t_1 \mid y_1, \tau_1)$. For instance, $P(x_1, t_1; x_2, t_2 \mid y_1, \tau_1) = P(x_1, t_1 \mid x_2, t_2; y_1, \tau_1) P(x_2, t_2 \mid y_1, \tau_1)$ and using the Markov property (1.4) we find

$$P(x_1, t_1; x_2, t_2; y_1, \tau_1) = P(x_1, t_1 \mid x_2, t_2) P(x_2, t_2 \mid y_1, \tau_1) \tag{1.5}$$

and for the general case it can be written

$$\begin{aligned} P(x_1, t_1; x_2, t_2; x_3, t_3; \ldots x_n, t_n) = \quad & P(x_1, t_1 \mid x_2, t_2) \quad P(x_2, t_2 \mid x_3, t_3) \ldots \\ \ldots \quad & P(x_{n-1}, t_{n-1} \mid x_n, t_n) \quad P(x_n, t_n) \end{aligned} \tag{1.6}$$

provided that $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_n$.

There are many processes in nature where this property appears. One of the most studied processes successfully described using this Markov property is the Brownian motion, presented in more detail in the next Section.

### 1.2.1 Brownian motion

The botanist Robert Brown discovered in 1827 that small particles suspended in water were found to be in a very animated and irregular motion. Initially it was supposed to represent some manifestation of life, though after some studies this option was rejected, as the same behavior was also observed in other fine particles suspension –minerals, glass .... The solution to this mysterious movement had to await a few decades, until a satisfactory explanation came through the work of Albert Einstein in 1905 [43]. The same explanation was independently developed by Smoluchowski [44], who was responsible for much of the later systematic development and for much of the experimental verification of Brownian motion theory.

Einstein's work had primarily two main premises:

- The motion of the particles is caused by the exceedingly frequent impacts on the pollen grain of the incessantly moving molecules of liquid in which it is suspended.
- The motion of these particles can only be described probabilistically in terms of frequent and statistical independent impacts, due to the erratic and irregular (and so complicated) movement that the particles carry out.

This process is the best known example of Markov process. We have the picture of a particle that makes random jumps back and forth over a given set of coordinates, for instance over the X–axis in one dimension. The jumps may have any length, but the probability for large jumps falls off rapidly. Moreover, the probability is symmetrical in space and independent of the starting point.

Hence, we can summarize the basic steps that Einstein took in order to derive his Brownian motion theory.

The first point to consider is that each individual particle executes a motion which is totally uncorrelated from the motion of all other particles; it will also be considered that the displacement of the same particle, but taken at different time intervals, are also independent processes – as long as these time intervals are not taken too small.

Then a characteristic time interval $\tau$ can be introduced, which is small compared to the observation time intervals, but large enough so that the approximation of independent successive time intervals $\tau$ is correct.

Now we consider $n$ particles suspended in a liquid. In a time interval $\tau$, the $x$–coordinate of the particles will increase by an amount $\Delta$, where this quantity may have different values – either positive or negative – for different particles in the same time interval. We will also consider that there exists a certain distribution law for $\Delta$, given by the function $\phi(\Delta)$. The number of particles that will shift their position with an interval between $\Delta$ and $\Delta + d\Delta$ will be given by the expression

$$dn = n\,\phi(\Delta)\,d\Delta \tag{1.7}$$

where

$$\int_{-\infty}^{\infty} \phi(\Delta)d\Delta = 1 \tag{1.8}$$

The function $\phi$ is only distinct from zero for small values of $\Delta$, and it also follows the property

$$\phi(\Delta) = \phi(-\Delta) \tag{1.9}$$

which implies that there exists no preferred direction of movement for the particles.

We can now study how the diffusion coefficient depends on $\phi$. Let $P(x,t)$ be the number of particles per unit volume at $(x,t)$. We compute the distribution of particles at time $t+\tau$ from the distribution at time $t$. From the definition of the function $\phi(\Delta)$, we can obtain the number of particles which at time $t+\tau$ are found between the points $x$ and $x+dx$. One obtains

$$P(x,t+\tau) = \int\limits_{-\infty}^{\infty} P(x-\Delta,t)\phi(\Delta)d\Delta. \tag{1.10}$$

But since $\tau$ is very small, we can Taylor expand $P(x,t+\tau)$

$$P(x,t+\tau) = P(x,t) + \tau\,\frac{\partial P}{\partial t}. \tag{1.11}$$

Besides, we can also Taylor expand the function $P(x-\Delta,t)$ in powers of $\Delta$

$$P(x-\Delta,t) = P(x,t) - \Delta\frac{\partial P(x,t)}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 P(x,t)}{\partial x^2} + \ldots \tag{1.12}$$

Introducing the results from Eq. (1.11,1.12) into the integral Eq. (1.10) we obtain the following expression

$$P + \frac{\partial P}{\partial \tau}\tau = P\int\limits_{-\infty}^{\infty}\phi(\Delta)\,d\Delta - \frac{\partial P}{\partial x}\int\limits_{-\infty}^{\infty}\Delta\phi(\Delta)\,d\Delta + \frac{\partial^2 P}{\partial x^2}\int\limits_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)\,d\Delta. \tag{1.13}$$

Due to the symmetry property Eq. (1.9), the odd terms of Eq. (1.13) – second term, fourth term, etc. – vanish, whereas for the remaining terms, that is, first term, third term, etc. each one is very small compared to the previous one. Introducing Eq. (1.8) in the last equation, setting

$$\frac{1}{\tau}\int\limits_{-\infty}^{\infty}\frac{\Delta^2}{2}\,\phi(\Delta)\,d\Delta = D, \tag{1.14}$$

and keeping only the first and third terms on the right hand side,

$$\frac{\partial P}{\partial t} = D\,\frac{\partial^2 P}{\partial x^2}. \tag{1.15}$$

We can clearly identify the latter equation as the diffusion equation, and $D$ as the diffusion coefficient. The solution for an initial condition at $t = 0$ given by $n(x) = n\,\delta(x)$ is

$$P(x,t) = \frac{n}{\sqrt{4\pi D}}\frac{\mathrm{e}^{-\frac{x^2}{4Dt}}}{\sqrt{t}} \tag{1.16}$$

which is a Gaussian function centered at the origin. Using this result we calculate the averages

$$\begin{aligned} \langle x \rangle &= 0 \qquad (1.17) \\ \langle x^2 \rangle &= 2Dt. \qquad (1.18) \end{aligned}$$

This result was derived by Einstein assuming a discrete–time assumption, that is , that the impacts occurred only at times $0, \tau, 2\tau, \ldots,$ and both Eqs. (1.15,1.16) are to be regarded as only approximations, where $\tau$ is considered so small that $t$ can be thought as being continuous.

### 1.2.2 Langevin's equation

After Einstein presented his theory about Brownian motion, Langevin [45] presented another method quite different from Einstein's work. In brief, his theory can be explained as follows.

From statistical mechanics it was already known that the mean kinetic energy of a Brownian particle at equilibrium should reach a value

$$< \frac{1}{2} m v^2 >= \frac{1}{2} kT \qquad (1.19)$$

where $T$ denotes the absolute temperature, $k$ is the Boltzmann constant, $m$ the mass and $v$ the velocity of the Brownian particle.

We can distinguish two different forces acting on the particle, namely,

- A viscous drag. Assuming that the expression of the force is analogous to the macroscopic hydrodynamic equation, for a low Reynolds number we can write down the following expression for the drag force $-6\pi\eta a \frac{dx}{dt}$, $\eta$ being the viscosity and $a$ the diameter of the particle, assuming it to be spherical.
- A fluctuating force $\xi$ coming from the consideration of the impacts of the fluid particles upon the Brownian particle. The unique consideration about this force is that it can be either positive or negative with the same probability. The ensemble may consist on many particles in the same field, far enough from each other so that they cannot influence mutually. Or it may also be considered as a unique particle, where the time intervals between measurements are large enough not to influence each other.

The stochastic properties of $\xi$ are given regardless of the velocity $v$ of the particle. Its average vanishes, $< \xi >= 0$, and its autocorrelation function reads

$$< \xi(t)\xi(t') >= \delta(t - t') \qquad (1.20)$$

The latter expression comes from the consideration that successive collisions are uncorrelated and practically instantaneous.

Writing down Newton's equation of motion for the particle we get

$$m\frac{d^2x}{dt^2} = -6\pi\eta a\frac{dx}{dt} + \xi \tag{1.21}$$

This equation is usually known as Langevin equation. Multiplying Eq. (1.21) by $x$, and after a little algebra we obtain

$$\frac{m}{2}\frac{d^2}{dt^2}(x^2) - mv^2 = -3\pi\eta a\frac{d(x^2)}{dt} + x\,\xi \tag{1.22}$$

where $v = \frac{dx}{dt}$. Averaging over a large number of particles and making use of Eq. (1.19) we obtain an equation for $< x^2 >$

$$\frac{m}{2}\frac{d^2}{dt^2} < x^2 > +3\pi\eta a\frac{d}{dt} < x^2 >= kT, \tag{1.23}$$

where the term $< x\xi >$ has been set to zero due to the irregularity of the fluctuating force $\xi$. This assumption implies that the variation suffered by the x variable can be considered as independent from the variation that the fluctuating force $\xi$ experiences[1]

$$< x\xi >=< x >< \xi > \tag{1.24}$$

The general solution to Eq. (1.23) is

$$\frac{d}{dt} < x^2 >= \frac{kT}{3\pi\eta a} + C\,\mathrm{e}^{\frac{-6\pi\eta a t}{m}} \tag{1.25}$$

where $C$ is an arbitrary constant.

Considering that the exponential in Eq. (1.25) decays very rapidly, we can dismiss this term and so the solution for the average square distance $< x^2 >$ reads

$$< x^2 > - < x_0^2 >= \left(\frac{kT}{3\pi\eta a}\right) t \tag{1.26}$$

Now we can compare Eq. (1.26) with Eq. (1.18) to obtain the following relation

$$D = \frac{kT}{6\pi\eta a} = \mu kT \tag{1.27}$$

where $\mu$ is the mobility of the Brownian particle.

This important result, known as the *fluctuation–dissipation* theorem, relates a quantity $D$ pertaining to statistically unpredictable dynamical fluctuations to a quantity which involves deterministic, steady state properties.

[1]This can be thought as equivalent to the assumption made by Einstein when he considers that for a sufficiently large time interval $\tau$, the displacements $\Delta$ suffered by the Brownian particle within two successive time intervals are independent.

## 1.3 The Fokker–Planck equation

This section aims to be a brief explanation on how to obtain the time evolution of the probability density function for the system under consideration. Its name comes from the work of Fokker [46] and Planck [47]. The former studied Brownian motion in a radiation field and the latter attempted to build a complete theory of fluctuations based on it.

### 1.3.1 Derivation of the Fokker–Planck equation

If we consider a Markov process, we can write a master equation as

$$\frac{\partial P(x,t)}{\partial t} = \int \Big\{ W(x \mid x')\, P(x',t) - W(x' \mid x)\, P(x,t) \Big\}\, dx' \tag{1.28}$$

where the term $W(x \mid x')$ denotes the transition probability between states $x$ and $x'$. $P(x,t)$ denotes the probability of finding the system at position $x$ at time $t$, and must be normalized, that is

$$\int_{-\infty}^{\infty} dx\, P(x,t) = 1 \tag{1.29}$$

If $x$ corresponds to a discretized variable, the master equation takes the form

$$\frac{dP_n(t)}{dt} = \sum_n \left\{ W_{nn'} P_{n'}(t) - W_{n'n} P_n(t) \right\}. \tag{1.30}$$

Written in this form clearly the master equation is a gain–loss equation. The first term on the right hand side of Eq. (1.30) corresponds to the gain of state $n$ due to transitions from different states $n'$ to $n$, whereas the second term is a loss term due to the transitions from the state $n$ to other states $n'$.

Planck derived the Fokker–Planck equation as an approximation to the master equation (1.28). He expressed the transition probability $W(x \mid x')$ as a function of the size $r$ of the jump and of the starting point

$$W(x \mid x') = W(x'; r), \qquad r = x - x'. \tag{1.31}$$

Then (1.28) can be rewritten in the form

$$\frac{\partial P(x,t)}{\partial t} = \int W(x-r; r) P(x-r,t) dr - P(x,t) \int W(x; -r) dr \tag{1.32}$$

At this stage two assumptions are made,

- Only small jumps occur, i.e., $W(x';r)$ is a sharply peaked function of $r$ but varies slowly with $x'$. Then there will exist some $\delta > 0$ such that

$$W(x';r) \approx 0 \qquad for \mid r \mid > \delta \tag{1.33}$$

$$W(x'+\Delta x;r) \approx W(x';r) \quad for \mid \Delta x \mid < \delta. \tag{1.34}$$

- The second assumption is that the solution $P(x,t)$ also varies slowly with $x$, making possible a Taylor expansion of the term $P(x-r,t)$ in terms of $P(x,t)$ obtaining

$$\frac{\partial P(x,t)}{\partial t} = \int W(x;r)P(x,t)\,dx - \int r\frac{\partial}{\partial x}\{W(x;r)P(x,t)\}\,dr + \frac{1}{2}\int r^2\frac{\partial^2}{\partial x^2}\{W(x;r)P(x,t)\}\,dr - P(x,t)\int W(x;-r)dr. \tag{1.35}$$

The first and fourth terms on the right hand side of Eq. (1.35) vanish, whereas the other two remaining terms are named as

$$F(x) = \int_{-\infty}^{\infty} r\,W(x;r)dr \tag{1.36}$$

$$D(x) = \int_{-\infty}^{\infty} r^2\,W(x;r)dr, \tag{1.37}$$

and they correspond to the first and second jump moments of $W(x;r)$, respectively. The first jump moment corresponds to the so called *drift term* –$F(x)$–, and the second moment to the *diffusion term* –$D(x)$ –. Then the final result is

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x)P(x,t)] \tag{1.38}$$

In conclusion, we have derived the Fokker–Planck equation starting from the master equation governing the transitions between different states from the system.

### 1.3.2 The Fokker–Planck equation in one dimension

For a one dimension we can write the following Fokker–Planck equation – as derived in the previous section –

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x,t)\,P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[D(x,t)\,P(x,t)\right]. \tag{1.39}$$

Here $D(x,t)$ is the diffusion term and $F(x,t)$ is the drift term usually written as $F(x,t) = -\frac{\partial V(x,t)}{\partial x}$, introducing a potential function $V(x,t)$. The stochastic process whose probability density function obeys Eq. (1.39) is equivalent to the stochastic process described by the Ito stochastic differential equation

$$\dot{x} = F(x,t) + \sqrt{D(x,t)}\,\xi(t) \tag{1.40}$$

where $\xi(t)$ is a gaussian white noise of mean zero and correlation given by $<\xi(t)\xi(t')>= \delta(t-t')$.

Defining a *probability current* $J(x,t)$ as

$$J(x,t) = F(x,t)\,P(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[D(x,t)\,P(x,t)] \tag{1.41}$$

Eq. (1.39) can be rewritten in the form of a continuity equation

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0 \tag{1.42}$$

### 1.3.3 Boundary conditions

The Fokker–Planck equation is a second–order parabolic differential equation, and in order to find its solution we need an initial condition as well as some boundary conditions where the variable $x$ is constrained. For a more general case, in more than one dimension, we can write

$$\partial_t P(\mathbf{x},t) = -\sum_i \frac{\partial}{\partial x_i} F(\mathbf{x},t)\,P(\mathbf{x},t) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 D(\mathbf{x},t)}{\partial x_i \partial x_j} \tag{1.43}$$

which can also be written as a continuity equation

$$\frac{\partial P(\mathbf{x},t)}{\partial t} + \sum_i \frac{\partial J_i(\mathbf{x},t)}{\partial x_i} = 0 \tag{1.44}$$

The previous equation has the form of a local conservation law, and so it can be rewritten in an integral form. Considering a region $R$ with boundary **S** we have

$$\frac{\partial P(R,t)}{\partial t} = -\int_{\mathbf{S}} dS\,\hat{\mathbf{n}}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x},t) \tag{1.45}$$

where we have defined the total probability in region $R$ as $P(R,t) = \int_R d\mathbf{x}\,P(\mathbf{x},t)$, and $\hat{\mathbf{n}}(\mathbf{x})$ is an outward vector pointing normal to **S**. Eq. (1.45) indicates that the total loss of probability in the region $R$ is given by the surface integral of $\mathbf{J}(\mathbf{x},t)$ over region $R$. The current $\mathbf{J}(\mathbf{x},t)$ also has the property that a surface integral over any surface **S** gives us the net flow of probability across that surface. Depending on the existing boundary conditions, we will impose different conditions, such as

**Reflecting barrier** In this case there is no flow of probability through surface **S**, which can be thought of as the particle not leaving region $R$. In this case it is required that

$$\hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x}, t) = 0, \, \forall \, \mathbf{x} \in \mathbf{S} \tag{1.46}$$

**Absorbing barrier** For this case when the particle reaches the boundary, it is removed from the system. As a consequence, the probability of finding the particle in the boundary is strictly zero,

$$P(\mathbf{x}, t) = 0, \forall \mathbf{x} \in \mathbf{S} \tag{1.47}$$

**Periodic boundary conditions** The process takes place in a closed interval $[\mathbf{a}, \mathbf{b}]$, where the two end points are identified with each other. This implies the following set of conditions to be fulfilled

$$\begin{aligned} \lim_{\mathbf{x} \to \mathbf{b}^-} P(\mathbf{x} + \mathbf{m}L, t) &= \lim_{\mathbf{x} \to \mathbf{a}^+} P(\mathbf{x} + \mathbf{m}L, t) \\ \lim_{\mathbf{x} \to \mathbf{b}^-} J(\mathbf{x} + \mathbf{m}L, t) &= \lim_{\mathbf{x} \to \mathbf{a}^+} J(\mathbf{x} + \mathbf{m}L, t). \end{aligned} \tag{1.48}$$

where the quantity $\mathbf{m}L$ accounts for a displacement in any direction equal to the periodicity of the system.

### 1.3.4 Stationary properties

Given a stochastic process $\mathbf{X}(t)$, we say that $\mathbf{X}(t)$ is a stationary process if $\mathbf{X}(t)$ and the process $\mathbf{X}(t + t_0)$ have the same statistics for any $t_0$. This property is equivalent to saying that all joint probability densities satisfy time translation invariance, that is

$$P(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = P(x_1, t_1 + t_0; x_2, t_2 + t_0; \ldots; x_n, t_n + t_0) \tag{1.49}$$

and therefore such probabilities are only functions of the time differences $t_i - t_j$. In the particular case of the one–time probability, it is independent of time $t$ and it can be written as $P_s(\mathbf{x})$. Furthermore, if the stationary Markov process satisfies

$$\lim_{t \to \infty} P(x, t | x_0, 0) = P_s(x) \tag{1.50}$$

then we can construct from the stationary Markov process a nonstationary process whose limit as time becomes large is the stationary process. It can be defined for $t, t' > t_0$ by

$$P(x, t) = P_s(x, t | x_0, t_0) P(x, t | x', t') = P_s(x, t | x', t') \tag{1.51}$$

So if Eq. (1.50) is satisfied, we find that as $t \to \infty$ or $t_0 \to -\infty$, $P(x, t) \to P_s(x)$ and the rest of probabilities become stationary because the conditional probability is also stationary. This process is known as a *homogeneous process*.

For a homogeneous process, the drift and diffusion terms of the Fokker–Planck equation are time independent. Then, returning to the $1D$ case, in the stationary state $\frac{\partial P(x,t)}{\partial t} = 0$ and so $P(x,t) = P^s(x)$ becomes independent of time. From Eq. (1.39) we have

$$\frac{d}{dx}[F(x)P(x)] - \frac{1}{2}\frac{d^2}{dx^2}[D(x)P(x)] = 0. \tag{1.52}$$

And using Eq. (1.42) we have $\frac{dJ(x)}{dx} = 0$, or $J(x) = J = Constant$.

If the process takes place in the interval $(a,b)$, it must be satisfied that $J(a) = J(x) = J(b) = J$; so if one of the boundary conditions is reflecting, it means that both of them must be reflecting, and then $J = 0$.

If the boundaries are not reflecting, the condition of constant current requires them to be periodic. In that case we may use the boundary conditions given by (1.48).

#### *1.3.4.a Zero–current case*

If $J = 0$, Eq. (1.52) can be rewritten as

$$F(x)\,P^s(x) = \frac{1}{2}\frac{d}{dx}[D(x)P^s(x)] \tag{1.53}$$

with solution

$$P^s(x) = \frac{\mathcal{N}}{D(x)}\,\mathrm{e}^{2\int_a^x dx'\,\frac{F(x)}{D(x)}} \tag{1.54}$$

$\mathcal{N}$ being a normalization constant ensuring that $\int_a^b dx\,P^s(x) = 1$.

#### *1.3.4.b Periodic boundary conditions*

For the case where we have a non–zero current Eq. (1.52) can be written as

$$F(x)\,P^s(x) - \frac{1}{2}\frac{d}{dx}[D(x)\,P^s(x)] = J. \tag{1.55}$$

In this case the current $J$ is completely determined by the boundary conditions

$$P^s(a) = P^s(b) \tag{1.56}$$

$$J(a) = J(b). \tag{1.57}$$

For calculating the stationary probability density function $P^s(x)$ we can integrate Eq. (1.55) to obtain

$$P^s(x) = P^s(a) \left[ \frac{\int_a^x \frac{dx'}{\psi(x')} \frac{D(b)}{\psi(b)} + \int_x^b \frac{dx'}{\psi(x')} \frac{D(a)}{\psi(a)}}{\frac{D(x)}{\psi(x)} \int_a^b \frac{dx'}{\psi(x')}} \right] \tag{1.58}$$

and the current is determined through

$$J = \left[ \frac{D(b)}{\psi(b)} - \frac{D(a)}{\psi(a)} \right] \frac{P^s(a)}{\int_a^b \frac{dx'}{\psi(x')}} \tag{1.59}$$

### 1.3.5 Particle current

Once the stationary probability density function (1.58) and the probability current (1.59) are obtained, the next quantity of interest is the *particle current* $< \dot{x} >$, defined as the ensemble average over the velocities. Its relation with the probability current $J(x,t)$ is

$$J(x,t) :=< \dot{x}(t)\delta(x - x(t)) > \tag{1.60}$$

from where we derive

$$< \dot{x} >= \int_{-\infty}^{\infty} dx J(x,t) \tag{1.61}$$

and using Eq. (1.42) can be written as

$$< \dot{x} >= \frac{d}{dt} \int_{-\infty}^{\infty} dx\, x P(x,t). \tag{1.62}$$

## 1.4 Markov–chain theory

This section is devoted to a class of Markov processes in discrete–time and discrete space. We call such processes Markov chains. We may define a Markov chain as a sequence $\mathbf{X}_0$, $\mathbf{X}_1$,... of discrete random variables with the property that the conditional distribution of $\mathbf{X}_{n+1}$ given $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n$ depends only on the value of $\mathbf{X}_n$ but not further on $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$; i.e., for any set of values $h, j, \ldots, k$ belonging to the discrete state space,

$$\mathrm{prob}(\mathbf{X}_{n+1} = k | \mathbf{X}_0 = h, \ldots, \mathbf{X}_n = j) = \mathrm{prob}(\mathbf{X}_{n+1} = k | \mathbf{X}_n = j) \tag{1.63}$$

Thus the conditional probability distribution for $\mathbf{X}_n$ depends only on the value of $\mathbf{X}$ at the latest time $n-1$.

### 1.4.1 A two–state Markov chain

We will consider a simple example of a two–state Markov chain. This is the simplest non–trivial state space. Let's denote by $1$ and $0$ the two states of the Markov chain. If the system is found in state $0$, there will be a probability $\alpha$ of a transition to state $1$, and a probability $1-\alpha$ of remaining in the same state. Similarly, when the system is found in state $1$, there will be a probability $\beta$ of a transition to state $0$, and $1-\beta$ of remaining in $1$. These probabilities are called *transition probabilities*, and can be represented by a *transition matrix* $\mathbb{T}$ as

$$\mathbb{T} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \tag{1.64}$$

The matrix element in position $(j,k)$ denotes the conditional probability of a transition to state $k$ at time $n+1$ given that the system is in state $j$ at time $n$. The transition probabilities considered here do not depend on time.

Let the column vector $\mathbb{P}^n = (P_0^n, P_1^n)^T$ denote the probabilities of finding the system in states $0$ or $1$ at time $n$ when the initial probabilities of the two states are given by $\mathbb{P}^0 = (P_0^0, P_1^0)^T$. Consider the system to be in state $0$ at time $n$. This state can be reached in two mutually exclusive ways: either state $0$ was occupied at time $n-1$ and no transition occurred at time $n$; this event may happened with probability $P_0^{n-1}(1-\alpha)$. Alternatively, the system could happenned to be in state $1$ at time $n-1$ followed by a transition from state $1$ to state $0$ at time $n$; this latter event has probability $P_1^{n-1}\beta$.Thus we can obtain the following recurrence relations

$$P_0^n = (1-\alpha)P_0^{n-1} + \beta P_1^{n-1} \tag{1.65}$$

$$P_1^n = \alpha P_0^{n-1} + (1-\beta)P_1^{n-1} \tag{1.66}$$

$$\tag{1.67}$$

which can be put in matrix form as $\mathbb{P}^n = \mathbb{T}\cdot\mathbb{P}^{n-1}$, and iterating we obtain

$$\mathbb{P}^n = \mathbb{T}^2\cdot\mathbb{P}^{n-2} = \ldots = \mathbb{T}^n\cdot\mathbb{P}^0. \tag{1.68}$$

Thus, given the initial probabilities $\mathbb{P}^0$ and the transition matrix $\mathbb{T}$, we can find the state occupation probabilities at any time $n$ by means of Eq. (1.68). One question arises naturally, and it concerns the possibility of whether the system reaches a situation of statistical equilibrium after a sufficiently long time, where the occupation probabilities $\mathbb{P}^n$ are independent of the initial conditions. If this happens there exists an equilibrium probability distribution $\pi = (\pi_0, \pi_1)$ when $n\to\infty$ satisfying

$$\pi = \mathbb{T}\pi \;\longrightarrow\; (\mathbb{I}-\mathbb{T})\pi = 0 \tag{1.69}$$

From where it follows through the normalization condition $\pi_0 + \pi_1 = 1$ that

$$\pi_0 = \frac{\beta}{\beta+\alpha}\ , \quad \pi_1 = \frac{\alpha}{\beta+\alpha} \tag{1.70}$$

Therefore, if the initial probability distribution coincides with $\pi$, the distribution $\mathbb{P}^n$ is stationary, i.e., it does not change in time.

If we want to find the time dependent probabilities $\mathbb{P}^n$ given a set of initial probabilities $\mathbb{P}^0$, we need to evaluate the matrix $\mathbb{T}^n$. For this purpose we can use the spectral representation of $\mathbb{T}$. Let us assume that $\mathbb{T}$ has distinct eigenvalues $\lambda_1$, $\lambda_2$. Then, we can find a $2\times 2$ matrix $\mathbb{Q}$ such that

$$\mathbb{T} = \mathbb{Q}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbb{Q}^{-1} \tag{1.71}$$

where the columns $q_1, q_2$ of $\mathbb{Q}$ are solutions of the equations $\mathbb{T}q_i = \lambda_i q_i$. Hence we have

$$\mathbb{T}^n = \mathbb{Q}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\mathbb{Q}^{-1} \tag{1.72}$$

The eigenvalues of $\mathbb{T}$ are the solutions of the equation $|\mathbb{T}-\lambda\mathbb{I}| = 0$, from where it follows the equation $(1-\alpha-\lambda)(1-\beta-\lambda)-\alpha\beta = 0$. The solutions are $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$ and $\lambda_1 \neq \lambda_2$ provided that $\alpha+\beta \neq 0$. We obtain for matrices $\mathbb{Q}$ and $\mathbb{Q}^{-1}$

$$\mathbb{Q} = \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}, \qquad \mathbb{Q}^{-1} = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} \tag{1.73}$$

Thus,

$$\mathbb{T} = \mathbb{Q}\begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix}\mathbb{Q}^{-1} \tag{1.74}$$

Recall that $\lambda_2 = 1-\alpha-\beta$ is less than one in modulus, unless $\alpha+\beta = 0$ or $\alpha+\beta = 2$. For a general time $n$ we have

$$\begin{aligned} \mathbb{T}^n &= \frac{1}{\alpha+\beta}\begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{pmatrix}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\beta-\alpha)^n}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}. \end{aligned} \tag{1.75}$$

We can easily identify the first term in Eq. (1.75) as $\begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}$, while the second term tends rapidly to zero with increasing $n$, as long as $|1-\alpha-\beta| < 1$. Thus as $n\to\infty$,

$$\mathbb{T}^n \to \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix} \tag{1.76}$$

and from Eq. (1.68) we obtain

$$\mathbb{P}^n \to \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix} \mathbb{P}^0 = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \pi \tag{1.77}$$

Therefore the limiting state occupation probability exist and are independent of the initial conditions $\mathbb{P}^0$.

### 1.4.2 General case of a Markov chain

We turn to a more general case in which our Markov chain may be composed of either a finite or infinite number of states. We have a sequence of discrete random variables $\mathbf{X}_0, \mathbf{X}_1, \ldots$ having the property that given the value of $\mathbf{X}_m$ for any instant time $m$, then for any later time instant $m + n$ the probability distribution of $\mathbf{X}_{m+n}$ is completely determined and the values of $\mathbf{X}_{m-1}, \mathbf{X}_{m-2}, \ldots$ at times earlier than $m$ are irrelevant to its determination[2]. Thus, if $m_1 < m_2 < \ldots < m_r < m < m+n$

$$\mathrm{prob}(\mathbf{X}_{m+n} = k | \mathbf{X}_{m_1}, \ldots, \mathbf{X}_{m_r}, \mathbf{X}_m) = \mathrm{prob}(\mathbf{X}_{m+n} = k | \mathbf{X}_m). \tag{1.78}$$

Besides, we will consider the case of *homogeneous* Markov chains, which are characterized by possesing a stationary state when the conditional probability (1.78) depends only on the time interval $n$, not on $m$. For this kind of chains we can define the *n–step transition probabilities*

$$p_{jk}^n = \mathrm{prob}(\mathbf{X}_{m+n} = k | \mathbf{X}_m = j) = \mathrm{prob}(\mathbf{X}_n = k | \mathbf{X}_0 = j) \qquad (m, n = 1, 2, \ldots). \tag{1.79}$$

Particularly we are interested in the *one-step transition probabilities*

$$p_{jk}^1 = p_{jk} = \mathrm{prob}(\mathbf{X}_{m+1} = k | \mathbf{X}_m = j). \tag{1.80}$$

Since our system must realize a transition to some state from any state $j$ (in this case we also include the possibility of a transition to the same state $j$), we have

$$\sum_{k=0}^{\infty} p_{jk} = 1. \tag{1.81}$$

Based on the previous results, a general transition matrix $\mathbb{T}$ would read

$$\mathbb{T} = \begin{pmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ . & . & . \\ . & . & . \end{pmatrix}, \tag{1.82}$$

[2]i.e., if we have complete knowledge of the present state of the system, we can determine the probability of any future state without reference to the past.

and it is known as a *stochastic matrix*, with the properties that its elements are non-negative and that its row sums are unity If the Markov chain defining $\mathbb{T}$ has a finite number of states $l$, then the stochastic matrix $\mathbb{T}$ is a $l$x$l$ square matrix.

Let $\mathbb{P}^0 = \{p_0^0, p_1^0, \ldots\}^T$ denote the column vector for the initial state occupation, and $\mathbb{P}^n = \{p_0^n, p_1^n, \ldots\}^T$ the vector of occupancy probabilities at time $n$. It can be shown, using arguments similar to those used for the two-state Markov chain, that

$$\mathbb{P}^n = \mathbb{T}\mathbb{P}^{n-1} = \ldots = \mathbb{T}^n\mathbb{P}^0 \ \ (n = 1, 2, \ldots). \tag{1.83}$$

### 1.4.3 Classification of states

According to their limiting behavior, we can classify the states of a Markov chain. Suppose that initially we are in state $j$; we call $j$ a *recurrent* state if the ultimate return to this state is a certain event, that is, if the probability of returning to state $j$ after some finite length of time is one. In this case the time of first return will be a random variable called the *recurrence time* and the state is called *positive-recurrent* or *null-recurrent* according as the mean recurrence time is finite or infinite respectively. On the other hand, if the ultimate return to state $j$ has probability less than one the state is called *transient*. At this point we define $f_{jj}^n$ as the probability that the next occurrence of state $j$ is at time $n$, i.e., $f_{jj}^1 = p_{jj}$, and for $n > 1$

$$f_{jj}^n = \text{prob}(\mathbf{X}_r \neq j, r = 1, \ldots, n-1; \mathbf{X}_n = j | \mathbf{X}_0 = j). \tag{1.84}$$

In other words we can say that conditional on state $j$ being occupied initially, $f_{jj}^n$ is the probability that state $j$ is avoided at times $1, 2, \ldots, n-1$ and entered at time $n$. Given that the chain starts in state $j$ the sum

$$f_j = \sum_{n=1}^{\infty} f_{jj}^n \tag{1.85}$$

is the probability that state $j$ is eventually re-entered. If $f_j = 1$ then state $j$ is recurrent while if $f_j < 1$ state $j$ is transient. Thus, conditional on starting in a transient state $j$, there is a positive probability $1 - f_j$ that state $j$ will never be re-entered, while for a recurrent state re-entrance is a certain event. For a recurrent state, therefore, $\{f_{jj}^n, n = 1, 2, \ldots\}$ is a probability distribution and the mean of this distribution

$$\mu_j = \sum_{n=1}^{\infty} n f_{jj}^n, \tag{1.86}$$

is the mean recurrence time.

Similarly, given that the chain starts in state $j$ the sum

$$f_{jk} = \sum_{n=1}^{\infty} f_{jk}^n \tag{1.87}$$

is the probability of ever entering state $k$, and is known as *first passage probability* from state $j$ to state $k$. If $f_{jk} = 1$ then $\sum_{n=1}^{\infty} n f_{jk}^{n}$ is the *mean first passage time* from state $j$ to state $k$.

Let us suppose that when the chain starts in state $j$, subsequent occupations of that state can only occur at times $t$, $2t$, $3t$,... where $t$ is an integer greater than 1; choose $t$ to be the largest integer with this property. Then state $j$ is called *periodic* with period $t$ and $p_{jj}^{n}$ vanishes except when $n$ is an integral multiple of $t$. A state which is not preiodic is called *aperiodic*. Essentially it has period 1. An aperiodic state which is positive-recurrent is called *ergodic*.

An important property of ergodic systems concerns the existence of a unique row vector $\pi$ of limiting occupation probabilities called the *equilibrium distribution*, which is formed by the inverse of the mean recurrence times. Thus a finite ergodic system settles down in the long run to a condition of statistical equilibrium independent of the initial conditions.

Another important classification can be done regarding the *communication* between different states from a Markov chain. State $j$ is said to be *accessible* from state $i$ if for some integer $n \geq 0$, $f_{ij}^{n} > 0$: i.e., state $j$ is accessible from sate $i$ if there is positive probability that in a finite number of transitions state $j$ can be reached starting from state $i$. Two states $i$ and $j$, each accessible to the other, are said to *communicate*. If two states $i$ and $j$ do not communicate, then either

$$\begin{aligned} &f_{ij}^{n} = 0 \quad \forall n \geq 0, \\ \text{or}& \\ &f_{ji}^{n} = 0 \quad \forall n \geq 0, \end{aligned} \tag{1.88}$$

or both relations are true. This concept of communication is an equivalence relation.

We can now partition the totality of states into equivalence classes. The states in an equivalence class are those which communicate with each other. We say that the Markov chain is *irreducible* if the equivalence relation induces only one class, i.e., a process is irreducible if any state communicates with any other.

### 1.4.4 Existence of stationary distributions for stochastic matrices

The basic theorem that demonstrates the existence of a stationary probability distribution is the *Perron-Frobenius* theorem [48]. The basic results of this theorem are:

Given an irreducible matrix $\mathbb{A}$, if every matrix component $a_{ij}$ is nonnegative, we write $\mathbb{A} \geq 0$. Then

- i) $\mathbb{A}$ has a real positive eigenvalue $\lambda_1$ with the following properties;
- ii) corresponding to $\lambda_1$ there is an eigenvector $\mathbf{x}$ all of whose elements may be taken as positive, i.e., there exists a vector $\mathbf{x} > 0$ such that

$$\mathbb{A}\mathbf{x} = \lambda_1 \mathbf{x}; \tag{1.89}$$

- iii) if $\alpha$ is any other eigenvalue of $\mathbb{A}$ then

$$|\alpha| \leq \lambda_1; \tag{1.90}$$

- iv) $\lambda_1$ increases when any element of $\mathbb{A}$ increases;
- v) $\lambda_1$ is a simple root of the determinantal equation

$$|\lambda\mathbb{I} - \mathbb{A}| = 0. \tag{1.91}$$

- vi)

$$\lambda_1 \leq \max_j \left( \sum_k a_{jk} \right) \ , \quad \lambda_1 \leq \max_k \left( \sum_j a_{jk} \right) . \tag{1.92}$$

If $\lambda_1$ itself is the only eigenvalue of modulus $\lambda_1$ then matrix $\mathbb{A}$ is said to be *primitive*. Besides, note that point (vi) gives an upper bound to $\lambda_1$ as the largest row sum or largest column sum of matrix $\mathbb{A}$. Thus, if we are dealing with a stochastic matrix such as $\mathbb{T}$, we know from the previous theorem that has an eigenvalue $1$ since $\mathbb{T} \cdot \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a column vector of $1$'s. Besides, it follows from (1.90), (1.92) that no eigenvalue can exceed $1$ in modulus.

An important aspect we can extract is that the nature of a finite chain is determined by the properties of the eigenvalues of $\mathbb{T}$ which have unit modulus. Another point is that the limiting values of transition probabilities are approached exponentially fast, the rate of approach being determined in general by the eigenvalue of largest modulus less than unity.

If the finite Markov chain is ergodic then its transition matrix $\mathbb{T}$ is irreducible and primitive, with a simple eigenvalue $1$ which exceeds all other eigenvalues in modulus (conversely, if $\mathbb{T}$ is primitive and irreducible then the system is ergodic). According to the theorem of Perron and Frobenius described above, there is a positive column eigenvector $\pi = \{\pi_j\}$ satisfying $\mathbb{T}\pi = \pi$ and we can normalize this vector so that $\sum \pi_j = 1$. Besides, the system is ergodic and

$$\lim_{n\to\infty} p_{kk}^n = \frac{1}{\mu_k} = \pi_k > 0, \tag{1.93}$$

the limit approached exponentially fast and uniformly for all $j$ and $k$. Conversely if the system is ergodic then $\mathbb{T}$ is primitive and irreducible. Thus, given the necessary conditions for the stochastic matrix, there exists a limiting value for the occupancy probabilities, being that of the stationary probability distribution $\pi$.

## 1.5 Information Theory

The information theory was introduced in the seminal paper by Shannon [49] in $1948$. Basically this work studies certain problems of the transmission of messages through channels involving communication systems. These communication systems can be divided in three main categories: discrete, continuous and mixed. By a discrete system it is meant one where the signal and the message are a sequence of discrete symbols – for example, the telegraphy. A continuous system is one where the message and the signal are both continuous, e.g., the television. The last one is the mixed system, where both discrete and continuous variables appear, for instance the pulse code-modulation (PCM) for the transmission of speech.

The case of our interest here deals with discrete systems. Basically we can distinguish three main parts: the information source, the communication channel (through where the signal is transmitted) and the receiver. Generally, a discrete channel will mean a system where a sequence of choices from a finite set of elementary symbols $\alpha_1, \ldots, \alpha_n$ can be transmitted from one point to another.

### 1.5.1 Discrete and ergodic sources

We can think of the information source as generating the message, symbol by symbol. The source will choose successively symbols according to certain probabilities depending, in general, on preceding choices as well as the particular symbols in question.

We may define an *ergodic source* as a source that generates strings of symbols $\alpha_1, \alpha_2, \ldots$ with the same statistical properties. Thus the symbols frequencies obtained from particular sequences will, as the length of the message increase, approach definite limits independent of the particular sequence.

In some cases a message $L$ that is not homogeneous statistically speaking, can be considered as composed of pieces of messages coming from various pure ergodic sources $L_1, L_2, L_3, \ldots$ that is

$$L = \Pi_1\, L_1 + \Pi_2\, L_2 + \Pi_3\, L_3 + \ldots \tag{1.94}$$

where $\Pi_i$ corresponds to the probability of the component source $L_i$.

#### *1.5.1.a Shannon Entropy*

For a single source we may define the entropy as

$$H = -\sum_i p^j \log(p^j) \tag{1.95}$$

where $p^j$ denotes the probability of emitting a given symbol $\alpha_j$. This quantity was introduced by Shannon for measuring, in some sense, how much lack of information is produced by such a source. It can also be regarded as a measure of how much “choice” is involved in the selection of the symbol emitted by the source or of the uncertainty of the outcome.

The information entropy represents the average information content of a message. Some of its most interesting properties are

1. $H = 0$ if and only if all the $p^i$ but one are zero, this one having the value unity. Thus only when we are certain of the outcome does $H$ vanish. Otherwise $H$ is positive.
2. For a given $n$, $H$ is a maximum and equal to $\log n$ when all the $p^i$ are equal, i.e.: $\frac{1}{n}$.
3. Any change toward equalization of the probabilities $p^1, p^2, \ldots, p^n$ increases $H$.

The Shannon entropy gives the minimum transfer rate – bit rate – at which a message can be transmitted without losing any information content. For instance, we can consider an information source that emits only two symbols, either $1$ or $0$ with probability $p$ and $q = 1 - p$ respectively. The corresponding expression for the entropy of the source reads

$$H = -p \log p - q \log q = -p \log p - (1-p) \log(1-p) \tag{1.96}$$

In Fig. 1.1 we plot the entropy as a function of the probability $p$ of emitting the symbol $1$. It can be appreciated how the entropy of the message generated by the source acquires its maximum when $p = \frac{1}{2}$, corresponding to the value where both symbols have the same probability of being emitted, and therefore the uncertainty of the resulting message is maximum.

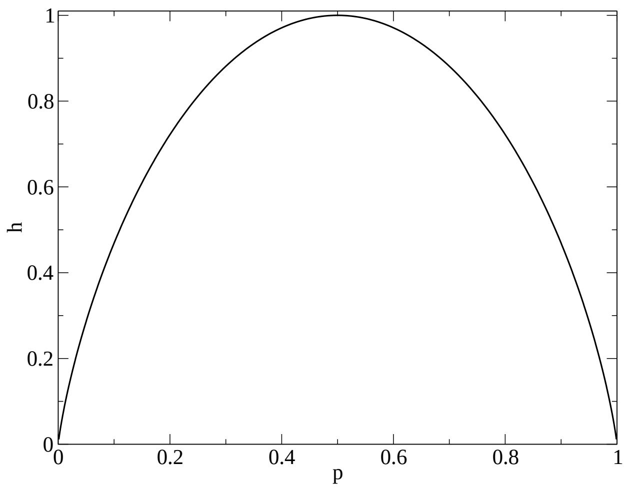


**Figure 1.1.** Plot of the variation of the entropy of the source when varying the probability $p$ of emitting symbol $1$.

If we now consider a source $L$, composed itself of a mixture of different sources $L_i$ with probability $\Pi_i$, the resulting entropy of the system will depend on the entropy of each individual source in the following way

$$H = \sum_i \Pi_i H_i = -\sum_{i,j} \Pi_i p_i^j \log p_i^j \tag{1.97}$$

where $p_i^j$ denotes the probability of emitting a symbol $\alpha_j$ by the source $L_i$.

#### *1.5.1.b Entropy of a message*

Given a message composed of a set of symbols $\alpha_1, \alpha_2, \ldots$, successive approximations of the actual entropy of the message can be obtained. As a first step, it can be considered that all the symbols have been emitted by the source with a fixed and independent probability. Therefore, we can measure the frequencies of all the symbols of the alphabet present in the message, estimating from them their probabilities using Eq. (1.95).

Next thing to consider are the so–called *block entropies*. We must calculate the probabilities of words constructed with symbols from the alphabet $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, and thereafter obtain their corresponding block entropies

$$H_n = -\sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1,\ldots,\alpha_n) \log[p(\alpha_1,\ldots,\alpha_n)]. \tag{1.98}$$

This quantity measures the average amount of information contained in a word of length $n$. From Eq. (1.98) we can then evaluate the differential entropy

$$\begin{aligned} h_n &= H_n - H_{n-1} \\ &= -\sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1,\ldots,\alpha_n) \log[p(\alpha_n|\alpha_1,\ldots,\alpha_{n-1})], \end{aligned} \tag{1.99}$$

that gives the new information of the *n*-th symbol if the preceding $(n-1)$ symbols are known; $p(\alpha_n|\alpha_1,\ldots,\alpha_{n-1})$ is the conditional probability for $\alpha_n$ being conditioned on the previous symbols $\alpha_1,\ldots,\alpha_{n-1}$. The Shannon entropy is then

$$h = \lim_{n\to\infty} h_n \tag{1.100}$$

The latter expression gives the average amount of information per symbol if all correlations are taken into account, and the limit approaches monotonically the actual value of $h$ from above, i.e., all the $h_n$ are upper bounds on $h$.

For a numerical estimation of Eq. (1.98) we must count the number of times $n$ that the word $\alpha_1,\ldots,\alpha_n$ is contained in the message, and then obtain its probability with $\frac{n}{N}$, where $N$ is the total length of the message.

The actual problem of evaluating the Shannon entropy in this way is that the number of possible words increases exponentially as the length of the word $n$ increases. In order to obtain good statistical results when calculating the word probabilities we must have a sufficiently long message when evaluating the probabilities of large words[3], which in fact is a considerable inconvenient.

There exist other ways of evaluating the entropy of a message. An interesting algorithm developed by A. Lempel and J. Ziv [50] permits the calculation of the entropy of a message, and it will be explained in the next section.

### 1.5.2 Lempel and Ziv algorithm

In $1977$, Abraham Lempel and Jakob Ziv created the lossless[4] compressor algorithm LZ77. This algorithm is present in programs such as `gzip`,`arj`, etc. It was later modified by Terry Welch in $1978$ becoming the LZW algorithm, and this is the algorithm commonly found today.

It was originally designed to obtain the algorithmic complexity of a binary string[5] [51]. Basically it is a dictionary based or substitutional encoding/decoding algorithm, creating a dictionary during the process of encoding and decoding of a certain message.

For a useful example of how the algorithm[6] works, we will encode/decode the following binary string $10010110100111011100101$, of length $n = 23$.

#### *1.5.2.a Encoding process*

First, we will partition the chain into words $B_1, B_2, ..$ of variable block length –Lempel & Ziv parsing–

$$10010110100111011100101 \tag{1.101}$$

So we obtain the following words: $B_1 = 1$, $B_2 = 0$, $B_3 = 01$, $B_4 = 011$, $B_5 = 010$, $B_6 = 0111$, $B_7 = 01110$, $B_8 = 0101$.

This words are then coded as *(prefix+newbit) =(pointer to the last occurrence, newbit)*: $(01) = (0 + 1) = (2, 1)$, $(011) = (01 + 1) = (3, 1)$, $(010) = (01 + 0) = (3, 0)$, $(0111) = (011 + 1) = (4, 1)$, $(01110) = (0111 + 0) = (6, 0)$, $(0101) = (010 + 1) = (5, 1)$. We have then the following pairs

$$(0,1) \quad (0,0) \quad (2,1) \quad (3,1) \quad (3,0) \quad (4,1) \quad (6,0) \quad (5,1) \tag{1.102}$$

Once the pairs for each $B_j$ are obtained, we replace each pair *(i,s)* by the integer $I_j = 2i + s$.

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[3] the necessary length of the message also increments exponentially with $n$

[4] it assures that the original information can be exactly reproduced from the compressed data

[5] *Algorithmic complexity* of a binary string is the length in bits of the shortest computer program able to reproduce the string and to stop afterward

[6] we will make use of the LZ78 algorithm, which is simpler than its original LZ77

$$\begin{array}{ll} (0,1) \to I_1 = 20+1 = 1 & (3,0) \to I_5 = 23+0 = 6 \\ (0,0) \to I_2 = 20+0 = 0 & (4,1) \to I_6 = 24+1 = 9 \\ (2,1) \to I_3 = 22+1 = 5 & (6,0) \to I_7 = 26+0 = 12 \\ (3,1) \to I_4 = 23+1 = 7 & (5,1) \to I_8 = 25+1 = 11 \end{array} \tag{1.103}$$

Each integer $I_j$ is then expanded to base two, and the binary expansions are padded with zeroes on the left so that the total length of bits is $\lceil \log_2(2j) \rceil$, where the brackets $\lceil \, \rceil$ denote the upper integer value of $log_2(2j)$. We obtain in this way the strings $W_j$.

| $j$ | $I_j$ | **Binary string** | $\lceil \log_2(2j) \rceil$ | $W_j$ | **Binary string** |
|---|---|---|---|---|---|
| 1 | 1 | 1 | $\lceil \log_2(2) \rceil = 1$ | $W_1$ | 1 |
| 2 | 0 | 0 | $\lceil \log_2(4) \rceil = 2$ | $W_2$ | 00 |
| 3 | 5 | 101 | $\lceil \log_2(6) \rceil = 3$ | $W_3$ | 101 |
| 4 | 7 | 111 | $\lceil \log_2(8) \rceil = 3$ | $W_4$ | 111 |
| 5 | 6 | 110 | $\lceil \log_2(10) \rceil = 4$ | $W_5$ | 0110 |
| 6 | 9 | 1001 | $\lceil \log_2(12) \rceil = 4$ | $W_6$ | 1001 |
| 7 | 12 | 1100 | $\lceil \log_2(14) \rceil = 4$ | $W_7$ | 1100 |
| 8 | 11 | 1011 | $\lceil \log_2(16) \rceil = 4$ | $W_8$ | 1011 |

Finally we just need to concatenate the binary words $W_j$ to obtain the encoded string: 1001011110110100111001011. Clearly, the length of the encoded string is not much shorter than the original in this case, but it must be kept in mind that the algorithm becomes *optimal* as the length of the string increases[7]

#### *1.5.2.b Decoding process*

The decoding process is much simpler than the encoding. We just need to know the size alphabet of the source that created the string. From the previous section we obtained the encoded string 1001011110110100111001011 with an alphabet equal to 2.

The first thing to do is to divide the string in blocks of size $\lceil \log_2(2j) \rceil$ : $1 \cdot 00 \cdot 101 \cdot 111 \cdot 0110 \cdot 1001 \cdot 1100 \cdot 1011$; then convert these blocks into integer form :$1, 0, 5, 7, 6, 9, 12, 11$ ; we divide by the size alphabet, 2 in this case, and we keep the quotient $q$ and remainder $r$, $(q, r)$ : $(0, 1)$, $(0, 0)$, $(2, 1)$, $(3, 1)$, $(3, 0)$, $(4, 1)$, $(6, 0)$, $(5, 1)$.

Finally we convert these pairs into words using the same formalism than in the encoding process, and we join them to obtain the original binary string 1001011010011101110010 1.

#### *1.5.2.c Properties of the LZ algorithm*

An important property of the LZ algorithm is that it relates the compression factor to the entropy of the compressed string.

[7]Because the length of the words $B_j$ that will be substituted increases linearly with the binary string, whereas the length of the words $W_j$ increases logarithmically.

The *compression factor* (CF) of strings is the ratio between the compression length $c$ and the original length $n$

$$CF = \frac{c}{n}. \tag{1.104}$$

The *optimality ratio* $\gamma(n)$ is defined as the ratio between the compression factor and the entropy per character $h$ of the source

$$\gamma(n) = \frac{CF}{h}, \tag{1.105}$$

it is said that the compression is *asymptotically optimal* if $\gamma(n) \to 1$ as $n \to \infty$.

Lempel and Ziv showed that their dictionary–based algorithms $LZ77$, $LZ78$ give asymptotically optimal compression for strings generated by an ergodic stationary process, that is, as the length of the file to compress $n \to \infty$ the ratio of the length of the compressed file with $n$ tends to the entropy per character $h$.

This algorithm together with the previous definitions explained above will be used in Sec. 5 for establishing a relation between Parrondo's games and information theory.

# Chapter 2

# The Brownian ratchet and Parrondo's games

In some physical and biological systems, combining processes may lead to counter-intuitive dynamics. For example, in control theory, the combination of two unstable systems can cause them to become stable [52]. In the theory of granular flow, drift can occur in a counter-intuitive direction [53, 54]. Also the switching between two transient diffusion processes in random media can form a positive recurrent process [55]. Other interesting phenomena where physical processes drift in a counter-intuitive direction can be found (see for example [56–60]). One part of the present chapter will be devoted to another example where a counter–intuitive result takes place: the flashing ratchet. This is characterized by directed motion obtained from the random or periodic alternation of two relaxation potentials acting on a Brownian particle, none of each producing any net flux.

Parrondo's paradox [5–7] shows that the combination of two losing games, can give rise to a winning game. This paradox is a qualitative translation of the physical model of the flashing ratchet into game-theoretic terms. These games were first devised in 1996 by the Spanish physicist Juan M.R. Parrondo, who presented them in unpublished form in Torino, Italy [3], as a pedagogical illustration of the flashing ratchet.

The first part of this chapter, Sec. 2.1, will be devoted to the explanation of the Brownian ratchet, also including the original model of the Smoluchowski–Feynman ratchet, that brought the idea of the ratchet effect; and finally we will focus on a detailed explanation of the flashing ratchet model.

Afterwards, in Sec. 2.2 we present the original Parrondo games as they were designed and explain thoroughly the basics of the so-called *Parrondo paradox*, unraveling the mechanism behind it. Furthermore, some other versions of Parrondo games that appeared later on will be also shown at the end of this section.

## 2.1 Smoluchowski–Feynman ratchet

Is it possible to obtain useful work out of unbiased random fluctuations? In the case of macroscopic devices we can find many ways of accomplishing this task, for example a wind–mill, the self–winding wristwratch, etc. But when dealing with the microscopic world, this case becomes more subtle. A clear example of this problem was illustrated in the conference talk by Smoluchowski in Münster 1912 (and published as a proceedings–article in [61]) and later popularized and extended in Feynman's Lectures on Physics [62].

### 2.1.1 Ratchet and pawl

The ratchet and pawl model consists on an axis with a paddle located at one end, and a circular saw with a ratchet–like shape on the other end, see Fig. 2.1 for details. This device is surrounded by a thermal bath at equilibrium at temperature $T$. If left alone, the system would perform a rotatory, random, Brownian motion due to the collisions of the gas molecules with the paddles.

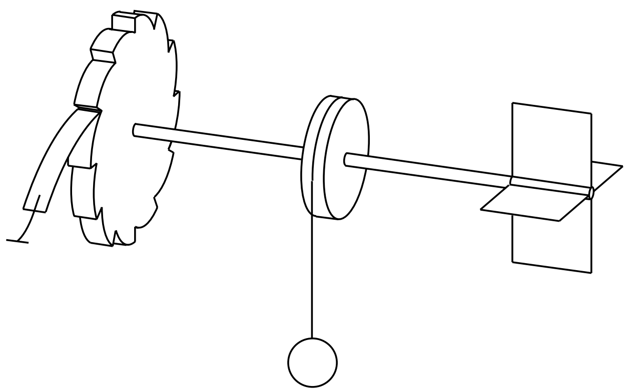


**Figure 2.1.** Plot of the ratchet and pawl device.

We can modify this picture by introducing a pawl in order to rectify this random fluctuations. Hence in this way rotations would be favored in one precise direction, allowing the saw–teeth to rotate clockwise – as depicted in Fig. 2.1 –, whereas it would block the saw–teeth to rotate in the counter–clock direction. So intuitively it seems that this gadget would perform a net rotation clockwise, and if a weight is added to the axis it could even perform some work lifting the weight.

Based on the previous reasoning we could conclude that the device constructed this way would constitute a *perpetuum mobile* of the second kind, therefore violating the second law of thermodynamics. However, this naive expectation is wrong. In spite of the asymmetry of the device, no preferred motion is possible. The reason is the following: due to the microscopic size of the machine, not only the paddles are subjected to the fluctuations due to the collisions with the gas particles, but also the pawl is exposed to them. These collisions of the particles with the pawl would, occasionally, lift the pawl. Then the ratchet could rotate counter–clockwise as it would not have any opposing force.

As a result the ratchet and pawl device would have no preferred direction of rotation. This Smoluchowski–Feynman's ratchet and pawl device was introduced as a pedagogic example of the second law of thermodynamics.

We can modify the previous picture by considering that the gas surrounding the paddles and the gas surrounding the ratchet have different temperatures. In this case an equilibrium situation no longer exists. This second model was introduced by Feynman [62], and later revised by Parrondo [63].

A simplified stochastic model known as Brownian ratchet will be presented in the next section, capturing the essential features of the Smoluchowski–Feynman's ratchet and pawl device.

### 2.1.2 Brownian ratchet

We will consider the motion of a Brownian particle of mass $m$ under the effect of a potential $V(x,t)$ that can be time–dependent, a friction force $-\eta\dot{x}(t)$, a force $F(t)$ exerted by an external agent and a stochastic force $\sqrt{D(x,t)}\xi(t)$, where $D(x,t) = 2\eta kT(x,t)$ is the noise strength or noise intensity, proportional to the temperature. Newton's equation of motion for this system can be expressed as

$$m\ddot{x}(t) + V'(x,t) = -\eta\dot{x}(t) + F(t) + \sqrt{D(x,t)}\xi(t). \tag{2.1}$$

The terms on the left hand side account for the deterministic, conservative part, whereas the terms on the right hand side account for the dissipative terms due to the interaction of the Brownian particle with its environment and the external agent. Usually the time–dependent external force $F(t)$ is split in two terms, a constant term $F$ and a time–dependent term $y(t)$, and so it can be written as $F(t) = F + y(t)$.

The potential $V(x,t)$ used in Eq. (2.1) must fulfill the following conditions

- *Periodicity*. It must be periodic with period L, that is, $V(x,t) = V(x+L,t)$ for all $x$ and $t$.
- *Asymmetry*. This asymmetry can be established in many ways, the simplest consisting on spatial asymmetry, that occurs when for any value of $x$ there exists no $\Delta x$ such that $V(-x,t) = V(x+\Delta x,t)$, in some sense this condition accounts for some kind of spatial anisotropy. A typical example of an asymmetric potential is

  $$V(x,t) = V_0\left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{4\pi x}{L}\right)\right]\cdot[1+W(t)], \tag{2.2}$$

  where the function $W(t)$ represents the time dependence of the potential, if there is any.

The stochastic force or thermal noise $\xi(t)$ generally is considered to be *Gaussian white noise* of zero mean $<\xi(t)>= 0$ and correlations

$$< \xi(t)\xi(s) >= \delta(t-s) \tag{2.3}$$

For the systems we will study, the inertia term $m\ddot{x}(t)$ is negligible, and so Eq. (2.1) can be written as

$$\eta\dot{x}(t) = -V'(x(t),t) + F + y(t) + \sqrt{D(x,t)}\xi(t). \tag{2.4}$$

The latter equation can be considered as a generalized equation describing the dynamics of an overdamped Brownian particle.

#### 2.1.2.a Reduced probability variables

As our interest is focused mainly on transport in periodic systems, we can introduce the *reduced probability density* and *reduced probability current* as

$$\hat{P}(x,t) := \sum_{n=-\infty}^{\infty} P(x+nL,t), \tag{2.5}$$

$$\hat{J}(x,t) := \sum_{n=-\infty}^{\infty} J(x+nL,t). \tag{2.6}$$

And from Eqs. (1.29,1.61) we get

$$\hat{P}(x+L,t) = \hat{P}(x,t), \tag{2.7}$$

$$\int_0^L dx\hat{P}(x,t) = 1, \tag{2.8}$$

$$< \dot{x} > = \int_0^L dx\hat{J}(x,t) \tag{2.9}$$

As $P(x,t)$ is solution of the Fokker–Planck equation (1.39), it follows from the periodic condition introduced above, $V(x,t) = V(x+L,t)$, that $P(x+nL,t)$ is also solution for any integer value $n$. Introducing expressions (2.5) and (2.6) into the Fokker–Planck equation (1.39), it can be rewritten as a continuity equation for the reduced probabilities

$$\frac{\partial\hat{P}(x,t)}{\partial t} + \frac{\partial\hat{J}(x,t)}{\partial x} = 0, \tag{2.10}$$

where

$$\hat{J}(x,t) = F(x,t)\,\hat{P}(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[D(x,t)\,\hat{P}(x,t)]. \tag{2.11}$$

Therefore, in order to obtain the particle current is sufficient to solve the Fokker–Planck equation (1.39) with periodic boundary conditions, together with the initial conditions. Besides, operating with $\int_{x_0}^{x_0+L} dx\, x \ldots$ on both sides of Eq. (2.10) we obtain

$$< \dot{x} >= \frac{d}{dt}\left[\int_{x_0}^{x_0+L} dx\, x\hat{P}(x,t)\right] + L\hat{J}(x_0,t), \tag{2.12}$$

where $x_0$ denotes the initial position of the particle. Essentially, we distinguish two contributions to the particle current: the first term on the right hand side of Eq. (2.12) accounts for the motion of the center of mass, and the second term is $L$ times the reduced probability current $\hat{J}(x_0,t)$ measured at the reference point $x_0$. If the reduced dynamics reaches a steady state, characterized by $\frac{d\hat{P}(x,t)}{dt} = 0$, then the reduced probability current $\hat{J}(x_0,t) = \hat{J}^{st}$ becomes independent of $x_0$ and $t$, and the particle current becomes

$$< \dot{x} >= L\hat{J}^{st}. \tag{2.13}$$

The particle current can also be calculated through the time-averaged velocity of a single realization $x(t)$ of the stochastic process described by Eq. (2.1), i.e.

$$< \dot{x} >= \lim_{t\to\infty} \frac{x(t)}{t}, \tag{2.14}$$

independent of the initial condition $x(0)$.

#### *2.1.2.b Ratchet effect*

The so–called ratchet effect takes place when a given set of conditions are accomplished.

- First, we must have a spatially periodic system.

- Second, there must be some asymmetry in the system, for example spatial asymmetry.

- Last but not least, the system must be out of equilibrium.

Depending on the way these conditions are accomplished, we may distinguish different types of ratchets.

### 2.1.3 Classes of ratchets

There are two main groups of ratchets that can be derived from Eq. (2.4). The first group considers those systems where the term $y(t) = 0$, these are the *pulsating ratchets*; the second group considers those where there is no time dependence in the potential $V(x,t)$, i.e. $W(t) = 0$, and they are known as *tilting ratchets*.

#### *2.1.3.a Pulsating ratchets*

Within this group, we can also distinguish the following types of ratchets

**Fluctuating potential ratchets** They are obtained when the time dependence of the potential $W(t)$ is additive, that is $V(x,t) = V(x)[1 + W(t)]$. This group contains as a special case the *on–off* ratchet, also known as *flashing ratchet*, consisting on $W(t)$ having only two possible values: $0$ (ON state) and $-1$ (OFF state).

**Traveling potential ratchets** They have potentials of the form $V(x,t) = V(x - W(t))$.

#### *2.1.3.b Tilting ratchets*

This group is characterized by $W(t) = 0$, and so the potential is time–independent $V(x,t) = V(x)$. Within this group we will distinguish three types of ratchets depending on the time dependence of $y(t)$ in Eq. (2.4)

**Fluctuating force ratchets** They are obtained when $y(t)$ is a stationary stochastic process. It can be another Gaussian white noise, hence we are dealing with an effective Smoluchowski-Feynman ratchet, or it can be a Gaussian colored noise. The former case needs a correlated (non-white), Gaussian or non-Gaussian noise (colored noise) in order to obtain directed transport. The latter case is represented by a Ornstein–Uhlenbeck noise with an exponentially decaying correlation.

**Rocking ratchet** It is obtained when $y(t)$ is periodic.

**Asymmetrically tilting ratchet** We explained before that one essential ingredient for the ratchet effect was the existence of an asymmetry in the system. If our potential $V(x)$ is symmetric the source of asymmetry can be introduced through the term $y(t)$, imposing it to be non–symmetric.

From all these different kinds of ratchets, we will now focus on the *flashing ratchet* model and analyze it a little closer.

### 2.1.4 The flashing ratchet

This system is characterized by a Brownian particle subjected to a potential that is switched on and off either periodically or stochastically – depending on the time dependence of the function $W(t)$. This scheme was introduced by Ajdari and Prost [56]. The model can be described through the equation

$$\eta \dot{x}(t) = -V'(x(t))\,[1 + W(t)] + \sqrt{D(x,t)}\xi(t), \tag{2.15}$$

where $V(x)$ is a spatially periodic and asymmetric potential, and usually a potential such as the one in Eq. (2.2) is used – in Fig. 2.2 we can see a plot of the potential for the parameters $L = 3$ and $V_0 = 1$. The function $W(t)$ is restricted to two values $0$, $-1$, switching on and off the potential, and $D(x,t) = 2\eta kT(x,t)$ is the noise strength.

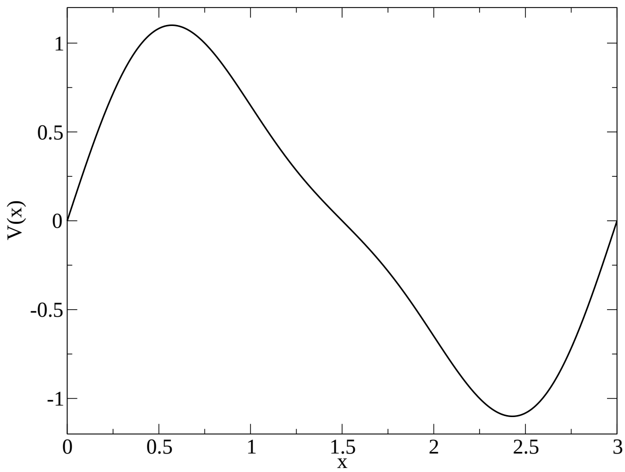



**Figure 2.2.** Plot of the asymmetric potential (2.2) with the parameters $L = 3$ and $V_0 = 1$.

The *ratchet mechanism* (or *ratchet effect*) can be explained as follows. Imagine a landscape with a few Brownian particles moving freely. At a given instant, a ratchet–like potential is switched on: $W(t) = 1$, and the particles (assuming the thermal energy $kT$ to be much smaller than the potential amplitude) are eventually confined to one of the potential wells located at $x_0$, see Fig. 2.3. When the potential is switched off: $W(t) = -1$, the particles are subjected only to the thermal noise $\xi(t)$ and start to diffuse.

If we let the particles diffuse for a large enough time interval, a small fraction of them will reach the vicinity of the next potential well [1] at $x_0 + L$. Repeating this cycle many times, a net current of particles is obtained $\langle \dot{x} \rangle > 0$. In Fig. 2.4 – left panel – we see the plot of the net current vs the natural logarithm of the flip rate[2] $\gamma$ for a single Brownian particle. It can be clearly identified the existence of an optimal switching rate that produces the maximum current.

[1] due to the asymmetry in the potential of Fig. 2.3, is more likely that the particles will reach the potential well located on the right than the one on the left, as the distance is shorter in the former case.

[2] The flip rate $\gamma$ accounts for the probability of switching the potential *on* or *off* per time unit.

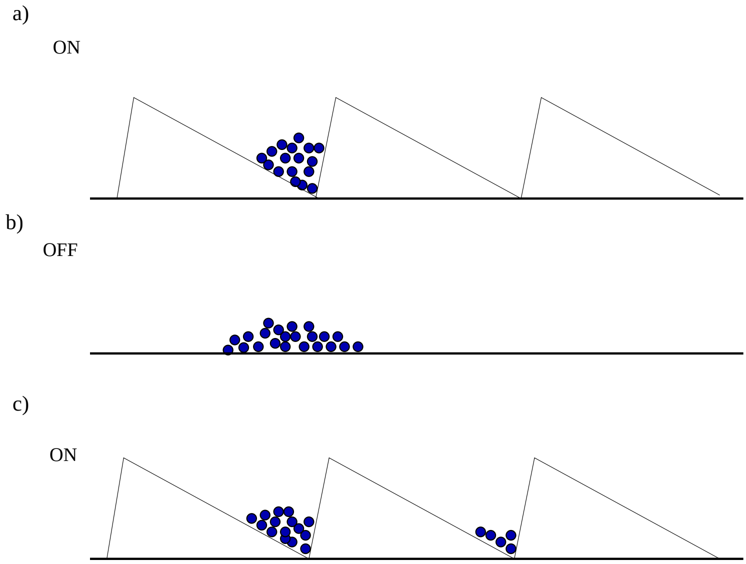



**Figure 2.3.** Three different stages of the *on-off* cycle for the case of the flashing ratchet. In case a) the potential is **on** and the particles get trapped in a potential well; in stage b) the potential is **off** and the particles spread due to diffusion; finally, in stage c) some particles have diffused up to the vicinity of the next potential well, and so when the potential is **on** again, there are a certain number of particles located in the next potential well. The flux of particles due to the asymmetry in the potential in this case is to the right.

We can modify this picture introducing an external force $F$ acting against the particle. Even with this opposing force applied on the particle, the *ratchet effect* is still present for sufficiently small values of $F$. We see in Fig. 2.4 – right panel – how the current is positive and different from zero up to a value of the applied force $F = F_0$, being $F_0$ the so–called *stopping force*. It is worth noting that for this case, the particle is doing work against the external force applied.

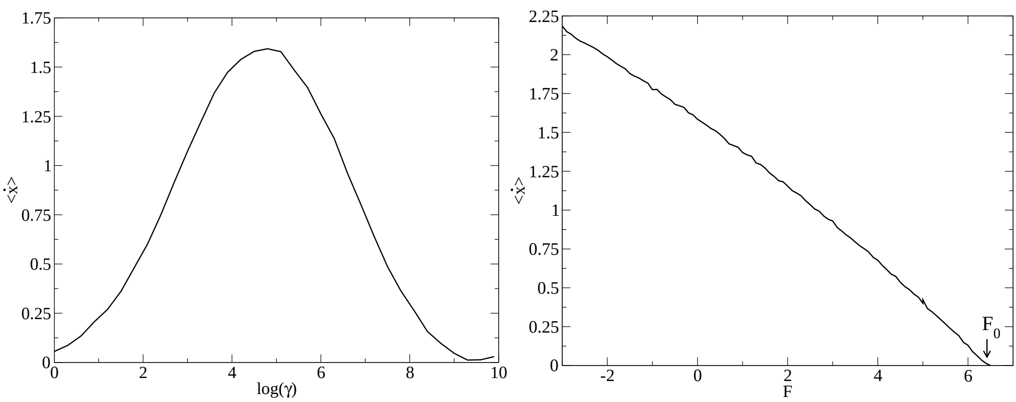



**Figure 2.4.** Left panel: Plot of the average particle current versus the logarithm of the flip rate. Right panel: Plot of the average particle current versus the applied external force $F$.

### 2.1.5 A *temperature* ratchet

A variation of the flashing ratchet, which also may lead to the same result, consists on the *temperature* ratchet. In this case the Brownian particles are exposed to an alternation between a *hot* and a *cold* temperature (for instance, we could expose the Brownian particles to temperatures such that $\frac{V}{kT_{hot}} \ll 1$, and $\frac{V}{kT_{cold}} \gg 1$), and simultaneously a ratchet–like potential as the one depicted in Fig. 2.2. Thus, when the particles are exposed to the *cold* temperature, the particles are pinned at a potential minimum due to the relatively high amplitude of the potential compared to the low temperature. In a second stage, when temperature now is increased to $T_{hot}$, the particles effectively do not *feel* the potential and begin to diffuse. Afterwards, when the temperature is cold again, there will be a certain number of particles that will have diffused up to the vicinity of the potential well on the right, and on average that number will be greater than those that got to the vicinity of the potential well on the left. On average, as in the case of the flashing ratchet model, there will be a net flux of particles to the right (as long as the asymmetry in the potential is the one depicted in Fig. 2.3).

## 2.2 A discrete–time flashing ratchet : Parrondo's games

### 2.2.1 Description of the games

Parrondo's two original games are as follows. Game A is a simple coin tossing game, where a player increases (decreases) his capital in one unit if heads (tails) show up. The probability of winning is denoted by $p$ and the probability of losing is $q = 1 - p$.

Game B is a capital dependent game, where the probability of winning depends upon the actual capital of the player, modulo a given integer $M$. Therefore if the capital is $i$ the probability of winning $p_i$ is taken from the set $\{p_0, p_1, \ldots, p_{M-1}\}$ as $p_i = p_{i \bmod M}$. In the original version of game B, the number $M$ is set equal to three and the probability of winning can take only two values, $p_1,\ p_2$, i.e. game B uses two different coins according to whether the capital of the player is multiple of three or not. The two games are represented diagrammatically in Fig. 2.5 using branches to represent wins and losses with probabilities given by the terms in brackets.

The numerical values corresponding to the original Parrondo's games [5] are:

$$\begin{cases} p = & \frac{1}{2} - \epsilon, \\ p_1 = & \frac{1}{10} - \epsilon, \\ p_2 = & \frac{3}{4} - \epsilon, \end{cases} \tag{2.16}$$

where $\epsilon$ is a small biasing parameter introduced to control the three probabilities. For a value of $\epsilon$ equal to zero, both games are fair games, whereas if $\epsilon$ is small and positive both games are losing. In both cases, the combined game results in a winning game.

Intuitively, we could think of a potential representing games A and B – for the simplest case of $\epsilon = 0$ – through the following reasoning: the winning and losing proba-

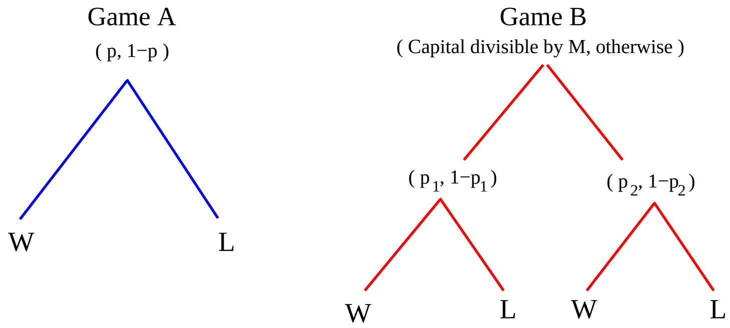



**Figure 2.5**: Probability trees for games A and B.

bilities for game A are independent of the site and equal to $\frac{1}{2}$. Therefore it would be equally likely a forward or a backward transition. Then the barriers of the potential that one would find would be of equal height, as depicted in Fig. 2.6a.

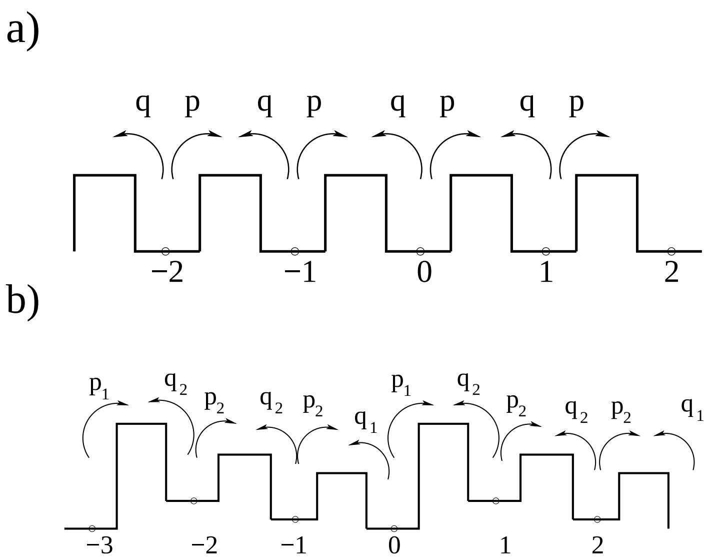



**Figure 2.6.** a) Schematic potential related to game A. b) Schematic potential related to game B.

For the case of game B, we must take into account the dependence of the winning probabilities with the current capital of the player. When the capital is multiple of three the winning probability is very small, i.e. $p_1 = \frac{1}{10}$, this translates into a high potential barrier between this site and the one located on the right. However, for the sites that correspond to the capital of the player not being multiple of three the winning probability is rather high, $p_2 = \frac{3}{4}$, and so the potential barriers must be placed in a way that it is favored a forward transition than a backward transition. One possible way of depicting the potential is found in Fig. 2.6b.

In Fig. 2.7 we can see a plot of the average gain for a player that alternates between games A and B, either periodically or stochastically. For both kind of alternations, it can be seen that the resulting game is a winning game. When the player alternates periodically between games A and B, it follows a fixed sequence of plays for game A and B. For example, the sequence $[3, 2]$ implies that the player will play game A three times in a row, followed by game B two times. The case of random mixing between games is obtained as follows: the player will decide on each time step if he plays game A or B with probability $\gamma$ and $1 - \gamma$ respectively[3]. In Fig. 2.7 we have plotted the random case for a value of $\gamma = \frac{1}{2}$.

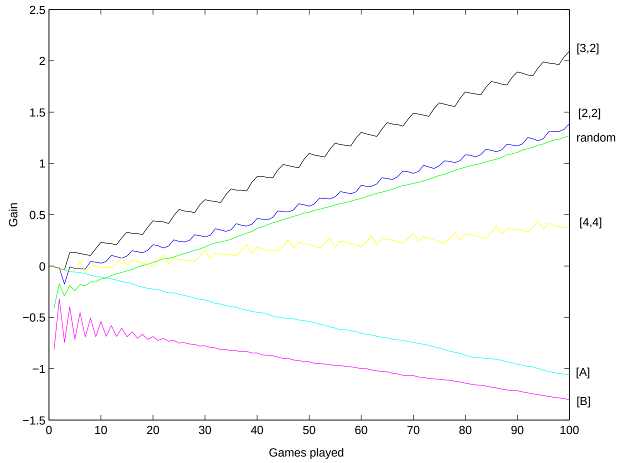



**Figure 2.7.** Plot of the average gain over 100 plays of either game A or B alone – both of them losing games –, although any combination of them, either periodic or stochastic results in a winning game. The notation $[a, b]$ indicates, for the periodic case, that we play $a$ times game A, followed by $b$ times game B. For the random case games A and B are alternated with a probability $\gamma = \frac{1}{2}$.

## 2.2.2 Theoretical analysis of the games

One way of analyzing these games is through discrete–time Markov chains [64]. Each value of capital is represented by a state, and the transition probabilities between these states are determined by the rules of the games. In this section we will analyze the games A, B and the randomized game AB with this technique in order to obtain the stationary probability distributions.

### *2.2.2.a Analysis of game B*

Either game A and B can be represented through discrete–time Markov chains. When playing game B alone, we could represent the evolution of the capital with an infinite Markov chain as the one depicted in Fig. 2.8. However, this Markov chain can be simplified inasmuch as there exists a periodicity in the system (we can see how the transition probabilities repeat each $M = 3$ states). Thus, game B can be reduced to a Markov chain with three states $0$, $1$ and $2$ –see Fig. 2.9 for details– representing the value of the capital modulo three. The transition probabilities between states will be given by the winning ($p_i$) and losing ($q_i$) probabilities for each state.

[3]From now on the randomized game will be referred to as game AB.

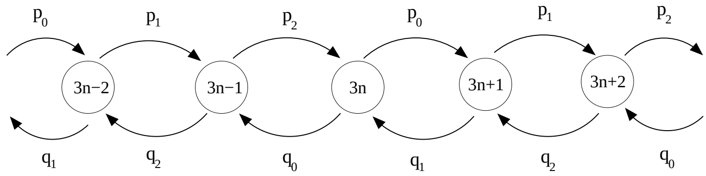



**Figure 2.8.** Discrete–time Markov chain corresponding to game B.

We are interested in obtaining the probabilities of finding the capital of the player in each of these states. We can write down a set of equations that describe the evolution with the number $n$ of games played – which in some sense would be equivalent to the time– of the probabilities $\Pi_0^B$, $\Pi_1^B$ and $\Pi_2^B$ of finding the capital of the player in states $0$, $1$ and $2$ respectively. These equations are

$$\Pi_0^B(n+1) = p_2\,\Pi_2^B(n) + (1-p_2)\,\Pi_1^B(n), \tag{2.17}$$

$$\Pi_1^B(n+1) = p_1\,\Pi_0^B(n) + (1-p_2)\,\Pi_2^B(n), \tag{2.18}$$

$$\Pi_2^B(n+1) = p_2\,\Pi_1^B(n) + (1-p_1)\,\Pi_0^B(n). \tag{2.19}$$

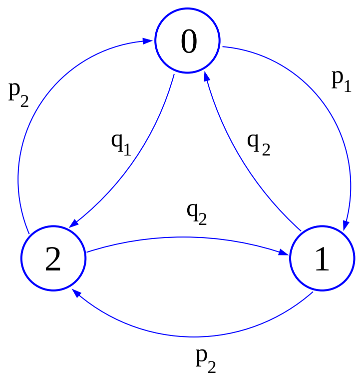



**Figure 2.9.** Diagram representing the different states of game B, as well as the allowed transitions between these states.

We can explain how these evolution equations are obtained through the following example: imagine that we are in state $1$ at time $n+1$. We could have got to this state by two ways: one would be if we were in state $0$ at a previous time step $n$ (with probability $\Pi_0^B(n)$) and we had won with probability $p_1$; on the other hand, we could have been in state $2$ at time $n$ (with probability $\Pi_2^B(n)$) and lost with probability $(1-p_2)$.

Eq. (2.18) is obtained through this reasoning, and the rest of equations can be obtained following the same procedure.

Defining the column vector $\Pi^B(n) = \left[\Pi_0^B(n), \Pi_1^B(n), \Pi_2^B(n)\right]^T$ we can rewrite the previous set of equations in a matrix form as $\Pi^B(n+1) = \mathbb{T}_B \Pi^B(n)$, where we have defined a transition matrix for game B as

$$\mathbb{T}_B = \begin{bmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{bmatrix}. \tag{2.20}$$

Our objective is to obtain the stationary probabilities, that occurs when the distribution of capital in the states $0$, $1$ and $2$ does not change from one game to the next. This implies that the distribution of probabilities is independent of the number of games played $n$ and invariant under the action of the matrix $\mathbb{T}_B$, i.e. $\Pi^B = \mathbb{T}_B \Pi^B$. Matrix $\mathbb{T}_B$ is a stochastic matrix as the elements of each column sum up to one, and from Sec. 1.4.4 we do know that there must be a stationary solution fulfilling the equation $(\mathbb{I} - \mathbb{T}_B)\Pi^B = 0$. The solution for vector $\Pi^B$ that corresponds to the eigenvalue $\lambda = 1$ is

$$\Pi^B = \frac{1}{D} \begin{bmatrix} 1 - p_2 + p_2^2 \\ 1 - p_2 + p_1 p_2 \\ 1 - p_1 + p_1 p_2 \end{bmatrix}, \tag{2.21}$$

and where $D = 3 - p_1 - 2p_2 + 2p_1 p_2 + p_2^2$ is a normalization constant. Introducing the probabilities for game B described in (2.16) when $\epsilon = 0$ we obtain

$$\Pi^B = \frac{1}{13} \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}. \tag{2.22}$$

#### *2.2.2.b Analysis for game A*

For the simplest case of game A we can make use of the previous result obtained for game B, as we need only to substitute the winning probabilities $p_1$ and $p_2$ by $p$. The result for the stationary probabilities $\Pi^A$ obtained when $\epsilon = 0$ reads

$$\Pi^A = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{2.23}$$

a logical result as all transition probabilities are equal.

#### *2.2.2.c Analysis for the randomized game*

Recalling that the randomized game is based on the combination of games A and B with probability $\gamma$ and $1-\gamma$ respectively, we can define an equivalent set of probabilities $p_1', p_2'$ characterizing this mixed game AB. The transition probabilities thus are

$$p_1' = \gamma\, p + (1-\gamma)\, p_1, \tag{2.24}$$
$$p_2' = \gamma\, p + (1-\gamma)\, p_2. \tag{2.25}$$

In order to solve for the vector of stationary probabilities $\Pi^{AB}$ we can introduce the previous expressions for $p_1'$ and $p_2'$ into Eq. (2.21). For the case of $\epsilon = 0$ and a mixing probability $\gamma = \frac{1}{2}$ we obtain

$$\Pi^{AB} = \frac{1}{709} \begin{bmatrix} 245 \\ 180 \\ 284 \end{bmatrix}. \tag{2.26}$$

#### 2.2.2.d Average winning probabilities

There are different ways of obtaining the average winning probabilities for these games, or equivalently, the conditions under which the games are losing, fair or winning. One of them makes use of the stationary probability distribution obtained in previous sections for games A, B and the randomized game AB. The average winning probability $p_{win}$ over all the states is then defined as

$$p_{win} = \sum_{i}^{M-1} p_i \, \Pi_i. \tag{2.27}$$

Thus, a game will be fair on average if $p_{win} = \frac{1}{2}$, losing if $p_{win} < \frac{1}{2}$ and winning if $p_{win} > \frac{1}{2}$. Substituting the set of winning probabilities (2.16) for $\epsilon = 0$ and the stationary probabilities for games A, B and AB given by Eqs. (2.22),(2.23) and (2.26) respectively, we obtain

$$p_{win}^{A} = \frac{1}{2}, \tag{2.28}$$

$$p_{win}^{B} = \frac{1}{2}, \tag{2.29}$$

$$p_{win}^{AB} = 0.5144. \tag{2.30}$$

This reflects what has been previously presented, namely, that games A and B are fair and the combined game AB is winning. For arbitrary values of $\{p, p_1, p_2\}$, we can easily obtain the set of conditions to be fulfilled in order to reproduce the same effect imposing that $p_{win}^{A} < \frac{1}{2}$ (losing game A), $p_{win}^{B} < \frac{1}{2}$ (losing game B) and $p_{win}^{AB} > \frac{1}{2}$ (winning game AB),

$$\frac{1-p}{p} > 1, \tag{2.31}$$

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1, \tag{2.32}$$

$$\frac{(1-p_1')(1-p_2')^2}{p_1' p_2'^2} < 1. \tag{2.33}$$

Perhaps another way of envisioning the appearance of this paradox is by looking at the parameter space of the winning probabilities $\{p_1, p_2\}$. In Fig. 2.10 we plot the curve in parameter space $\{p_1, p_2\}$ separating the winning –upper part– from the losing –lower part– region. The point marked as A corresponds to the set of values of a fair game A ($\epsilon = 0$), whereas the point marked as B corresponds to that of fair game B. The line joining both points shows the evolution of the winning probabilities $\{p_1', p_2'\}$ of the randomized game AB when increasing $\gamma$ from zero (point A) to one (point B). It can be seen that due to the local concavity of the losing region, when moving from one point to another we cross the winning region, i.e., the values obtained by mixing these two sets A and B give as a result a winning game. Therefore, if we want to reproduce the paradox for any other two sets of values, we need only two points in the fair/losing region where the line between them crosses the winning region.

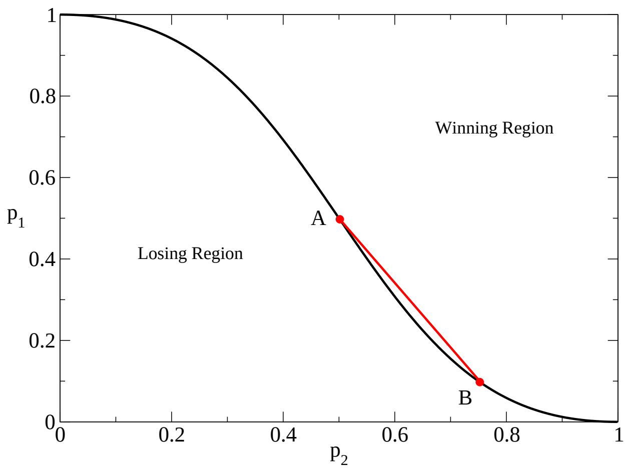



**Figure 2.10.** Parameter space $\{p_1, p_2\}$ where it is plotted the division line between a winning and a losing game. The evolution of the transition probabilities $\{p'_1, p'_2\}$ when varying the mixing probability $\gamma$ is represented with the red line. When $\gamma = 0$ the probabilities correspond to game A, whereas for $\gamma = 1$ correspond to game B.

Besides, we can also obtain the region in parameter space $\{p_1, p_2, p\}$ where the paradox occurs. Fig. 2.11 shows the three surfaces $\Pi_a$, $\Pi_b$ and $\Pi_{ab}$ delimiting the winning and losing regions either for game A, game B and the randomized game AB (for a mixing probability $\gamma = \frac{1}{2}$) respectively. In case of game A the losing region corresponds to the lower half of surface $\Pi_a$, where $p < \frac{1}{2}$; for game B the losing volume is located on the right side of surface $\Pi_b$, and for the randomized game the winning volume is located on the upper side of surface AB. Thus the unique region fullfiling all conditions at once corresponds to the small volume on the front left side of Fig. 2.11, which is also bounded by the plane $p_1 = 0$.

### *2.2.2.e Rates of winning*

With the stationary probabilities obtained for the games it is possible to find the rate of winning as a function of the number of games played, $r(n)$. The rate of winning can be obtained by subtracting the probability of losing from the probability of winning. Thus, we have

$$\frac{d\langle X_n \rangle}{dn} \equiv r = \sum_{i=0}^{M-1} 2\, \Pi_i \, p_i - 1. \tag{2.34}$$

For the simplest case of game A, the rate of winning is $r_A = 2p - 1$. For game B the corresponding rate of winning is $r_B = 2\, p_2 - 1 + 2\, \Pi_0 \, (p_1 - p_2)$. Substituting the set of

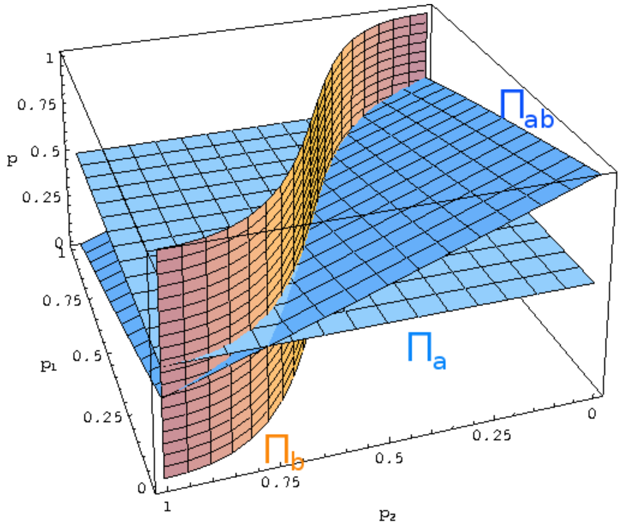



**Figure 2.11.** Probability space $\{p_1, p_2, p\}$ where surfaces $\Pi_A$, $\Pi_B$ and $\Pi_{AB}$ delimiting winning and losing regions are plotted. The region where the paradox is obtained is located on the front left side, that is, the small triangle bounded also by the $p_1 = 0$ plane.

probabilities (2.16) with $\epsilon \neq 0$ we obtain

$$r_A = 2p - 1 = -2\,\epsilon, \tag{2.35}$$

$$\begin{aligned} r_B &= \frac{3(p_1p_2^2 - q_1q_2^2)}{2 + p_1p_2 + q_1q_2 - p_2q_2} = -\frac{6\epsilon\left(80\epsilon^2 - 8\epsilon + 49\right)}{240\epsilon^2 - 16\epsilon + 169} \\ &= -1.74\,\epsilon + 0.119\epsilon^2 - 0.358\epsilon^3 + \mathcal{O}(\epsilon^4), \end{aligned} \tag{2.36}$$

$$\begin{aligned} r_{AB} &= -\frac{6(\epsilon - 0.01311)\left(\epsilon^2 - 0.0369\epsilon + 0.7151\right)}{3\epsilon^2 - 0.1\epsilon + 2.216} \\ &= 0.0254 - 1.9368\epsilon + 0.01361\epsilon^2 - 0.085\epsilon^3 - \mathcal{O}(\epsilon^4). \end{aligned} \tag{2.37}$$

It can be checked that for values of small and positive values of $\epsilon$ both rates of winning of games A and B are negative, whereas for the randomized game AB is positive.

#### *2.2.2.f Other ways of evaluating the rates of winning*

Besides the method described in the previous section, we can also derive the same result with another approach based on existing results from continuous–time random walks [65] on the set of integers. In this system we can calculate which is the first passage time $\mathcal{T}(i \to i+1)$ to go from site $i$ to site $i+1$, considering all transitions shown in Fig.2.12, through the following expression

$$\mathcal{T}(i \to i+1) = <\tau_i> p_i + [<\tau_i> + \mathcal{T}(i \to i+1)]r_i + [<\tau_i> + \mathcal{T}(i-1 \to i+1)]q_i. \tag{2.38}$$

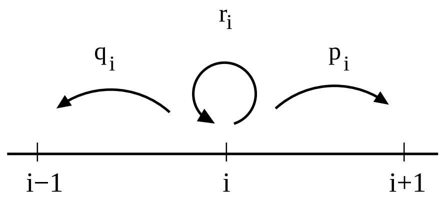



**Figure 2.12.** Diagram corresponding to the allowed transitions from state $i$ to states $i+1$ and $i-1$.

where $<\tau_i>$ is the average residence time at site $i$. The first term on the *rhs* of the previous equation accounts for the probability that after a time $<\tau_i>$ the particle has made a transition to site $i+1$ with probability $p_i$; the second term accounts for the probability that the particle after a time $<\tau_i>$ remains in site $i$ with probability $r_i$ and then jumps to $i+1$ in a time $\mathcal{T}(i \to i+1)$; finally the last term considers the probability that the particle makes a transition to site $i-1$ with probability $q_i$ after a time $<\tau_i>$ plus the time it takes for the particle then to jump to site $i+1$, i.e., $\mathcal{T}(i-1 \to i+1)$.

After some algebra manipulation, and recalling that $\mathcal{T}(i-1 \to i+1) = \mathcal{T}(i-1 \to i) + \mathcal{T}(i \to i+1)$, we obtain the following expression

$$\mathcal{T}(i \to i+1) = \frac{<\tau_i>}{p_i} + \mathcal{T}(i-1 \to i)\frac{q_i}{p_i}. \tag{2.39}$$

Iterating Eq. (2.39) we can obtain a general expression for the mean first passage time from site $i$ to site $i+1$ as

$$\mathcal{T}(i \to i+1) = \frac{<\tau_i>}{p_i} + \sum_{j=-\infty}^{i} \left[\frac{q_i}{p_i} \dots \frac{q_j}{p_j}\frac{<\tau_{j-1}>}{p_{j-1}}\right]. \tag{2.40}$$

and since $\mathcal{T}(i \to i+n) = \mathcal{T}(i \to i+1) + \mathcal{T}(i+1 \to i+2) + \ldots + \mathcal{T}(i+n-1 \to i+n)$ we obtain

$$\mathcal{T}(n_0 \to n) = \sum_{r=n_0}^{n-1} \left[\frac{<\tau_r>}{p_r} + \sum_{j=-\infty}^{r} \frac{q_r}{p_r} \dots \frac{q_j}{p_j}\frac{<\tau_{j-1}>}{p_{j-1}}\right]. \tag{2.41}$$

Therefore, once we have obtained a general expression for the mean first passage time considering continuous–time, we can simply particularize it to the discrete–time case recalling that $<\tau_i>= 1 \,\forall i$ ,

$$\mathcal{T}(n_0 \to n) = \sum_{r=n_0}^{n-1} \left[\frac{1}{p_r} + \sum_{j=-\infty}^{r} \frac{q_r}{p_r} \dots \frac{q_j}{p_j p_{j-1}}\right]. \tag{2.42}$$

From a previous section we know that the Parrondo games can be described through a Markov chain that eventually is reduced to a three–state Markov chain due to the periodicity in the transition probabilities. Thus, we are interested in particularizing the previous result (2.42) for a periodic system with arbitrary period $L$, obtaining

$$\mathcal{T}(i \to i+1) = \frac{\sum_{j=0}^{L} \prod_{k=i-j+1}^{i} q_k \prod_{k=i+1}^{i-j+L-1} p_k}{\prod_{k=1}^{L} p_k - \prod_{k=1}^{L}}. \tag{2.43}$$

Finally we can obtain the general expression for the rate of winning –or equivalently the velocity– through

$$r = \frac{L}{\sum_{i=1}^{L} \mathcal{T}(i \to i+1)} = \frac{L\left[\prod_{k=1}^{L} p_k - \prod_{k=1}^{L}\right]}{\sum_{i=1}^{L}\left\{\sum_{j=0}^{L} \prod_{k=i-j+1}^{i} q_k \prod_{k=i+1}^{i-j+L-1} p_k\right\}}, \tag{2.44}$$

which after some manipulation leads to the same expression of the current obtained through discrete–time Markov chain analysis, c.f. Eq. (2.36). Eq. (2.44) has a simple interpretation if we think of the rate of winning as a velocity, thus it is nothing but a quotient between a distance ($L$) and the time it takes to cover it ($\sum_i^L \mathcal{T}_i$). The general result (2.44) agrees with other studies of one–dimensional hopping models with arbitrary period $L$ [66].

## 2.3 Other classes of Parrondo's games

We have seen in previous sections that Parrondo's paradox appears when one combines a simple coin tossing game, either unbiased or negatively biased, with another unbiased (or negatively biased) game where the coin to be used depends on the actual capital of the player. Whatever sort of alternation between these games, either stochastically or randomly, leads to a positively biased game. However, we might wonder if there exist different games giving a similar effect, without considering the modulo rule introduced in game B.

Parrondo et al. [4] introduced a new version for game B, where a player uses four different coins depending on its previous history of wins and losses. On the other hand, effects of cooperation between players in Parrondo's games have been considered by Toral [38, 67]. In the following sections we will briefly present the basics of these games.

### 2.3.1 History dependent games

As already mentioned, Parrondo *et al* [4] devised a new game B (which we will refer to as game B') where the winning probabilities of a player depend on his/her previous history of wins and losses. Therefore we have two games: game A is identical to the original game, that is, there is a winning probability $p$ and a losing probability $q = 1 - p$. For game B' there are four probabilities $\{p_1, p_2, p_3, p_4\}$ that will be used depending on whether the player won or lost in the two previous rounds. If subscript $n$ denotes the round played we can summarize the probabilities in the following table,

| $n-2$ | $n-1$ | Winning probability |
|---|---|---|
| Loss | Loss | $p_1$ |
| Loss | Win | $p_2$ |
| Win | Loss | $p_3$ |
| Win | Win | $p_4$ |

(2.45)

Originally they were assigned the following set of values

$$\begin{cases} p = \frac{1}{2} - \epsilon, \\ p_1 = \frac{9}{10} - \epsilon, \\ p_2 = p_3 = \frac{1}{4} - \epsilon, \\ p_4 = \frac{7}{10} - \epsilon. \end{cases} \tag{2.46}$$

where $\epsilon$ accomplishes the same task than in the original games, i.e. when $\epsilon = 0$ both games are fair and when $\epsilon > 0$ they are losing games; however, any sort of combination between both (either periodic or stochastic) gives rise to a winning game, see for example Fig. 2.13.

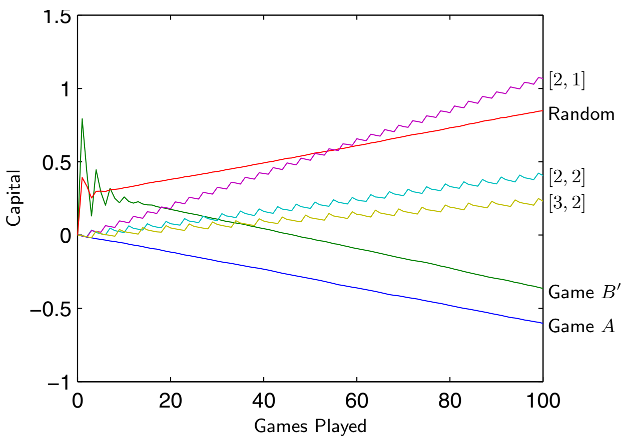



**Figure 2.13.** Plot of the average gain of a single player versus the number of plays for Parrondo's history dependent games A and B', as well as a periodic and a random combination of them. Simulations were performed using the probabilities defined in (2.46) together with $\epsilon = 0.003$.

Furthermore, even when two games like game B' are combined, we still reproduce the paradox [68].

### *2.3.1.a Analysis of the games*

These games can also be described through discrete–time Markov chains. For game B' we may distinguish four different states: $\{LL, LW, WL, WW\}$. As a result we obtain the Markov chain represented in Fig. 2.14.

We can write down a set of evolution equations for the set of probabilities $\{\Pi_{ll}^{B'}, \Pi_{lw}^{B'}, \Pi_{wl}^{B'}, \Pi_{ww}^{B'}\}$ as

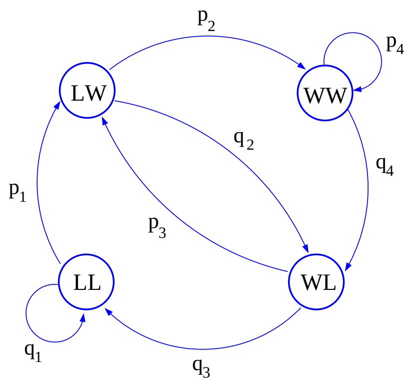



**Figure 2.14.** Diagram representing the different states of the history dependent game B', as well as the allowed transitions between these states.

$$\Pi^{B'}_{ll}(n+1) = (1-p_3)\,\Pi^{B'}_{wl}(n) + (1-p_1)\,\Pi^{B'}_{ll}(n), \tag{2.47}$$
$$\Pi^{B'}_{lw}(n+1) = p_3\,\Pi^{B'}_{wl}(n) + p_1\,\Pi^{B'}_{ll}(n), \tag{2.48}$$
$$\Pi^{B'}_{wl}(n+1) = (1-p_2)\,\Pi^{B'}_{lw}(n) + (1-p_4)\,\Pi^{B'}_{ww}(n), \tag{2.49}$$
$$\Pi^{B'}_{ww}(n+1) = p_2\,\Pi^{B'}_{lw}(n) + p_4\,\Pi^{B'}_{ww}(n). \tag{2.50}$$

Which can be put in matrix form as $\mathbf{\Pi}^{B'}(n+1) = \mathbb{T}_{B'}\mathbf{\Pi}^{B'}(n)$, where $\mathbb{T}_{B'}$ accounts for the transition matrix between these states and is given by

$$\mathbb{T}_{B'} = \begin{bmatrix} 1-p_1 & 0 & 1-p_3 & 0 \\ p_1 & 0 & p_3 & 0 \\ 0 & 1-p_2 & 0 & 1-p_4 \\ 0 & p_2 & 0 & p_4 \end{bmatrix}, \tag{2.51}$$

and $\mathbf{\Pi}^{B'}(n+1) = \{\Pi^{B'}_{ll}(n+1), \Pi^{B'}_{lw}(n+1), \Pi^{B'}_{wl}(n+1), \Pi^{B'}_{ww}(n+1)\}^T$ is the column vector of occupancy probabilities for the states. As already explained in Sec. 1.4.4 we know there exists a stationary probability distribution for $\mathbf{\Pi}^{B'}$ such that $(\mathbb{I}-\mathbb{T}_{B'})\mathbf{\Pi}^{B'} = 0$, and whose solution reads

$$\mathbf{\Pi}^{B'} = \frac{1}{D'}\begin{bmatrix} (1-p_3)(1-p_4) \\ p_1(1-p_4) \\ p_1(1-p_4) \\ p_1 p_2 \end{bmatrix}, \tag{2.52}$$

where $D' = p_1 p_2 + (1+2p_1-p_3)(1-p_4)$. Once we have obtained the stationary probability distribution for game B' we can easily obtain that of game A and the randomized AB'; the former case would be equivalent to setting $p_i = p\ \forall i$, whereas for the second case we would substitute $p_i$ by $p'_i = \gamma\, p + (1-\gamma)\, p_i$ for $i = 1, \ldots, 4$.

Using the probabilities for game A and B' with $\epsilon = 0$, and for the randomized AB' with $\gamma = \frac{1}{2}$, we obtain for the stationary distributions

$$\Pi^A = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \Pi^{B'} = \frac{1}{22}\begin{bmatrix} 5 \\ 6 \\ 6 \\ 5 \end{bmatrix}, \quad \Pi^{AB'} = \frac{1}{429}\begin{bmatrix} 100 \\ 112 \\ 112 \\ 105 \end{bmatrix}. \tag{2.53}$$

The average winning probabilities (2.27) are $p^A_{win} = p^{B'}_{win} = \frac{1}{2}$, $p^{AB'}_{win} = 0.512$. Thus, the necessary conditions for the paradox to occur are accomplished, that is, we have two fair/losing games that when combined give as a result a winning game.

Finally, for arbitrary values $\{p, p_1, p_2, p_3, p_4\}$ the following set of conditions needs to be fulfilled in order to reproduce the Parrondo effect

$$\begin{aligned} \frac{1-p}{p} &> 1, \\ \frac{(1-p_3)(1-p_4)}{p_1 p_2} &> 1, \\ \frac{(1-p'_3)(1-p'_4)}{p'_1 p'_2} &< 1. \end{aligned} \tag{2.54}$$

### 2.3.2 Collective games

Once reviewed an alternative group of Parrondo games where the capital dependent rules of game B have been substituted for history rules, we turn to another sort of Parrondo games introduced by Toral [38, 67] where the Parrondo effect is also obtained but for a set of $N$ players (collective games). In one of these games [67], game B is substituted by another game that depends on the state of a player's neighbor. We refer to this state as whether a player has win or lost the previous game. The other version [38] considers a redistribution of capital between a set of $N$ players. We will now briefly explain both games.

#### *2.3.2.a Cooperative games*

A group of N players with capitals $C_i$, $i = 1, \ldots, N$ are arranged in a circle so that each player has two neighbors. A player chosen randomly for playing either can play game A with probability $\gamma$ or game B with probability $1-\gamma$. These players are labelled as winners/losers depending on whether they have won/lost the previous round played. Game A is the same as the original, where a player has a winning probability $p$ and a losing probability $1-p$. Probabilities for game B depend on the state of the neighbors $i-1$ and $i+1$ of player $i$. In the following table we have summarized the different combinations available with their corresponding winning probabilities

| player $i-1$ | player $i+1$ | Winning probability |
|---|---|---|
| Loser | Loser | $p_1$ |
| Loser | Winner | $p_2$ |
| Winner | Loser | $p_3$ |
| Winner | Winner | $p_4$ |

(2.55)

The games are classified according to the behavior of the total capital $C(t) = \sum_i C_i(t)$. Thus, a winning game is one for which the average value of the total capital $C(t)$ increases with time, and similarly for losing and fair games. Fig. 2.15 shows the average gain per player $\frac{\langle C(t)\rangle}{N}$ for the set of probabilities $p = 0.5$, $p_1 = 1$, $p_2 = p_3 = 0.16$, and $p_4 = 0.7$. We can see how the Parrondo effect is again reproduced: playing either game A or B reports no winnings on average, whereas an alternation between both increases the average capital per player with time.

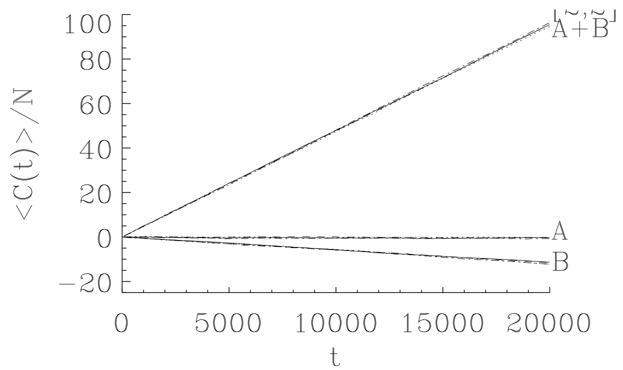



**Figure 2.15.** Average capital per player, $\frac{\langle C(t)\rangle}{N}$ versus time $t$. The probabilities defining the games are: $p = 0.5$, $p_1 = 1$, $p_2 = p_3 = 0.16$, $p_4 = 0.7$. These results show that game A is fair, game B is a losing game, but when games A and B are combined (AB) or in the [2, 2] alternation $AABBAABB\ldots$, the result is a winning game. Results are shown for $N = 50, 100$, and 200 players.

### *2.3.2.b Capital redistribution between players*

The other version of collective games [38] substitutes the randomizing effect of game A by a game that redistributes the capital between the players. Depending on the way the capital is redistributed we may distinguish different versions for this new game A:

- Game A': A unit of capital is given to a randomly selected player (with probability $\frac{1}{N}$).
- Game $A''$: A unit of capital is given to a nearest neighbor with probabilities that depend on the capital difference between players $p(i \to i \pm 1) \propto \max[C_i - Ci \pm 1, 0]$, with $p(i \to i+1) + p(i \to i-1) = 1$ and $C_i$ denotes the current capital of player $i$.

These versions of game A are clearly fair, as they only redistribute the capital between different players, keeping the total amount of capital constant. The mechanism of plays

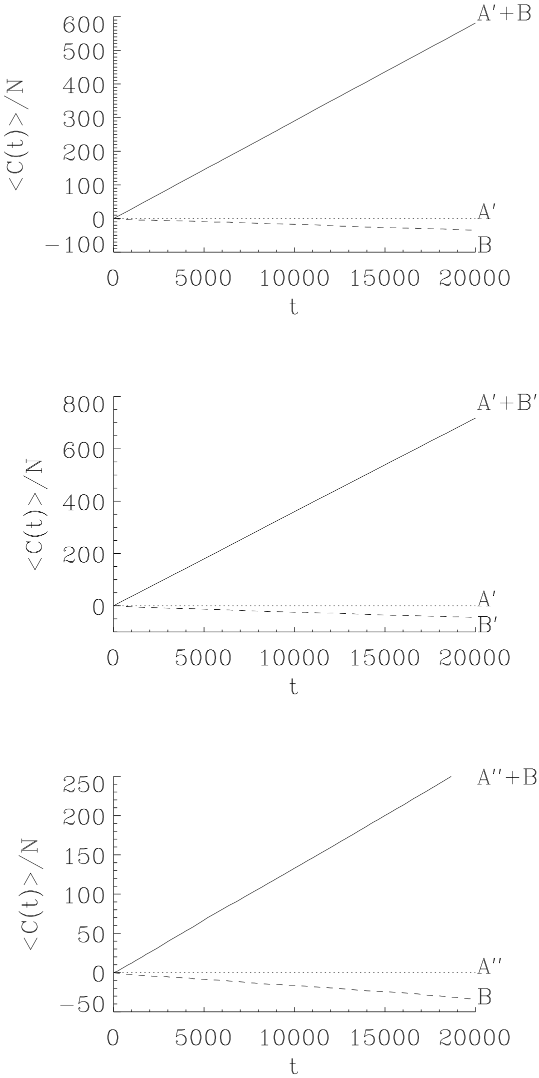



**Figure 2.16.** Average capital per player, $\langle C(t)\rangle/N$, versus time, $t$ (in units of games per player) for different combinations of games A and B. We find the evolution of the capital when players only play game A ($\gamma = 1$), game B ($\gamma = 0$) and a combination of both ($\gamma = \frac{1}{2}$). Upper panel: Combination of the new game A' with the original game B with probabilities: $p_1 = 0.1 - \epsilon$, $p_2 = 0.75 - \epsilon$. Middle panel: Alternation between the new game A' and game B' with probabilities: $p_1 = 0.9 - \epsilon$, $p_2 = p_3 = 0.25 - \epsilon$, $p_4 = 0.7 - \epsilon$, with $\epsilon = 0.01$. Lower panel: Alternation between the game $A''$ and game B with probabilities: $p_1 = 0.9 - \epsilon$, $p_2 = p_3 = 0.25 - \epsilon$, $p_4 = 0.7 - \epsilon$, with $\epsilon = 0.01$. These figures have been obtained considering an ensemble of $N = 200$ players; the results have been averaged over 10 realizations of the games. In all cases, the initial condition is that of zero capital, $C_i(0) = 0\,, \forall i = 1, \dots N$.

can be described as follows: we have a set of $N$ players, and each time step a random player $i$ is chosen for playing. In one version of these collective games, the player chooses to play either game A' or the original capital dependent game B; another version involves an alternation between game A' and the history dependent game B', already explained in a previous section. A third version includes an alternation between game $A''$ and the capital dependent game B.

Fig. 2.16 shows the evolution of the average capital per player versus time for the three different versions explained previously. In all cases, the Parrondo effect is again reproduced, i.e., the resulting game from the combination of any version of game A with any other game B turns to be a winning game. This result emphasizes the fact that it is better, collectively speaking, for an individual player to redistribute part of its capital between other players, in order to increase on average the total amount of capital.

Chapter 3

# Parrondo's games with self–transition

The aim of this Chapter is to study a new version of Parrondo's games, where a new transition probability is taken into account. We introduce a *self-transition* probability, that is, now the capital of the player can remain the same after a game played with a probability that will be denoted by $r_i$, $i = 0, \ldots, M-1$ (for simplicity the case of $M = 3$ will be considered).

As we will show, the significance of this new version is a natural evolution of Parrondo's games, which will be of particular interest in a succeeding chapter, when the quantitative relation between Parrondo's games and the Brownian ratchet is established.

## 3.1 Analysis of the new Parrondo games with self-transitions

### 3.1.1 Game A

We start with the new game A, where the probability of winning is $p$, the probability of remaining with the same capital will be denoted as $r$, and the losing probability is given by $q = 1 - r - p$ .

Following the same reasoning as [7] we will calculate the probability $f_j$ that the capital reaches zero in a finite number of plays. Let us assume that initially we have a given capital of $j$ units. From Markov chain analysis [64] we find

- $f_j = 1$ for all $j \geq 0$, and so the game is either fair or losing; or
- $f_j < 1$ for all $j > 0$, in which case the game can be winning because there is a certain probability that the capital can grow indefinitely.

We are looking for the set of numbers $\{f_j\}$ that correspond to the minimal non-negative solution of the equation

$$f_j = p \cdot f_{j+1} + r \cdot f_j + q \cdot f_{j-1}\,, \qquad j \geq 1 \tag{3.1}$$

with the boundary condition

$$f_0 = 1\,. \tag{3.2}$$

Eq.(3.1) can be put in the following form

$$f_j = \frac{p}{1-r} \cdot f_{j+1} + \frac{q}{1-r} \cdot f_{j-1}\,, \tag{3.3}$$

whose solution, for the initial condition (3.2), is $f_j = A \cdot [(\frac{1-p-r}{p})^j - 1] + 1$, where A is a constant. For the minimal non-negative solution we obtain

$$f_j = \min\left[1, \left(\frac{1-p-r}{p}\right)^j\right]. \tag{3.4}$$

So we can see that the new game A is a winning game for

$$\frac{1-p-r}{p} < 1, \tag{3.5}$$

is a losing game for

$$\frac{1-p-r}{p} > 1, \tag{3.6}$$

and is a fair game for

$$\frac{1-p-r}{p} = 1. \tag{3.7}$$

### 3.1.2 Game B

We now analyze the new game B. Like game A, we have introduced the probabilities of a self-transition in each state, that is, if the capital is a multiple of three we have a probability $r_1$ of remaining in the same state, whereas if the capital is not a multiple of three then the probability is $r_2$. The rest of the probabilities will follow the same notation as in the original game B, so we have the following scheme

$$\begin{cases} \mathrm{mod}(\mathrm{capital}, 3) = 0 \rightarrow p_1, r_1, q_1 \\ \mathrm{mod}(\mathrm{capital}, 3) \neq 0 \rightarrow p_2, r_2, q_2. \end{cases} \tag{3.8}$$

Now let $g_j$ be the probability that the capital will reach the zeroth state in a finite number of plays, supposing an initial capital of $j$ units. Again, from Markov chain theory we have

- $g_j = 1$ for all $j \geq 0$, so game B is either fair or losing; or
- $g_j < 1$ for all $j > 0$, in which case game B can be winning because there is a certain probability for the capital to grow indefinitely.

For $j \geq 1$, the following set of recurrence equations must be solved:

$$\begin{aligned} g_{3j} &= p_1 \cdot g_{3j+1} + r_1 \cdot g_{3j} + (1-p_1-r_1) \cdot g_{3j-1}, & j \geq 1 \\ g_{3j+1} &= p_2 \cdot g_{3j+2} + r_2 \cdot g_{3j+1} + (1-p_2-r_2) \cdot g_{3j}, & j \geq 0 \\ g_{3j+2} &= p_2 \cdot g_{3j+3} + r_2 \cdot g_{3j+2} + (1-p_2-r_2) \cdot g_{3j+1}, & j \geq 0\,. \end{aligned} \tag{3.9}$$

As in game A, we are looking for the set of numbers $\{g_j\}$ that correspond to the minimal non-negative solution. Eliminating terms $g_{3j-1}$, $g_{3j+1}$ and $g_{3j+2}$ from (3.9) we get

$$[p_1p_2^2+(1-p_1-r_1)(1-p_2-r_2)^2]\cdot g_{3j} = p_1p_2^2\cdot g_{3j+3}+(1-p_1-r_1)(1-p_2-r_2)^2\cdot g_{3j-3}\,. \tag{3.10}$$

Considering the same boundary condition as in game A, $g_0 = 1$, the last equation has a general solution of the form $g_{3j} = B \cdot \left[\left(\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1p_2^2}\right)^j - 1\right] + 1$, where B is a constant. For the minimal non-negative solution we obtain

$$g_{3j} = \min\left[1, \left(\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1p_2^2}\right)^j\right]. \tag{3.11}$$

It can be verified that the same solution (3.11) will be obtained solving (3.9) for $g_{3j+1}$ and $g_{3j+2}$, leading all them to the same condition for the probabilities of the games.

As with game A, game B will be winning if

$$\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1p_2^2} < 1, \tag{3.12}$$

losing if

$$\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1p_2^2} > 1, \tag{3.13}$$

and fair if

$$\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1p_2^2} = 1. \tag{3.14}$$

### 3.1.3 Game AB

Now we will turn to the random alternation of games A and B with probability $\gamma$. As in a previous chapter this game will be named as game AB. For this game AB we have the following (primed) probabilities

- if the capital is a multiple of three

$$\begin{cases} p'_1 = \gamma \cdot p + (1-\gamma) \cdot p_1, \\ r'_1 = \gamma \cdot r + (1-\gamma) \cdot r_1, \end{cases} \tag{3.15}$$

- if the capital is not multiple of three

$$\begin{cases} p'_2 = \gamma \cdot p + (1-\gamma) \cdot p_2, \\ r'_2 = \gamma \cdot r + (1-\gamma) \cdot r_2. \end{cases} \tag{3.16}$$

The same reasoning as with game B can be made but with the new probabilities $p'_1$, $r'_1$, $p'_2$, $r'_2$ instead of $p_1$, $r_1$, $p_2$, $r_2$. Eventually we obtain that game AB will be winning if

$$\frac{(1-p'_1-r'_1)(1-p'_2-r'_2)^2}{p'_1 p'^2_2} < 1, \tag{3.17}$$

losing if

$$\frac{(1-p'_1-r'_1)(1-p'_2-r'_2)^2}{p'_1 p'^2_2} > 1, \tag{3.18}$$

and fair if

$$\frac{(1-p'_1-r'_1)(1-p'_2-r'_2)^2}{p'_1 p'^2_2} = 1. \tag{3.19}$$

The paradox will be present if games A and B are losing, while game AB is winning. In this framework this means that the conditions (3.6), (3.13) and (3.17) must be satisfied simultaneously. In order to obtain sets of probabilities fulfilling theses conditions we have first obtained sets of probabilities yielding *fair* A and B games but such that AB is a winning game, and then introducing a small biasing parameter $\epsilon$ making game A and game B losing games, but still keeping a winning AB game. As an example, we give some sets of probabilities that fulfill these conditions:

$$\begin{array}{lllllll} (a) & p = \frac{9}{20}-\epsilon, & r = \frac{1}{10}, & p_1 = \frac{9}{100}-\epsilon, & r_1 = \frac{1}{10}, & p_2 = \frac{3}{5}-\epsilon, & r_2 = \frac{1}{5}, \\ (b) & p = \frac{9}{20}-\epsilon, & r = \frac{1}{10}, & p_1 = \frac{509}{5000}-\epsilon, & r_1 = \frac{1}{10}, & p_2 = \frac{7}{10}-\epsilon, & r_2 = \frac{1}{20}, \\ (c) & p = \frac{9}{20}-\epsilon, & r = \frac{1}{10}, & p_1 = \frac{3}{25}-\epsilon, & r_1 = \frac{2}{5}, & p_2 = \frac{3}{5}-\epsilon, & r_2 = \frac{1}{10}, \\ (d) & p = \frac{1}{4}-\epsilon, & r = \frac{1}{2}, & p_1 = \frac{3}{25}-\epsilon, & r_1 = \frac{2}{5}, & p_2 = \frac{3}{5}-\epsilon, & r_2 = \frac{1}{10}. \end{array} \tag{3.20}$$

## 3.2 Properties of the Games

### 3.2.1 Rate of winning

If we consider the capital of a player at play number $n$, $X_n$ modulo $M$, we can perform a discrete-time Markov chain analysis of the games with a state-space $\{0, 1, \ldots, M-1\}$[1]. For the case of Parrondo's games we have $M = 3$, so the following set of difference equations for the probability distribution can be obtained:

$$\begin{aligned} P_0(n+1) &= p_2 \cdot P_2(n) + r_1 \cdot P_0(n) + q_2 \cdot P_1(n), \\ P_1(n+1) &= p_1 \cdot P_0(n) + r_2 \cdot P_1(n) + q_2 \cdot P_2(n), \\ P_2(n+1) &= p_2 \cdot P_1(n) + r_2 \cdot P_2(n) + q_1 \cdot P_0(n), \end{aligned} \tag{3.21}$$

which can be put in a matrix form as $\mathbf{P}(n+1) = \mathbb{T} \cdot \mathbf{P}(n)$, where

$$\mathbb{T} = \begin{pmatrix} r_1 & q_2 & p_2 \\ p_1 & r_2 & q_2 \\ q_1 & p_2 & r_2 \end{pmatrix} \tag{3.22}$$

and

$$\mathbf{P}(n) = \begin{pmatrix} P_0(n) \\ P_1(n) \\ P_2(n) \end{pmatrix}. \tag{3.23}$$

In the limiting case where $n \to \infty$ the system will tend to a stationary state (*c.f.* Sec. 1.4.4) characterized by

$$\mathbf{\Pi} = \mathbb{T} \cdot \mathbf{\Pi}, \tag{3.24}$$

where $\lim_{n\to\infty} \mathbf{P}(n) = \mathbf{\Pi}$.

Solving (3.24) is equivalent to solving an eigenvalue problem. As we are dealing with Markov chains and the transition matrix obtained is a stochastic matrix, we know that there will be an eigenvalue $\lambda = 1$ and the rest will be under 1 (see the *Perron-Frobenius* theorem in Sec. 1.4.4 for further details). For $\lambda = 1$ we obtain the following eigenvector giving the stationary probability distribution in terms of the games' probabilities.

$$\mathbf{\Pi} \equiv \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} (1-r_2)^2 - p_2 \cdot (1-p_2-r_2) \\ (1-r_1)(1-r_2) - p_2 \cdot (1-p_1-r_1) \\ (1-r_1)(1-r_2) - p_1 \cdot (1-p_2-r_2) \end{pmatrix}, \tag{3.25}$$

where $D$ is a normalization constant given by

[1] As in the original Parrondo games, we can reduce the infinite state Markov chain to an $M$ finite Markov chain.

$$D = (1-r_2)^2 + 2(1-r_1)(1-r_2) - p_2(2-p_2-r_2-r_1-p_1) - p_1(1-p_2-r_2). \quad (3.26)$$

The rate of winning at the $n$–th step, has the general expression

$$r(n) \equiv E[X_{n+1}] - E[X_n] = \sum_{i=-\infty}^{\infty} i \cdot [P_{i,n+1} - P_{i,n}] \,. \quad (3.27)$$

Using these expressions it is possible to obtain the stationary rate of winning for the new games introduced in the previous section. The results are, for game A:

$$r_A^{st} = 2p + r - 1, \quad (3.28)$$

and for game B

$$\begin{aligned} r_B^{st} &= 2p_2 + r_2 - 1 + [q_2 - p_2 + p_1 - q_1] \cdot \Pi_0 \\ &= \frac{3}{D}\,(p_1 p_2^2 - (1-p_1-r_1)(1-p_2-r_2)^2), \end{aligned} \quad (3.29)$$

where $D$ is given by (3.26).

It is an easy task to check that when $r_1 = r_2 = 0$ we recover the well-known expressions for the original games obtained in [8]. To obtain the stationary rate for the randomized game AB we just need to replace in the above expression the probabilities from (3.15) and (3.16).

Within this context the paradox appears when $r_A^{st} \leq 0$, $r_B^{st} \leq 0$ and $r_{AB}^{st} > 0$. If, for example, we use the values from (3.20d) and a switching probability $\gamma = 1/2$, we obtain the following stationary rates for game A, game B and the random combination AB:

$$\begin{aligned} r_A^{st} &= -2\epsilon, \\ r_B^{st} &= \frac{-\epsilon\,(441 - 120\epsilon + 1000\epsilon^2)}{231 - 40\epsilon + 500\epsilon^2}, \\ r_{AB}^{st} &= \frac{93 - 9828\epsilon + 1920\epsilon^2 - 32000\epsilon^3}{2\,(2499 - 320\epsilon + 8000\epsilon^2)}\,. \end{aligned} \quad (3.30)$$

which yield the desired paradoxical result for small $\epsilon > 0$.

We can also evaluate the stationary rate of winning when both the probability of winning and the self-transition probability for the games vary with a parameter $\epsilon$ as $p = p - \frac{\epsilon}{2}$ and $r = r + \epsilon$, so that normalization is preserved. Using the set of probabilities derived from (3.20d), namely $p = \frac{1}{4} - \frac{\epsilon}{2}\,, r = \frac{1}{2} + \epsilon\,, p_1 = \frac{3}{25} - \frac{\epsilon}{2}\,, r_1 = \frac{2}{5} + \epsilon\,, p_2 = \frac{3}{5} - \frac{\epsilon}{2}\,, r_2 = \frac{1}{10} + \epsilon$, the result is:

$$
\begin{aligned}
r_A^{st} &= 0, \\
r_B^{st} &= \frac{-\epsilon\,(21-20\epsilon)}{2\,(77-200\epsilon+125\epsilon^2)}, \\
r_{AB}^{st} &= \frac{31-164\epsilon+160\epsilon^2}{2\,(833-2600\epsilon+2000\epsilon^2)}\,,
\end{aligned}
\tag{3.31}
$$

again a paradoxical result.

A comparison between the expressions for the rates of winning of the original Parrondo games [8] and the new games can be done in two ways. The first one consists in comparing two games with the same probabilities of winning, say original game A with probabilities $p = \frac{1}{2}$ and $q = \frac{1}{2}$ and the new game A with probabilities $p_{\rm new} = \frac{1}{2}$, $r_{\rm new} = \frac{1}{4}$ and $q_{\rm new} = \frac{1}{4}$. In this case we can think of the 'old' probability of losing $q$ as taking the place of the *self-transition* probability $r_{\rm new}$ and the new probability of losing $q_{\rm new}$. In this way we obtain a higher rate of winning in the new game A than in the original game – remember that the new game A has an extra term $r$ in the rate of winning compared to the original rate, and this extra term is what gives rise to the higher value. The same reasoning applies for game B, leading to the same conclusion.

The other possibility could be to compare the two games with the same probability of losing. In this case, we follow the same reasoning as before, but now we can imagine the 'old' probability of winning as replacing the winning and self-transition probabilities of the new game. What we now obtain is a lower rate of winning for the new game compared to the original one. An easy way of checking this is by rewriting (3.28) and (3.29) as

$$
r_A^{st} = p - q, \tag{3.32}
$$

$$
r_B^{st} = \frac{3}{D}\,(p_1 p_2^2 - q_1 q_2^2).
$$

So for the same value of $q$ but a lower value of $p$ we obtain a lower value for the rates of game A and B.

We now explore the range of probabilities in which the Parrondo effect takes place. We restrict ourselves to the case $M = 3$ and $\gamma = 1/2$ used in the previous formulae.

The fact that we have introduced three new probabilities complicates the representation of the parameter space as we have six variables altogether, two variables $\{p, r\}$ from game A and four variables $\{p_1, r_1, p_2, r_2\}$ coming from game B. In order to simplify this high number of variables, some probabilities must be set so that a representation in three dimensions will be possible. In our case we will fix the variables $\{r, r_1, r_2\}$ so that the surfaces can be represented in the parameter space $\{p, p_1, p_2\}$.

In Fig. 3.1 we can see the resulting region where the paradox exists for the variables $r = \frac{1}{4}$, $r_1 = \frac{1}{8}$ and $r_2 = \frac{1}{10}$. It is possible to show that the volume where the paradox

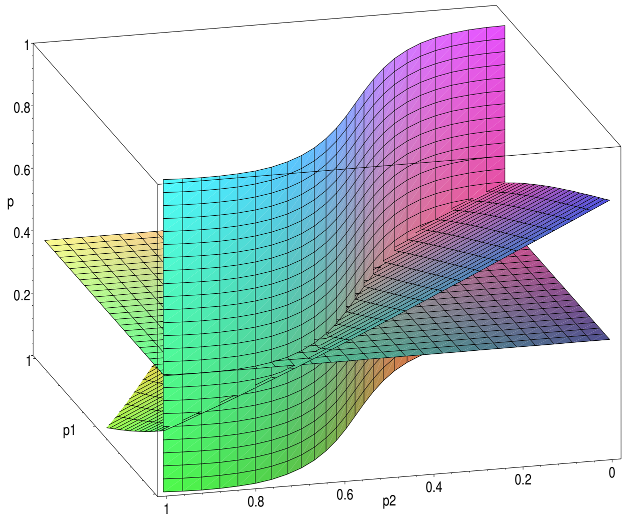



**Figure 3.1.** Parameter space corresponding to the values $r = \frac{1}{4}$, $r_1 = \frac{1}{8}$ and $r_2 = \frac{1}{10}$. The actual region where the paradox exists is delimited by the plane $p_1 = 0$ and the triangular region situated at the frontal face, where all the planes intersect.

takes place, gradually shrinks to zero as the variables $r$, $r_1$ and $r_2$ increase from zero to their maximum value of one.

Although it still remains an open question, we have not been able to obtain the equivalent parameter space to Fig. 3.1 with the fixed variables $p, p_1, p_2$ and with the parameter space variables $r, r_1, r_2$ instead – it is possible to obtain the planes for games A and B, but not for the randomized game AB.

### 3.2.2 Simulations and discussion

We have analyzed the new games A and B, and obtained the conditions in order to reproduce the Parrondo effect. We now present some simulations to verify that the paradox is present for a different range of probabilities – see Fig. 3.2. Some interesting features can be observed from these graphs. First it can be noticed that the performance of random or deterministic alternation of the games drastically changes with the parameters.

We use the notation $[a, b]$ to indicate that game A was played $a$ times and game B $b$ times. The performance of the deterministic alternations $[3, 2]$ and $[2, 2]$ remain close to one another, as can be seen in Fig. 3.2. However the alternation $[4, 4]$ has a low rate of winning because as we play each game four times, that causes the dynamics of games A and B to dominate over the dynamic of alternation, thereby considerably reducing the gain.

The performance of the random alternation is more variable, obtaining in some cases a greater gain than in the deterministic cases – see Fig. 3.2c.

In figures (3.3a) and (3.3b) a comparison between the theoretical rates of winning for

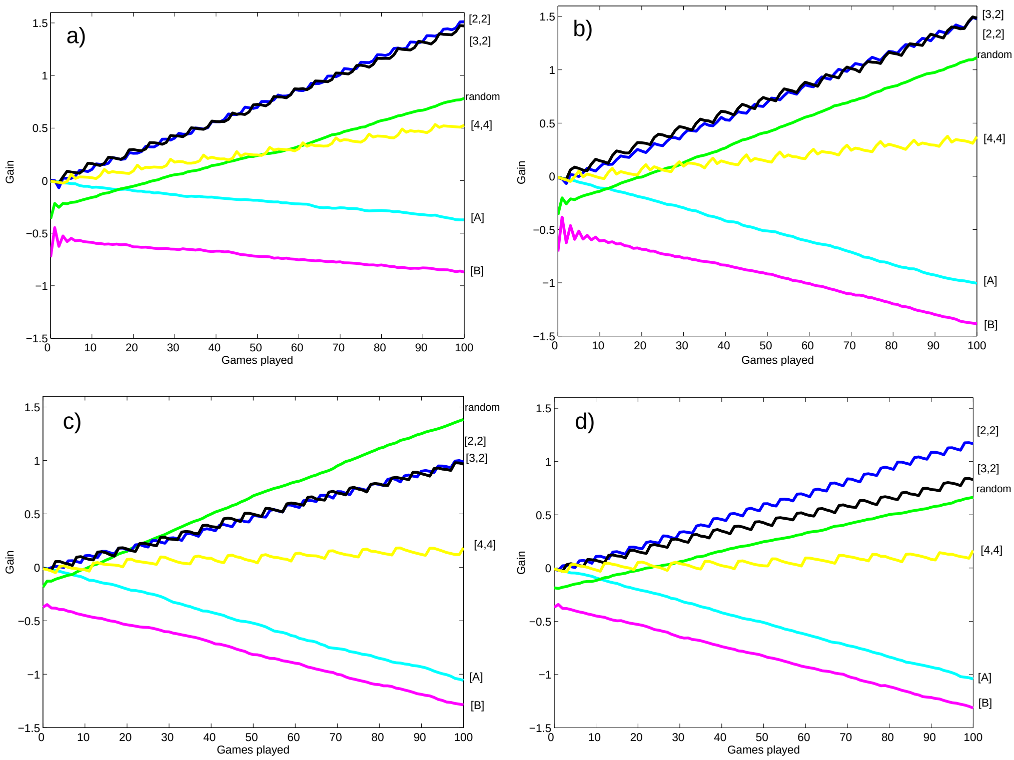



**Figure 3.2.** Average gain as a function of the number of games played coming from numerical simulation of Parrondo's games with different sets of probabilities. The notation $[a, b]$ indicates that game A was played $a$ times and game B $b$ times. The gains were averaged over 50 000 realizations of the games. a) Simulation corresponding to the probabilities (3.20a) and $\epsilon = \frac{1}{500}$; b) probabilities (3.20b) and $\epsilon = \frac{1}{200}$; c) probabilities (3.20c) and $\epsilon = \frac{1}{200}$; d) probabilities (3.20d) and $\epsilon = \frac{1}{200}$.

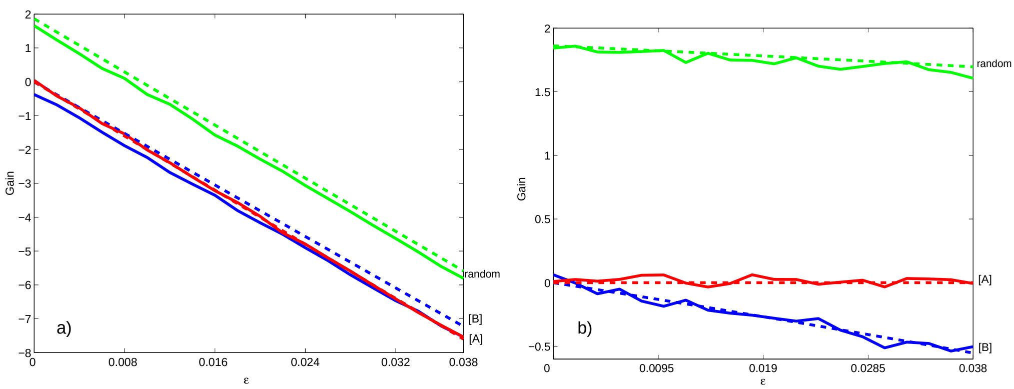



**Figure 3.3.** Comparison of the theoretical rates of winning – dashed lines – together with the rates obtained through simulations – solid lines. All the simulations were obtained by averaging over 50 000 trials and over all possible initial conditions. a) The parameters correspond to the ones used in equations (3.30). b) The parameters correspond to the ones used in equations (3.31).

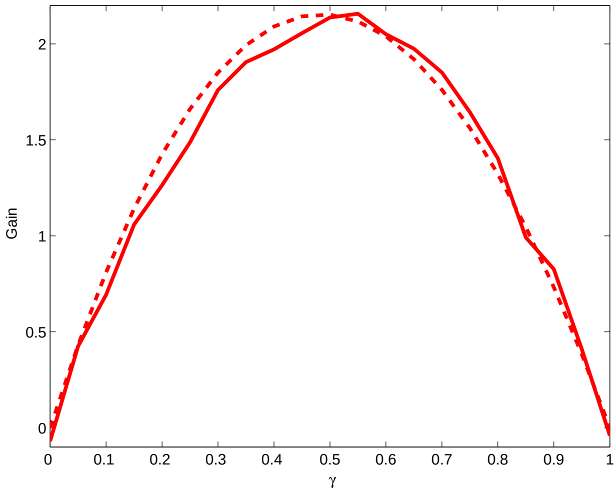



**Figure 3.4.** Comparison between the theoretical and the simulation for the gain vs gamma, for the following set of probabilities : $p = \frac{1}{3}$, $r = \frac{1}{3}$; $p_1 = \frac{3}{25}$, $r_1 = \frac{2}{5}$ and $p_2 = \frac{3}{5}$, $r_2 = \frac{1}{10}$. The simulations were carried out by averaging over 50 000 trials and all possible initial conditions.

games A, B and AB given by (3.30) and (3.31) and the rates obtained through simulations is presented. It is worth noting the good agreement between both results.

It is also interesting to analyze the evolution of the average gain obtained from the random alternation of game A and game B when varying the mixing parameter $\gamma$. In Fig. 3.4 we compare the theoretical curves and the ones obtained through simulations. As in the original games, the maximum gain obtained for this set of parameters is obtained for a value around $\gamma \sim \frac{1}{2}$ [69]. For other sets of the game probabilities, though, the optimal $\gamma$ differs from $\gamma = \frac{1}{2}$.

## Chapter 4

# Relation between Parrondo's games and the Brownian ratchet

Parrondo's games were originally inspired by the model of the flashing ratchet. However, no direct relation was ever established between both. In this chapter we address a quantitative relation between the variables defining a game, i.e., the winning and losing probabilities, and the physical variables defining a Brownian ratchet. Depending on the game considered, a different formulation will be obtained: it will be shown that the original Parrondo's games can be derived from a Langevin equation with additive noise, and Parrondo's games with self–transition can be related to a Langevin equation using multiplicative noise in the sense of Ito.

### 4.1 Additive noise

The evolution in time of the games can be described through a master equation with discrete time $\tau$. This time increases by one at every coin toss. If we denote by $P_i(\tau)$ the probability that at time $\tau$ the capital of the player is equal to $i$, we can write a general master equation as

$$P_i(\tau+1) = a^i_{-1} P_{i-1}(\tau) + a^i_0 P_i(\tau) + a^i_1 P_{i+1}(\tau), \tag{4.1}$$

where $a^i_{-1}$ is the probability of winning when the capital is $i-1$, $a^i_1$ is the probability of losing when the capital is $i+1$, and, for completeness, we have introduced $a^i_0$ as the probability that the capital $i$ remains unchanged (a possibility not considered in the original Parrondo games). In accordance with the rules of the game described in Sec. 2.2, the probabilities $\{a^i_{-1}, a^i_0, a^i_1\}$ do not depend on time and they satisfy the normalization condition $a^{i+1}_{-1} + a^i_0 + a^{i-1}_1 = 1$, which ensures the conservation of probability:

$\sum_{i=-\infty}^{+\infty} P_i(\tau+1) = \sum_{i=-\infty}^{+\infty} P_i(\tau) = 1$ if $\sum_{i=-\infty}^{+\infty} P_i(0) = 1$.

We can rewrite Eq. (4.1) by making use of the normalization condition for the transition probabilities:

$$\begin{aligned} P_i(\tau+1) - P_i(\tau) &= a_{-1}^i P_{i-1}(\tau) + (a_0^i - 1)P_i(\tau) + a_1^i P_{i+1}(\tau) \\ &= a_{-1}^i P_{i-1}(\tau) - (a_{-1}^{i+1} + a_1^{i-1})P_i(\tau) + a_1^i P_{i+1}(\tau) \\ &= a_{-1}^i P_{i-1}(\tau) - a_{-1}^{i+1} P_i(\tau) - a_1^{i-1} P_i(\tau) + a_1^i P_{i+1}(\tau) \\ &= -\left[J_{i+1}(\tau) - J_i(\tau)\right]. \end{aligned} \tag{4.2}$$

where the current $J_i(\tau)$ is given by:

$$J_i(\tau) = \frac{1}{2}\left[F_i P_i(\tau) + F_{i-1}P_{i-1}(\tau)\right] - \left[D_i P_i(\tau) - D_{i-1}P_{i-1}(\tau)\right], \tag{4.3}$$

and $F_i = a_{-1}^{i+1} - a_1^{i-1}$, $D_i = \frac{1}{2}(a_{-1}^{i+1} + a_1^{i-1})$. This form is a consistent discretization of the Fokker–Plank equation for a probability $P(x,t)$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \tag{4.4}$$

with a current

$$J(x,t) = F(x)P(x,t) - \frac{\partial [D(x)P(x,t)]}{\partial x}, \tag{4.5}$$

with an arbitrary drift $F(x)$, and diffusion $D(x)$. If $\Delta t$ and $\Delta x$ are, respectively, the time and space discretization steps, such that $x = i\Delta x$ and $t = \tau\Delta t$, it is clear the identification

$$F_i \longleftrightarrow \frac{\Delta t}{\Delta x}F(i\Delta x), \qquad D_i \longleftrightarrow \frac{\Delta t}{(\Delta x)^2}D(i\Delta x). \tag{4.6}$$

The discrete and continuum probabilities are related by $P_i(\tau) \leftrightarrow P(i\Delta x, \tau\Delta t)\Delta x$ and the continuum limit can be taken by considering that $M = \lim\limits_{\Delta t\to 0, \Delta x \to 0} \frac{(\Delta x)^2}{\Delta t}$ is a finite number. In this case $F_i \leftrightarrow M^{-1}\Delta x F(i\Delta x)$ and $D_i \leftrightarrow M^{-1}D(i\Delta x)$.

From now on, we restrict ourselves to the case $a_0^i = 0$ (which corresponds to the original Parrondo's games). Since $p_i = a_{-1}^{i+1}$ we can rewrite the terms $D_i$, $F_i$ as

$$D_i \equiv D = \frac{1}{2}, \tag{4.7}$$

$$F_i = -1 + 2p_i. \tag{4.8}$$

and the current $J_i(\tau) = -(1-p_i)P_i(\tau) + p_{i-1}P_{i-1}(\tau)$ is nothing but the probability flux from $i-1$ to $i$. We are interested in solving our system for the stationary case. In this regime we know that $P_i(\tau) \equiv P_i^{st}$ and the current does not depend on site $i$, acquiring a constant value $J_i \equiv J$. The stationary solutions for the probability $P_i^{st}$ are found solving the recurrence relation (4.3) for a constant current $J$ together with the boundary condition $P_i^{st} = P_{i+L}^{st}$:

$$P_i^{st} = N\mathrm{e}^{-V_i/D}\left[1 - \frac{2J}{N}\sum_{j=1}^{i}\frac{\mathrm{e}^{V_j/D}}{1-F_j}\right], \qquad J = N\frac{\mathrm{e}^{-V_L/D}-1}{2\sum_{j=1}^{L}\frac{\mathrm{e}^{V_j/D}}{1-F_j}}. \tag{4.9}$$

where $N$ is the normalization constant obtained from $\sum_{i=0}^{L-1} P_i^{st} = 1$. In these expressions we have introduced the potential $V_i$ in terms of the probabilities of the games[1]

$$V_i = -D\sum_{j=1}^{i}\ln\left[\frac{1+F_{j-1}}{1-F_j}\right] = -D\sum_{j=1}^{i}\ln\left[\frac{p_{j-1}}{1-p_j}\right], \tag{4.10}$$

The case of zero current $J = 0$, implies a periodic potential $V_L = V_0 = 0$. This latter condition leads to $\prod_{i=0}^{L-1} p_i = \prod_{i=0}^{L-1}(1-p_i)$ for a fair game, a requirement already obtained when analyzing the games with discrete–time Markov chains, *c.f.* Eq. (2.32). In this case, the stationary solution can be written as the exponential of the potential $P_i^{st} = N\mathrm{e}^{-V_i/D}$. Note that Eq. (4.10) reduces in the limit $\Delta x \to 0$ to $V(x) = -M^{-1}\int F(x)dx$ or $F(x) = -M\frac{\partial V(x)}{\partial x}$, which is the usual relation between the drift $F(x)$ and the potential $V(x)$ with a mobility coefficient $M$.

The inverse problem of obtaining the game probabilities in terms of the potential requires solving Eq. (4.10) for $F_i$ with the boundary condition $F_0 = F_L$ [2]:

$$F_i = (-1)^i\mathrm{e}^{V_i/D}\left[\frac{\sum_{j=1}^{L}(-1)^j[\mathrm{e}^{-V_j/D}-\mathrm{e}^{-V_{j-1}/D}]}{(-1)^L\mathrm{e}^{(V_0-V_L)/D}-1} + \sum_{j=1}^{i}(-1)^j[\mathrm{e}^{-V_j/D}-\mathrm{e}^{-V_{j-1}/D}]\right]. \tag{4.11}$$

These results allow us to obtain the stochastic potential $V_i$ (and hence the current $J$) for a given set of probabilities $\{p_0,\ldots,p_{L-1}\}$, using (4.10); as well as the inverse: obtain the probabilities of the games given a stochastic potential, using (4.11). Note that the game resulting from the alternation, with probability $\gamma$, of a *game A* with $p_i = 1/2$, $\forall i$ and a game B defined by the set $\{p_0,\ldots,p_{L-1}\}$ has a set of probabilities $\{p'_0,\ldots,p'_{L-1}\}$ with $p'_i = (1-\gamma)\frac{1}{2} + \gamma p_i$. For the $F_i$'s variables, this relation yields $F'_i = \gamma F_i$, and the related potential $V'$ follows from (4.10).

We give now two examples of the application of the above formalism. In the first one we compute the stochastic potentials of the fair game B and the winning game AB, the

[1] In this, as well as in other similar expressions, the notation is such that $\sum_{j=1}^{0} = 0$. Therefore the potential is arbitrarily rescaled such that $V_0 = 0$.

[2] The singularity appearing for a fair game $V_L = V_0$ in the case of an even number $L$ might be related to the lack of ergodicity explicitely shown in [31] for $L = 4$. In this case additional conditions on the potential are required for the existence of a fair game, and will be further explained in the next section.

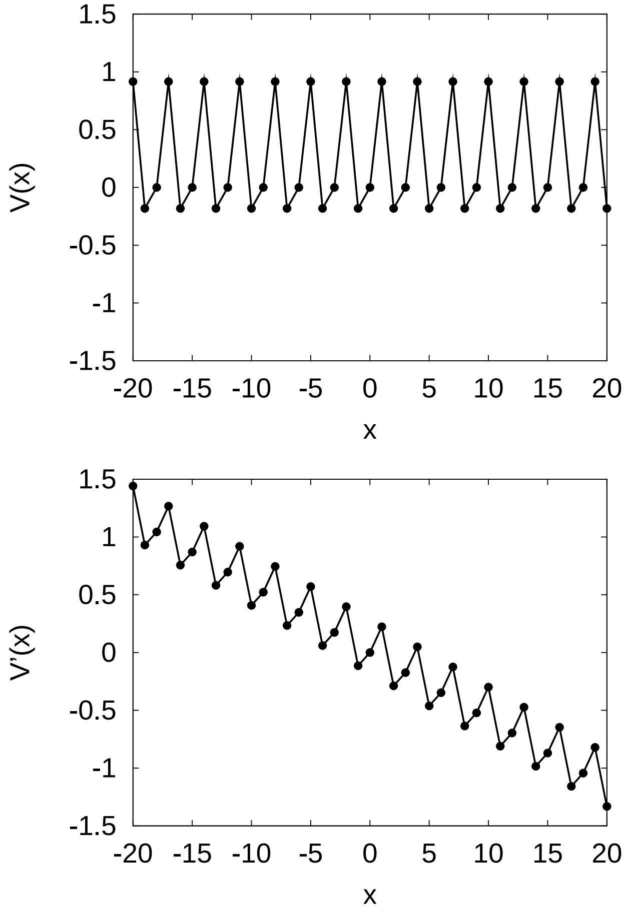



**Figure 4.1.** Upper panel: potential $V_i$ obtained from (4.10) for the fair game B defined by $p_0 = 1/10$, $p_1 = p_2 = 3/4$. Lower panel: potential for the randomized game AB, with $p'_0 = 3/10$, $p'_1 = p'_2 = 5/8$ resulting from the random alternation of game B with a game A with constant probabilities $p_i = p = 1/2$, $\forall i$.

random combination with probability $\gamma = 1/2$ of game B and a game A with constant probabilities, in the original version of the paradox [5]. The resulting potentials are shown in Fig. 4.1. Note that the potential for game B takes different values at each point $i \mod 3$ even though the probabilities were equal for $i = 1, 2 \mod 3$. The resulting asymmetry in the potential is the required one for the existence of the ratchet effect. On the other hand, the potential of the combined game AB has a non-zero and negative mean slope, as it corresponds to a winning game.

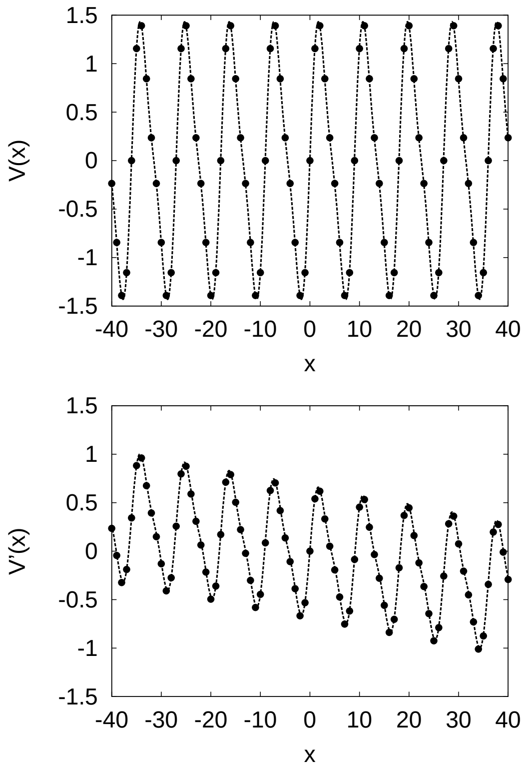



**Figure 4.2.** Upper panel: Ratchet potential (2.2) in the case $L = 9$, $A = 1.3$. The dots are the discrete values $V_i = V(i)$ used in the definition of game B. Lower panel: discrete values for the potential $V_i'$ for the combined game AB obtained by alternating with probability $\gamma = 1/2$ games A and B. The line is a fit to the empirical form $V'(x) = -\Gamma x + \alpha V(x)$ with $\Gamma = 0.009525$, $\alpha = 0.4718$.

The second application considers as input the potential (2.2), setting the time–dependent function $W(t) = 1$, which has been widely used as a prototype for ratchets [70, 71]. Using (4.11) we obtain a set of probabilities $\{p_0, \ldots, p_{L-1}\}$ by discretizing this potential with $\Delta x = 1$, i.e. setting $V_i = V(i)$. Since the potential $V(x)$ is periodic, the resulting game B defined by these probabilities is a fair one and the current $J$ is therefore zero. Game A, as always is defined by $p_i = p = 1/2$, $\forall i$. We plot in Fig. 4.2 the potentials for game B and for the randomized game AB, the random combination with probability $\gamma = 1/2$ of games A and B. Note again that the potential $V_i'$ is tilted as corresponding to a winning game AB. As shown in Fig. 4.3, the current $J$ depends on the probability $\gamma$ for the alternation of games A and B.

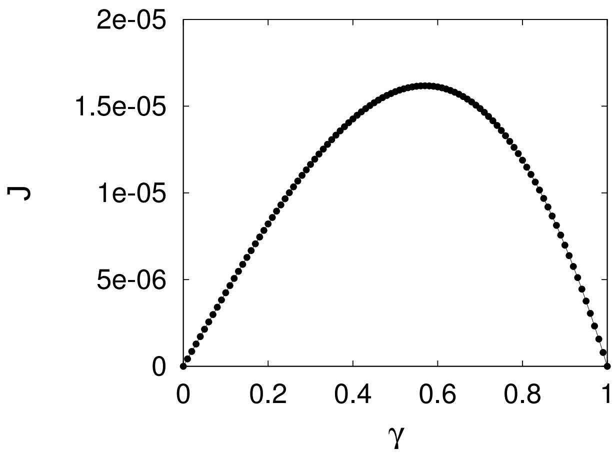



**Figure 4.3.** Current $J$ resulting from equation (4.9) for game AB as a function of the probability $\gamma$ of alternation of games A and B. Game B is defined as the discretization of the ratchet potential (2.2) in the case $A = 0.4$, $L = 9$. The maximum gain corresponds to $\gamma = 0.57$.

## 4.2 The case of $L$ even

A problem arises when finding the probabilities $p_i$ using (4.11) for a periodic potential (corresponding to a fair game) when the number of points $L$ is even. This is obvious since the periodicity condition $V_L = V_0$ gives a zero value for the denominator $(-1)^L \mathrm{e}^{2(V_0 - V_L)} - 1$ in (4.11). In order to be able to find solutions for the probabilities, the numerator has to vanish as well. This is equivalent to the condition:

$$\sum_k e^{-2V_{2k}} = \sum_k e^{-2V_{2k+1}}, \tag{4.12}$$

which, in terms of the stationary probabilities, becomes:

$$\sum_k P_{2k}^{st} = \sum_k P_{2k+1}^{st}. \tag{4.13}$$

This condition implies that one can have a fair game in the case of an even number $L$ only if the probability of finding an even value for the capital equals that of finding an odd value. To our knowledge, this curious property, which emerges naturally from the relation between the potential and the probabilities, has not been reported previously.

It turns out that one has to be careful when discretizing a periodic potential $V(x)$ in order to preserve this property. Otherwise, there will be no equivalent Parrondo game with zero current. The simple identification $V_i = V(i\lambda)$ might not satisfy this requirement, but we have found that a possible solution is to shift the origin of the $x$-axis, i.e. setting $V_i = V((i+\delta)\lambda)$ for a suitable value of $\delta$. For example, in Fig. 4.4 we plot the difference

$$d(\delta) = \sum_i e^{-2V((2i+\delta)\lambda)} - \sum_i e^{-2V((2i+1+\delta)\lambda)}, \tag{4.14}$$

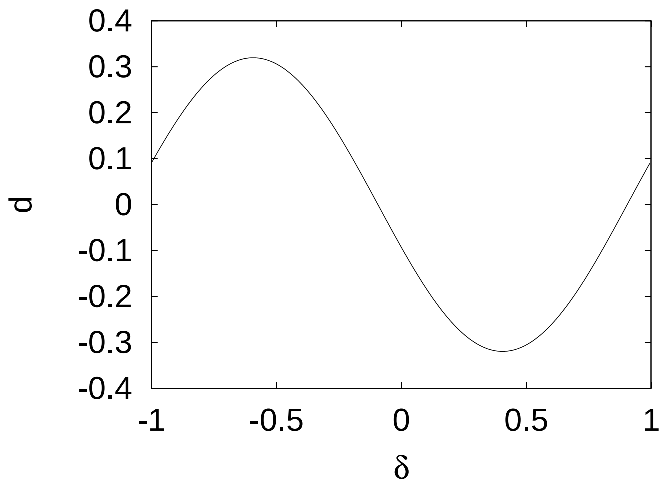



**Figure 4.4.** Plot of $d(\delta)$ as given by Eq. (4.14) versus displacement $\delta$. The unique zero crossing is at $\delta = -0.068616$.

as a function of $\delta$ in the case of the potential (2.2) and $\lambda = 1/4$ (which corresponds to $L = 4$ points per period). We see that there is only one value that accomplishes $d(\delta) = 0$, namely $\delta = -0.068616$.

Once the proper value of $\delta$ is found, it follows from Eq. (4.11) that there are infinitely many solutions for the probabilities. They can be found by varying, say, $p_0$, such that for each value of $p_0$ we will get a set of probabilities $\{p_0, \ldots, p_i, \ldots, p_{L-1}\}$. Solutions satisfying the additional requirement that $p_i \in [0, 1]$, $\forall i$, will exist only for a certain range of values of $p_0 \in [0.0025, 0.68]$. Some of the different solutions are plotted in Fig. 4.5. Some numerical values are :

- $p_0 = 0.125$, $p_1 = 0.8167766$, $p_2 = 0.3927740$, $p_3 = 0.7082539$
- $p_0 = 0.25$, $p_1 = 0.6335531$,$p_2 = 0.5289900$, $p_3 = 0.6070749$
- $p_0 = 0.3525$, $p_1 = 0.4833099$, $p_2 = 0.6406871$, $p_3 = 0.5241081$
- $p_0 = 0.50$, $p_1 = 0.2671062$, $p_2 = 0.8014221$, $p_3 = 0.4047168$

An additional criterion to choose between the different sets of probabilities is to impose the maximum “smoothness” in the distribution of the $p_i$’s. For instance, one could minimized the sum $\sum_{i=0}^{L-1}(p_{i+1} - p_i)^2$. In our example this criterion yields $p_0 = 0.3525$ and the other values follow from the previous table.

## 4.3 Multiplicative Noise

We go now a step forward, and calculate how these previous expressions obtained for the stationary probability, current and the defined potential vary when we consider the

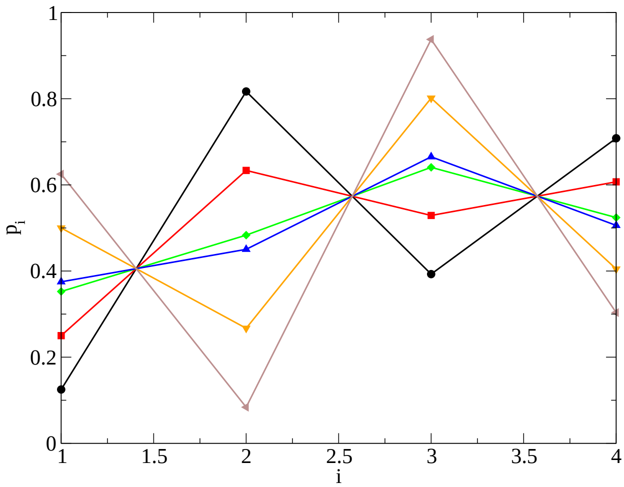



**Figure 4.5.** Multiple solutions for the probabilities $p_i$ obtained with equation (4.10) for a potential like (2.2) with $A = 0.3$, $\lambda = \frac{1}{4}$, $\delta = -0.068616$ varying the value of $p_0$. The continuous line corresponds to the "optimal" solution, $p_0 = 0.3525$ (see the text).

case $a_0^i \neq 0$ (which is equivalent to $r_i \neq 0$). As stated previously, considering this term implies that the player has a certain probability of remaining with the same capital after a round played.

The drift and diffusion terms now read

$$F_i = a_{-1}^{i+1} - a_1^{i-1} = 2p_i + r_i - 1, \tag{4.15}$$

$$D_i = \frac{1}{2}(1 - a_0^i) = \frac{1}{2}(1 - r_i). \tag{4.16}$$

It can be appreciated that both terms, the diffusion $D_i$ as well as the drift $F_i$, may vary on every site. Using Eq. (4.3) and considering the stationary case $P_i(\tau) = P_i^{st}$ together with a constant current $J_i = J$, we may solve for the probability distribution obtaining

$$P_i^{st} = \frac{J}{\frac{1}{2}F_i - D_i} - \left( \frac{\frac{1}{2}F_{i-1} + D_{i-1}}{\frac{1}{2}F_i - D_i} \right) P_{i-1}^{st}. \tag{4.17}$$

The previous equation can be put in a general form as $x_i = a_i + b_i x_{i-1}$, from which a solution can be derived solving recursively for $x_n$,

$$x_n = \left[ \prod_{k=1}^{n} b_k \right] \cdot x_0 + \sum_{j=1}^{n} a_j \cdot \left[ \prod_{k=j+1}^{n} b_k \right]. \tag{4.18}$$

Applying the latter result to the stationary probability we have

$$P_n^{st} = \left[ \prod_{k=1}^{n} \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k} \right] \cdot P_0^{st} - J \sum_{j=1}^{n} \frac{1}{D_j - \frac{1}{2}F_j} \left[ \prod_{k=j+1}^{n} \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k} \right]. \tag{4.19}$$

We can solve for the current $J$ using Eq. (4.17) together with the periodic boundary condition $P_L^{st} = P_0^{st}$

$$J = \frac{P_0^{st} \cdot \left(\prod_{k=1}^{L} \left[\frac{\frac{1}{2}F_{k-1}+D_{k-1}}{D_k-\frac{1}{2}F_k}\right] - 1\right)}{\sum_{j=1}^{L} \frac{1}{D_j-\frac{1}{2}F_j} \prod_{k=j+1}^{L} \left[\frac{\frac{1}{2}F_{k-1}+D_{k-1}}{D_k-\frac{1}{2}F_k}\right]}. \tag{4.20}$$

An *effective potential* can be defined in a similar way to its continuous analog as

$$V_i = -\sum_{j=1}^{i} \ln\left(\frac{1+\frac{1}{2}\frac{F_{j-1}}{D_{j-1}}}{1-\frac{1}{2}\frac{F_j}{D_j}}\right) = -\sum_{j=1}^{i} \ln\left(\frac{\frac{p_{j-1}}{1-r_{j-1}}}{\frac{1-p_j-r_j}{1-r_j}}\right). \tag{4.21}$$

It is important to note that, as in the previous case $a_0^i = 0$, the potential must verify periodic conditions $V_0 = V_L$ when the set of probabilities define a fair game. It is an easy task to check that using Eq. (4.21) together with a periodic boundary condition, what we obtain is the fairness condition for a given set of probabilities defining a Parrondo game with *self–transition* (*c.f.* (3.14)), that is

$$\prod_{k=1}^{L} p_i = \prod_{k=1}^{L} q_i = \prod_{k=1}^{L} (1-p_i-r_i). \tag{4.22}$$

By means of Eq. (4.21) we can obtain the stationary probability (4.19) and current (4.20) in terms of the defined potential as

$$P_n^{st} = \mathrm{e}^{-V_n}\left(\frac{D_0 \cdot P_0^{st}}{D_n} - J\sum_{j=1}^{n} \frac{\mathrm{e}^{V_j}}{D_n\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}\right), \tag{4.23}$$

where

$$J = \frac{P_0^{st}\left[D_0 - D_L \cdot \mathrm{e}^{V_L}\right]}{\sum_{j=1}^{L} \frac{\mathrm{e}^{V_j}}{\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}}. \tag{4.24}$$

These are the new expressions which, together with Eqs. (4.15) and (4.16) allow us to obtain the potential, current and stationary probability for a given set of probabilities $\{p_i, r_i, q_i\}$ defining a Parrondo game with *self–transition*. We will now show that the set of Eqs. (4.21),(4.23),(4.24) can be related in a consistent form with the continuous solutions corresponding to the Fokker–Planck equation of a process with multiplicative noise.

Given a Langevin equation with multiplicative noise

$$\dot{x} = F[x(t), t] + \sqrt{B[x(t), t]} \cdot \xi(t), \tag{4.25}$$

interpreted in the sense of Ito, we can obtain its associated Fokker–Planck equation given by Eq. (4.4) recalling that $D(x,t) = \frac{1}{2}B(x,t)$. The general solution for the stationary probability density function $P(x,t)$ is given by

$$P^{st}(x) = \frac{\mathrm{e}^{\int^{x} \Psi(x)dx}}{D(x)} \cdot \left[ \mathcal{N} - J \int^{x} \mathrm{e}^{-\int^{x'} \Psi(x'')dx''} dx' \right], \tag{4.26}$$

where $\mathcal{N}$ is a normalization constant and $\Psi(x) = \frac{F(x)}{D(x)}$. Making use of the periodicity and the normalization condition $P(0) = P(L)$ and $\int_0^x P(x)dx = 1$ we obtain the following expressions for $\mathcal{N}$ and $J$

$$\mathcal{N} = P(0) \cdot D(0) \qquad J = \frac{P(0) \cdot \left( D(0) - D(L)\mathrm{e}^{\int_0^L \Psi(x)dx} \right)}{\int_0^L \mathrm{e}^{-\int_0^{x'} \Psi(x'')dx''} dx'}. \tag{4.27}$$

Comparing the discrete equations for the current and stationary probability (4.23-4.24) with the continuous solutions (4.26-4.27) we have the following equivalences

$$P_0^{st} \cdot D_0 \equiv P(0) \cdot D(0), \tag{4.28}$$

$$D_j \equiv D(x), \tag{4.29}$$

$$\mathrm{e}^{V_n} \equiv \mathrm{e}^{\int^{x} \Psi(x)dx}, \tag{4.30}$$

$$\sum_{j=1}^{n} \frac{\mathrm{e}^{V_j}}{\left(1 - \frac{1}{2}\frac{F_j}{D_j}\right)} \equiv \int^{x} \mathrm{e}^{-\int^{x'} \Psi(x'')dx''} dx'. \tag{4.31}$$

It is clear the identification of the terms in Eqs. (4.28) and (4.29). Now we need to demonstrate the equivalence given by Eqs. (4.30) and (4.31). If we define a *discretised function* as $\psi_j = \frac{F_{j-1}}{D_{j-1}}$ and we use the Taylor expansion up to first order of the logarithm $\ln(1+x) \approx x$ we get

$$\begin{aligned} V_n &= -\sum_{j=1}^{n} \ln\left(\frac{1+\frac{1}{2}\psi_{j-1}}{1-\frac{1}{2}\psi_j}\right) \approx -\frac{1}{2}\sum_{j=1}^{n}(\psi_{j-1}+\psi_j) = \\ &= -\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1}\psi_k + \frac{1}{2}\psi_n\right), \end{aligned} \tag{4.32}$$

$$\begin{aligned} \sum_{j=1}^{n} \frac{\mathrm{e}^{V_j}}{1-\frac{1}{2}\psi_j} &= \sum_{j=1}^{n} \mathrm{e}^{V_j - \ln\left(1-\frac{1}{2}\psi_j\right)} \approx \sum_{j=1}^{n} \mathrm{e}^{-\frac{1}{2}\left(\sum_{k=1}^{j}[\psi_{k-1}+\psi_k]-\psi_j\right)} = \\ &= \sum_{j=1}^{n} \mathrm{e}^{-\left(\frac{1}{2}\psi_0+\sum_{k=1}^{j}\psi_k+\frac{1}{2}\psi_j\right)+\frac{1}{2}\psi_j}. \end{aligned} \tag{4.33}$$

It can be clearly seen that Eq. (4.32) corresponds to the numerical integration of the function $\Psi(x)$ defined previously, but with a $\Delta = 1$ (the difference in the sign is due to the way we have defined our potential). It can be demonstrated that when $\Delta \neq 1$ both expressions agree up to first order in $\Delta$,

$$V_{n\Delta} = -\Delta\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1}\psi_k + \frac{1}{2}\psi_n\right). \tag{4.34}$$

In the case of Eq. (4.33) what we obtain is nearly the Simpson's numerical integration method but for an extra term. As in the previous case, when $\Delta \neq 1$ then we have up to a first order an extra $\Delta$ term,

$$\sum_{j=1}^{n} \frac{\mathrm{e}^{V_{j\Delta}}}{1-\frac{1}{2}\psi_{j\Delta}} \approx \Delta \cdot \sum_{j=1}^{n} \mathrm{e}^{-\Delta\left(\frac{1}{2}\psi_0+\sum_{k=1}^{j}\psi_{k\Delta}+\frac{1}{2}\psi_{j\Delta}\right)+\frac{1}{2}\Delta\psi_{j\Delta}}. \tag{4.35}$$

So when $\Delta \to 0$ the contribution of the *extra* term can be neglected as compared to that of the sum.

We can also perform the inverse process, that is, to obtain the set of probabilities $\{p_i, r_i, q_i\}$ for a given potential $V_i$. If we define $A_n = \frac{1}{2}\frac{F_n}{D_n} = \frac{p_n - q_n}{p_n + q_n}$, we need only to solve Eq. (4.21) for $A_n$ obtaining

$$A_n = (-1)^n \cdot \mathrm{e}^{V_n}\left[\frac{\sum_{j=1}^{L}(-1)^j(\mathrm{e}^{-V_j}-\mathrm{e}^{-V_{j-1}})}{(-1)^L \cdot \mathrm{e}^{V_0 - V_L} - 1} + \sum_{j=1}^{n}(-1)^j \cdot \left(\mathrm{e}^{-V_j}-\mathrm{e}^{-V_{j-1}}\right)\right]. \tag{4.36}$$

Once these values are obtained, we must solve for the probabilities together with the normalization condition $p_i + r_i + q_i = 1$. Since we have a free parameter in the set of solutions, we can fix the $r_i$ values on every site and the rest of parameters can be obtained through

$$p_i = \frac{1}{2}(1 + A_i)(1 - r_i), \tag{4.37}$$

$$q_i = \frac{1}{2}(1 - A_i)(1 - r_i). \tag{4.38}$$

In this way what we have is a method for inverting an *effective* potential, fixing a parameter that in our case is the diffusion in every site (remember that the parameter $r_i$ is related to the diffusion coefficient by Eq. (4.16) or is also equivalent to the temperature).

The fact that we can obtain different sets of probabilities, both describing different dynamics but coming from the same potential $V(x)$, it is not surprising. We need only to remember that a system with multiplicative noise is equivalent, in the sense that both possess the same stationary probability distribution, to another system with additive noise

$$\dot{x} = F(x) + D(x) \cdot \xi(t) \longrightarrow \dot{x} = \bar{F}(x) + \xi(t), \tag{4.39}$$

but with a renormalized drift term $\bar{F}(x)$ given by $\bar{F}(x) = -\frac{\partial \bar{V}}{\partial x}$, where $F(x) = -\frac{\partial V}{\partial x}$ and $\bar{V} = \int \frac{F(x)}{D(x)} dx + \ln D(x)$.

## Chapter 5

# Parrondo's games and Information theory

Recently, Arizmendi *et. al* [37] quantified the transfer of information – negentropy – between a Brownian particle and the nonequilibrium source of fluctuations acting on it. These authors coded the particle motion of a flashing ratchet into a string of $0$'s and $1$'s according to whether the particle had moved to the left or to the right respectively, and then compressed the resulting binary file using the Lempel and Ziv algorithm (see Sec. 1.5.2 for details). They obtained in this way an estimation of the entropy per character $h$ as the ratio between the lengths of the compressed and the original file, for a sufficiently large file length. They applied this method to estimate the entropy per character of the ergodic source for different values of the flipping rate, with the result that there exists a close relation between the current in the ratchet and the net transfer of information in the system. The aim of the present Chapter is to apply this technique to the discrete–time and space version of the Brownian ratchet, i.e., Parrondo's games.

## 5.1 Parrondo's games and Information Theory

Some previous works in the literature have related Parrondo's games and information theory. Pearce [72] considers the relation between the entropy and the fairness of the games, and the region of the parameter space where the entropy of game A is greater than that of B and the randomized game AB. Harmer *et. al* [73] study the relation between the fairness of games A and B and the entropy rates considering two approaches. The first one calculates the entropy rates not taking into account the correlations present on game B, finding a good agreement between the region of maximum entropy rates and the region of fairness. The second approach introduces these correlations, obtaining lower entropy rates and no significant relation between fairness and entropy rates for game B.

The goal of this chapter is to relate the current or gain in Parrondo's games with the variation of information entropy of the binary file generated using techniques similar to those in [37]. In the next section we will present numerical results coming from simulations of different versions of Parrondo's games: in the cooperative games [38, 67], one considers an ensemble of interacting players; in the history dependent games [4, 68], the probabilities of winning depend on the history of previous results of wins and loses; finally, in the games with self–transition (*c.f.* Chapter 3), there is a non–zero probability $r_i$ that the capital remains unchanged (not winning or losing) in a given toss of the coins. Finally, we offer in Sec. 5.3, a theoretical analysis that helps to understand the behavior observed in the simulations.

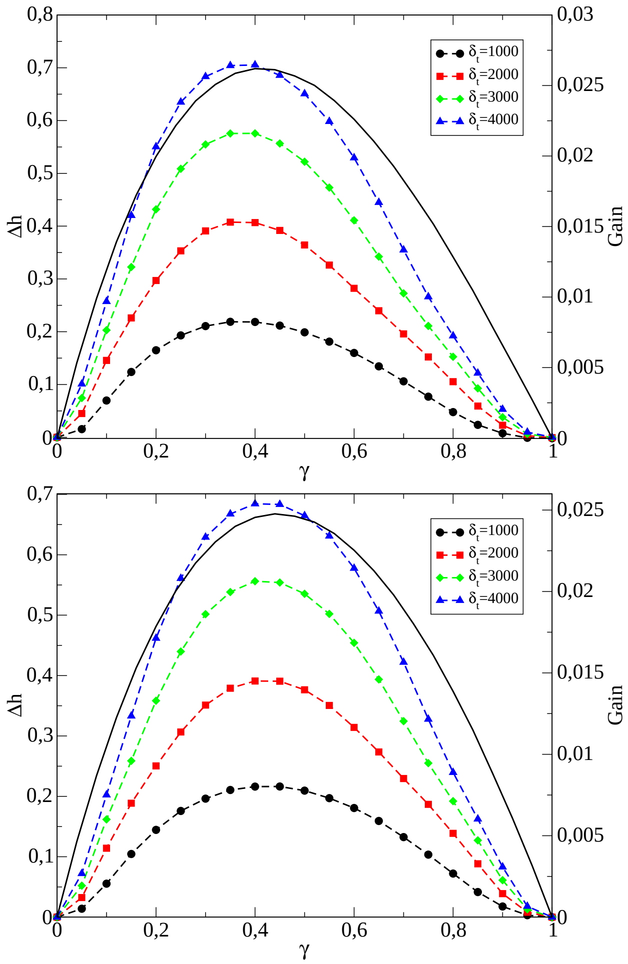



**Figure 5.1.** Comparison of the average gain per game (solid line) with the entropy difference $\Delta h$ (symbols) as a function of the switching rate $\gamma$, for several values of the delay time $\delta_t$, as shown in the legend, and the following versions of the Parrondo's games: Upper panel: Original Parrondo's combination of games A and B with probabilities: $p = \frac{1}{2}$, $p_0 = \frac{1}{10}$ and $p_1 = \frac{3}{4}$. Lower panel: Parrondo's combination of games $A$ and $B$ including self–transitions. The values for the probabilities are: $p = \frac{9}{20}$, $r = \frac{1}{10}$, $p_0 = \frac{3}{25}$, $r_0 = \frac{2}{5}$, $p_1 = \frac{3}{5}$ and $r_1 = \frac{1}{10}$.

## 5.2 Simulation results

We have performed numerical simulations of the different versions of the games. In every case, the evolution of the capital of the player has been converted to a string of bits where bit 0 (resp., 1) corresponds to a decrease (resp., increase) of the capital after $\delta_t$ plays of the games. It will be shown that the delay time $\delta_t$ between capital measurements is a relevant parameter.

An estimation of the entropy per character $h$, is obtained as the compression ratio obtained with the `gzip` (v. 1.3) program, that implements the Lempel and Ziv algorithm (although it has been stressed by some authors that this is not the best compressing algorithm one can find in the literature). The simplicity in the use of this algorithm (as it is already implemented "for free" in many operating systems) is an added value, as it will become apparent in the following when we consider strings of symbols generated by more than one ergodic source. As suggested in [37], we expect that the negentropy, $-h$, which accounts for the known information about the system, is related in some way with the average gain in the games.

In the upper panel of Fig. 5.1 we compare the average gain in the randomized game AB with the value of the entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ as a function of the probability $\gamma$ and for different delay times $\delta_t$. We find indeed a qualitative agreement between the increase in the gain and the decrease in entropy as the $\gamma$ parameter is varied. This decrease in the entropy of the system implies that there exists an increase in the amount of known information about the system. Notice that the compression rate depends on $\delta_t$, and that the $\gamma$ value for which there is the maximum decrease in entropy agrees with the value for the maximum gain in the games. This agreement is similar to the one observed when applying this technique to the Brownian flashing ratchet [37].

Similar results are obtained in other versions of Parrondo's games. For instance, in the lower panel of Fig. 5.1 we compare the average gain and the entropy difference in the games with self–transition [74]. Again in this case the maximum gain coincides with the $\gamma$ value for the minimum entropy per character for all values of $\delta_t$.

Finally, in Fig. 5.2 we present the comparison in the case of the history dependent games [4] (upper panel), and cooperative games [67] (lower panel), showing all of them the same features as in previous cases. We may conclude from these results that there exists, as it happens for the Brownian ratchet, a close relation between the entropy and the average gain. In the next section we will develop a simple argument that helps explaining this relation.

## 5.3 Theoretical analysis

As stressed in Sec. 5.1, the entropy per character of a text produced by an ergodic source is[1] $H = -\sum_i p_i \cdot \log(p_i)$, where $p_i$ denotes the probability that the source will emit a given symbol $\alpha_i$, and the sum is taken over all possible symbols that the source can emit. For instance, if we consider game A as a source of two symbols, $0$ (losing) and $1$ (winning) , the Shannon entropy according as a function of the probability $p$ of emitting symbol $1$ (i.e. the probability of winning) is given by Eq. (1.96). In Fig. 5.3 we compare

[1] Units are taken such that all logarithms are base 2.

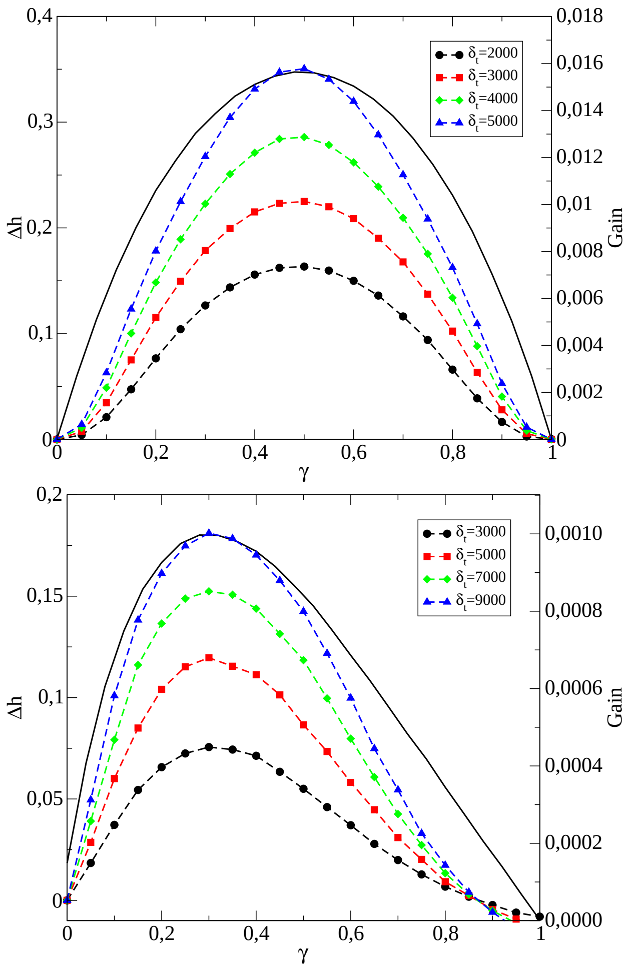



**Figure 5.2.** Same as Fig. 5.1 in other versions of Parrondo's games: Upper panel: History dependent games, alternating between two games with probabilities: $p_1 = \frac{9}{10}$, $p_2 = p_3 = \frac{1}{4}$, $p_4 = \frac{7}{10}$; $q_1 = \frac{2}{5}$, $q_2 = q_3 = \frac{3}{5}$ and $q_4 = \frac{2}{5}$.
Lower panel: Cooperative Parrondo's games with probabilities: $p = \frac{1}{2}$, $p_1 = 1$, $p_2 = p_3 = \frac{16}{100}$, $p_4 = \frac{7}{10}$ and $N = 150$ players

this expression with the compression factor $h$ obtained using the `gzip` algorithm. As shown in this figure for the case of a single source, the compression factor of the `gzip` algorithm does give a good approximation to the Shannon entropy.

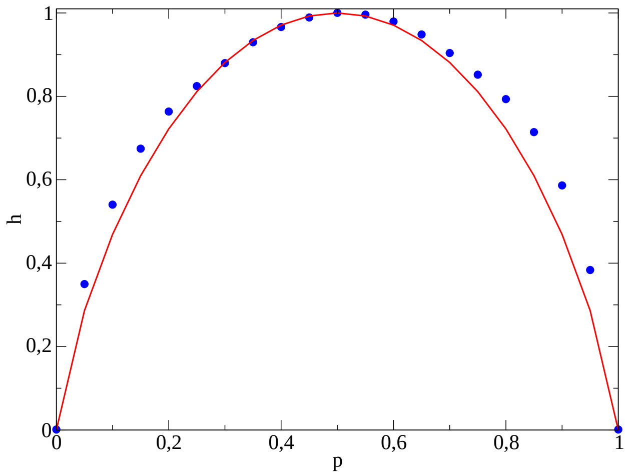



**Figure 5.3.** Comparison between the theoretical value obtained for the Shannon entropy – solid line – with the numerical values – circles – obtained with the `gzip` algorithm for a single source emitting two symbols with probability $p$.

From now on, we restrict our analysis to the case of the original Parrondo's paradox combining games A and B, as explained in Sec. 2.2. For the combined game AB we must distinguish two states, that is, when the capital is multiple of three and when it is not. Therefore, we can think of the randomized game AB as originated by two sources depending on whether the capital is multiple of 3 or not. The probability of emitting symbol 1 when using the first source will be denoted by $q_0$, whereas the same probability will be $q_1$ when using the second source.

Let us first consider the case $\delta_t = 1$, i.e. we store the capital after each single play of the games. According to the expression (1.97) for the entropy of a mixed source, the Shannon entropy for the combined game AB is:

$$H = -\Pi_0[q_0 \log(q_0) + (1-q_0)\log(1-q_0)] - (1-\Pi_0)[q_1 \log(q_1) + (1-q_1)\log(1-q_1)], \tag{5.1}$$

being $\Pi_0$ the stationary probability than in a given time the capital is a multiple of 3. From the Markov chain analysis in Sec. 2.2.2 we know that the stationary probability $\Pi_0$ is given by

$$\Pi_0 = \frac{1 - q_1 + q_1^2}{3 - q_0 - 2q_1 + 2q_0q_1 + q_1^2}. \tag{5.2}$$

In Fig. 5.4 we compare the Shannon entropy $H$ given by the previous formula with the numerical compression factor $h$ as a function of the probability $\gamma$ of mixing games A and B. Although certainly not as good as in the case of a single game, in this case, the `gzip` compression factor gives a reasonable approximation to the Shannon entropy of the combined game AB. It is worth noting that in this case of $\delta_t = 1$ the entropy increases with $\gamma$, corresponding to a decrease of the information known about the system. In order to relate the entropy difference with the current gain, we need to consider larger values for $\delta_t$.

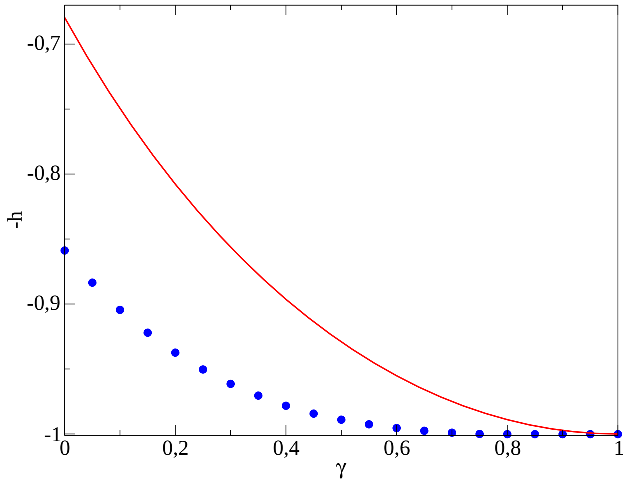



**Figure 5.4.** Plot of Shannon negentropy (solid line) for the combination game AB according to expression (5.1), together with the numerical values (circles) obtained with the compression factor of the `gzip` algorithm in the case when $\delta_t = 1$ step.

For $\delta_t \gg 1$ the system gradually loses its memory about its previous state. Therefore, the different measures are statistically independent and they can be considered as generated by a single ergodic source. For this single source, the probability of winning after one single play of the games is $p_w = \Pi_0\, q_0 + (1 - \Pi_0)\, q_1$. However, we are interested in calculating the winning probability $p_>$ after $\delta_t$ plays. In order to have a larger capital after $\delta_t$ plays it is necessary that the number of wins overcomes the number of losses in single game plays. The distribution of the number of wins follows a binomial distribution and the probability $p_>$ is given by:

$$p_> = \sum_{k=0}^{\frac{\delta_t}{2}} \binom{\delta_t}{k} \cdot p_w^{\delta_t - k} \cdot (1 - p_w)^k. \tag{5.3}$$

The corresponding Shannon entropy for this single source is:

$$H = -p_> \cdot \log(p_>) - (1 - p_>) \cdot \log(1 - p_>). \tag{5.4}$$

We compare in Fig. 5.5 the Shannon entropy coming from this formula and the one obtained by the compression ratio of the `gzip` program for two different values of $\delta_t = 500, 1000$. In both cases, there is a reasonable agreement between both results. Moreover, as shown in Figs. 5.1 and 5.2 the entropy follows closely the average gain of the combined games.

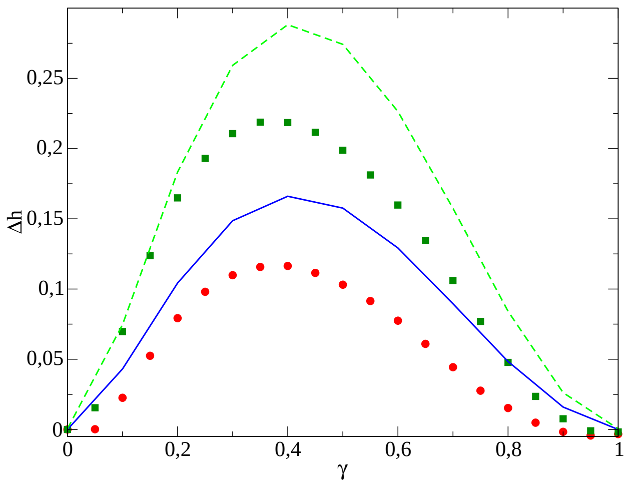



**Figure 5.5.** Plot of Shannon entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ according to formulas (5.4) and (5.3) for $\delta_t = 500$ (solid line) and $\delta_t = 1000$ (dashed line) together with the numerical curves obtained with the compression ratio of the `gzip` algorithm for the same values of $\delta_t = 500$ (circles) and $\delta_t = 1000$ (squares).

# Chapter 6

# Efficiency of discrete–time ratchets

Since the field of Brownian ratchets acquired its importance, there have been numerous studies on the energetics of these microscopic devices [70, 75–77]. However, finding the ratchet efficiency in the discrete case uptil now has been an outstanding open problem. We have shown earlier in Chapter 4 the connection established between the Fokker-Planck equation associated to a Brownian ratchet, and the master equation describing Parrondo's games. Therefore, Parrondo's games can be considered as being a discrete–time and discrete–space version of the continuous flashing ratchet model [56, 78].

While this approach gives much insight and allows straightforward development of games starting from suitable potentials, finding the correct formalism for describing the efficiency of the discrete ratchet and relating it back to the continuous case, has been problematic [8].

It is the aim of this Chapter to deepen this relationship in order to calculate the efficiency of the games. We develop an efficient method for obtaining the stationary probabilities and probability current for a discrete–time and space ratchet in terms of a potential function. We combine the new methods presented herein together with known results from ratchet theory in order to calculate the efficiency of the discrete ratchet. This allows to gain new insight into the games behavior by quantifying the relation between the gain and the dissimilarity between games A and B.

The Chapter is organized as follows: in Sec. 6.1 we present our theoretical model, followed in Sec. 6.2 of the calculation of the efficiency.

# 6.1 Theoretical model

## 6.1.1 Continuous model

We consider the following version of the flashing ratchet: let $x(t)$ represent the position of a Brownian particle whose dynamics can be described through the Langevin equation

$$\dot{x}(t) = -V'(x) \cdot \zeta(t) + f + D(x) \cdot \xi(t), \tag{6.1}$$

where

1. $\xi(t)$ accounts for white noise,
2. $\zeta(t)$ is a form of dichotomous noise that switches on (state B, $\zeta(t) = 1$) and off (state A, $\zeta(t) = 0$) the potential $V(x)$,
3. $f$ is a constant external force acting on the particle
4. $D(x)$ is the diffusion function.

If $V(x)$ is periodic $V(x + L) = V(x)$, then the individual dynamics corresponding to the off and on states both yield $\langle x(t)\rangle = 0$ (for $f = 0$). However, it is known that if the potential has a certain degree of spatial asymmetry, the combined dynamics can rectify the white noise fluctuations obtaining directed motion, $\langle x(t)\rangle \neq 0$, this is the case of the flashing ratchet. Without loss of generality, we will consider $V(x)$ to be of the form given by Eq. (2.2) but setting $W(t) = 0$, that is,

$$V(x) = V_0 \left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{4\pi x}{L}\right)\right], \tag{6.2}$$

although other similar potentials can perform the same task.

It can be demonstrated that the previous Langevin equation is equivalent to a set of Fokker-Planck equations describing the transitions of the particle between states A and B [56, 78] as:

$$\frac{\partial P_A(x,t)}{\partial t} = -\frac{\partial J_A(x,t)}{\partial x} - \omega_{A\to B} P_A(x,t) + \omega_{B\to A} P_B(x,t), \tag{6.3}$$

$$\frac{\partial P_B(x,t)}{\partial t} = -\frac{\partial J_B(x,t)}{\partial x} - \omega_{B\to A} P_B(x,t) + \omega_{A\to B} P_A(x,t), \tag{6.4}$$

where $P_A(x,t)$ (resp. $P_B(x,t)$) denotes the probability of finding the particle in state A (resp. B) at a given position $x$ and time $t$. The $\omega_{\alpha\to\beta}$ term accounts for the transition rate between states $\alpha$ and $\beta$. The probability currents $J_A$ and $J_B$ are given by the expressions

$$
\begin{aligned}
J_A(x,t) &= f\, P_A(x,t) - \frac{\partial[D(x)P_A(x,t)]}{\partial x},\\
J_B(x,t) &= [f - V'(x)]P_B(x,t) - \frac{\partial[D(x)P_B(x,t)]}{\partial x}.
\end{aligned} \tag{6.5}
$$

In this model, once attained the stationary regime we have $P_A(x,t) = P_A(x)$ and $P_B(x,t) = P_B(x)$; the total current in this regime is constant [79] and given by $J = J_A(x) + J_B(x)$.

### 6.1.2 Discrete model

Based on the previous model, we can elaborate a set of equations describing the evolution of the capital when alternating between games A and B, namely, they would be equivalent to the set (6.3),(6.4) but for discrete time and space. The set of master equations are

$$
\begin{aligned}
P_i^A(\tau+1) &= (1-\gamma_{A\to B})[p_{i-1}^A P_{i-1}^A(\tau) + r_i^A P_i^A(\tau) + q_{i+1}^A P_{i+1}^A(\tau)]+\\
&\qquad \gamma_{B\to A}[p_{i-1}^B P_{i-1}^B(\tau) + r_i^B P_i^B(\tau) + q_{i+1}^B P_{i+1}^B(\tau)], \qquad (6.6)\\
P_i^B(\tau+1) &= (1-\gamma_{B\to A})[p_{i-1}^B P_{i-1}^B(\tau) + r_i^B P_i^B(\tau) + q_{i+1}^B P_{i+1}^B(\tau)]+\\
&\qquad \gamma_{A\to B}[p_{i-1}^A P_{i-1}^A(\tau) + r_i^A P_i^A(\tau) + q_{i+1}^A P_{i+1}^A(\tau)], \qquad (6.7)
\end{aligned}
$$

where $P_i^A(\tau)$ is the probability that the player plays game A with a capital $i$ at time $\tau$; $p_i^A$, $r_i^A$ and $q_i^A$ are the probabilities of winning, drawing and losing, respectively, when playing game $A$ with a capital $i$, and a similar notation for game $B$. They satisfy the normalization condition $p_i^A + r_i^A + q_i^A = 1$. This notation generalizes the original games for which the *self–transition probabilities* are $r_i^A = r_i^B = 0$. Note that the probabilities $p_i^A$, $r_i^A$, $q_i^A$, $p_i^B$, $r_i^B$ and $q_i^B$ repeat periodically $p_{i+L}^A = p_i^A$, etc. with periodicity $L$.

For the original Parrondo games we know that period $L = 3$ and the winning probabilities are given by $p_i^A = \frac{1}{2} - \varepsilon$, $p_0^B = \frac{1}{10} - \varepsilon$, $p_1^B = p_2^B = \frac{3}{4} - \varepsilon$. Finally, $\gamma_{\alpha\to\beta}$ accounts for the transition probability between state $\alpha$ and $\beta$. The particular case considered in the original games in which the probability of playing game $A$ and $B$ is $\gamma$ and $1-\gamma$, respectively, independently of the previously played game, implies that $\gamma_{AB} = 1-\gamma_{BA} = 1-\gamma$.

Then, following the approach of Chapter 4, it is possible to rewrite Eqs. (6.3),(6.4) in the Fokker-Planck form as

$$
P_i^A(\tau+1) - P_i^A(\tau) = -\left[J_{i+1}^A(\tau) - J_i^A(\tau)\right] - \gamma_{A\to B}P_i^A(\tau) + \gamma_{B\to A}P_i^B(\tau), \tag{6.8}
$$

$$
P_i^B(\tau+1) - P_i^B(\tau) = -\left[J_{i+1}^B(\tau) - J_i^B(\tau)\right] + \gamma_{A\to B}P_i^A(\tau) - \gamma_{B\to A}P_i^B(\tau), \tag{6.9}
$$

with a current $J_i^A(\tau) = \frac{1}{2}\left[F_i^A P_i^A(\tau) + F_{i-1}^A P_{i-1}^A(\tau)\right] - \left[D_i^A P_i^A(\tau) - D_{i-1}^A P_{i-1}^A(\tau)\right]$, where $F_i^A = p_i^A - q_i^A$, $D_i^A = \frac{1}{2}(1 - r_i^A)$, and similarly for $J_i^B(\tau)$. This form stresses the similarity of the continuum and discrete descriptions (compare with Eqs. (6.3)-(6.5)) and it is easy to show that the currents are also given by the net flux between consecutive states, i.e., $J_i^A(\tau) = p_i^A \cdot P_i^A(\tau) - q_{i+1}^A \cdot P_{i+1}^A(\tau)$ and $J_i^B(\tau) = p_i^B \cdot P_i^B(\tau) - q_{i+1}^B \cdot P_{i+1}^B(\tau)$.

In general, it is not possible to solve the previous equations to obtain the probabilities $P_i^{A,B}(\tau)$ as a function of the set of probabilities $p_i^{A,B}$, $r_i^{A,B}$, $q_i^{A,B}$ and $\gamma_{\alpha\to\beta}$, even in the steady state where the left-hand-sides of (6.8) and (6.9) vanish. A remarkable exception is that of the case $\gamma_{AB} = 1 - \gamma_{BA} = 1 - \gamma$ discussed above. In this case it turns out that the total probability $P_i(\tau) = P_i^A(\tau) + P_i^B(\tau)$ satisfies a master equation

$$P_i(\tau+1) = p_{i-1}P_{i-1}(\tau) + r_iP_i(\tau) + q_{i+1}P_{i+1}(\tau), \tag{6.10}$$

where $p_i = \lambda p_i^A + (1-\lambda)p_i^B$, $r_i = \lambda r_i^A + (1-\lambda)r_i^B$ and $q_i = \lambda q_i^A + (1-\lambda)q_i^B$. Furthermore, it is possible to show that the steady state solutions satisfy $P_i^A = \gamma P_i$ and $P_i^B = (1-\gamma)P_i$. This result allows us to find an analytic solution to Eqs. (6.6),(6.7) for $P_i^A$ and $P_i^B$ in the stationary regime. The solution is based upon on the corresponding expression derived from Eq. (6.10) in the periodic steady-state regime for $P_i$ (see Chapter 4 for further details)

$$P_i = \mathrm{e}^{-V_i}\left(\frac{D_0\cdot P_0}{D_i} - J\sum_{j=1}^{i}\frac{\mathrm{e}^{V_j}}{D_i\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}\right), \tag{6.11}$$

where $F_i = \gamma F_i^A + (1-\gamma)F_i^B$, $D_i = \gamma D_i^A + (1-\gamma)D_i^B$ and the value of $P_0$ has to be found using the normalization condition $\sum_{i=0}^{L-1} P_i = 1$. The potential $V_i$ is given by Eq. (4.21), and the total current $J$ can be obtained from Eq. (4.24) and coincides with the net flux between states $i$ and $i+1$, $J = p_i\, P_i - q_{i+1}\, P_{i+1}$. Notice that although $J = J_i^A + J_i^B$ is a constant independent of $i$ in the steady state, it can not be assured that $J_i^A$ and $J_i^B$ are constant as well. Finally, the average gain is obtained by multiplying the current by the periodicity of the system, i.e. $G = JL$.

In Fig. 6.1 we have plotted the stationary probabilities $P_i^A$, $P_i^B$ for the case $L = 3$. We can see the agreement between the stationary probability distribution $\Pi_i$ obtained through the analysis with discrete–time Markov chains (*c.f.* 2.21) and their corresponding equivalents $P_i^A + P_i^B$ obtained with the current method.

Let us remark that in the case of playing a single game, either game $A$ or $B$ (corresponding formally to setting $\gamma = 1$ or $\gamma = 0$, respectively) it is possible to obtain the corresponding steady state solutions in term of potential functions $V_i^A$ and $V_i^B$, defined as in Eq.(4.21) but using the corresponding probabilities $(p_i^A, q_i^A, r_i^A)$ or $(p_i^B, q_i^B, r_i^B)$ instead of $(p_i, q_i, r_i)$.

So far, we have introduced a method that allows the calculation of the stationary properties, such as the probabilities $P_i^A$, $P_i^B$ and $P_i$, and the currents $J_i^A$, $J_i^B$ and $J$. We turn now to the problem of evaluating efficiency for our discrete–time system. This has been problematic [8] because there was no clear way of evaluating the energy input or energy output of the system when dealing only with probabilities defining the games.

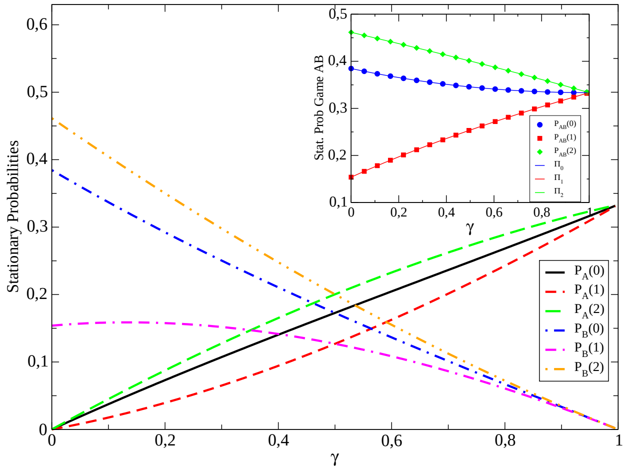



**Figure 6.1.** Plot of the stationary probabilities for games $A$ and $B$ versus the mixing probability $\gamma$. The inset shows that the sum of both probabilities $P_{AB}(i) = P_i^A + P_i^B$ agrees with the expressions obtained for the stationary probabilities $\Pi_i$ obtained for the mixed game AB through Markov chain analysis.

With the formalism introduced earlier, a direct relation can be established between these games and the physical model of the ratchets so as to obtain an estimation of the efficiency for the discrete–time case.

## 6.2 Efficiency

Let us now evaluate the efficiency of our system. We will use the definition of the efficiency as the ratio $\eta = \mathcal{E}_{\text{out}}/\mathcal{E}_{\text{in}}$, and we will provide with suitable definitions for the energy output, $\mathcal{E}_{\text{out}}$, and input, $\mathcal{E}_{\text{in}}$, of the system.

Let us begin by $\mathcal{E}_{\text{in}}$, defined as the energy that must be supplied to the system for switching between the two potentials. In order to evaluate this energy input in our system we need potential functions related to each of the two games. Therefore, if we are dealing with probabilities defining our games A and B, we will make use of Eq. (4.21) for obtaining the potential for each game.

The energy input can be calculated theoretically by means of a probability flux balance. In the stationary regime, the net flux from a given game, say game A, and state $i$, towards the other game B and the same state $i$ can be calculated through the difference equation $J_i^{A\to B} = J_{i-1}^A - J_i^A$. Clearly the net current $J_i^{A\to B}$ equals the opposite current from game B to game A, that is, $J_i^{A\to B} = -J_i^{B\to A}$, where $J_i^{B\to A} = J_{i-i}^B - J_i^B$ (see Fig.6.2). Therefore the input energy can now be obtained as $\mathcal{E}_{\text{in}} = \sum_{i=0}^{L-1} J_i^{A\to B} \cdot (V_i^B - V_i^A)$.

For the energy output, we will use the definition introduced in [80], where $\mathcal{E}_{out}$ is defined as the *minimum* energy input $\mathcal{E}_{in}$ required to accomplish the same task as the engine. The novelty of this definition is that it permits the evaluation of the efficiency for a Brownian particle even in the absence of an external load $f$ (it includes in the evaluation of the power output the work done by the Brownian particle against the friction force). This leads to $\mathcal{E}_{out} = fv + \Gamma v^2$, being $v$ the mean velocity of the Brownian particle and

$\Gamma$ the friction coefficient. In our system $\Gamma$ has been rescaled to 1 and the mean velocity corresponds to the average gain $G$. We thus obtain $\mathcal{E}_{\text{out}} = fJL + J^2L^2$ as the expression to be used for determining the energy output of our system. Once the expressions for the energy input and energy output of the system have been obtained, we can compute the efficiency for both fair and biased games.

In the case of fair games, $\epsilon = 0$ leads to $f = 0$ and $\mathcal{E}_{\text{out}} = J^2L^2$. We consider first the original Parrondo games as defined before. The results are shown in Fig.6.3 where we plot the energy input, energy output and the efficiency for those games, as a function of the mixing probability $\gamma$. Notice that the efficiency attains its maximum value, $\eta = 0.011$, at $\gamma = 0.362$ approximately, as seen in Fig. 6.3(c).

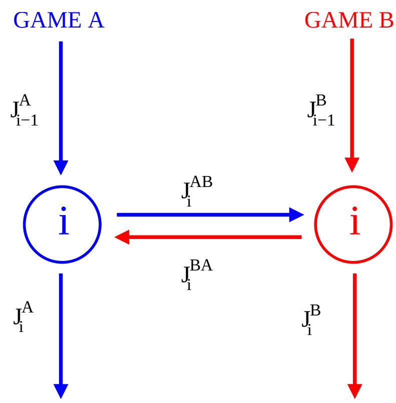



**Figure 6.2.** Diagram showing the net probability current $J^{AB}$ from game $A$ to game $B$.

We consider now still fair games $A$ and $B$, but now the probabilities $p_i^A$ and $p_i^B$ are obtained from suitable ratchet potentials $V^A(x)$ and $V^B(x)$. In particular we choose a flat potential $V^A(x) = 0$ while $V^B(x)$ is given by Eq. (6.2) with $L = 5$ and $V_0 = 0.35$. For the fair games considered here, the force is $f = 0$. The probabilities $p_i^A$ and $p_i^B$ are obtained by inverting Eq. (4.21) (recall the trivial result $p_i^A = 1/2$). The results are displayed in Fig. 6.4. Notice that the the maximum value for the efficiency $\eta = 3.554 \times 10^{-3}$ is obtained when $\gamma = 0.358$.

In these two cases of fair games the system possesses a low efficiency mainly because it works in an irreversible manner, far from its equilibrium state. It is worth remarking that the magnitude obtained for the efficiency agrees with other studies for the *on-off* ratchets [81, 82].

Now we turn to biased games and study the dependence of the efficiency on the parameter $f$. Given a set of probabilities $p_i$ defining a game it is possible to compute $f$ as the average slope, $(V_L - V_0)/L$, of the associated potential $V_i$ given by Eq. (4.21). Applying this method to games $A$ and $B$ of the original Parrondo paradox, it is possible to relate $f$ to the biasing parameter $\epsilon$. However, the average slope, $f_A$, resulting from game A is different from the slope $f_B$ resulting from game B. Since we want to study the effect that a common force $f$ has on the efficiency, we have chosen a different approach: we first compute the potentials $V_i^A$ and $V_i^B$ using the unbiased probabilities $p_i^A$ and $p_i^B$ with $\epsilon = 0$, then we modify the potentials by tilting them with a common slope, $V'^A_i = V_i^A - fi$ and $V'^B_i = V_i^B - fi$, and then compute the probabilities of the biased game $p'^A_i$ and $p'^B_i$ using the inverse of Eq. (4.21). The energy input, output and efficiency are then computed using the above defined formalism with the potentials $V'^A_i$ and $V'^B_i$.

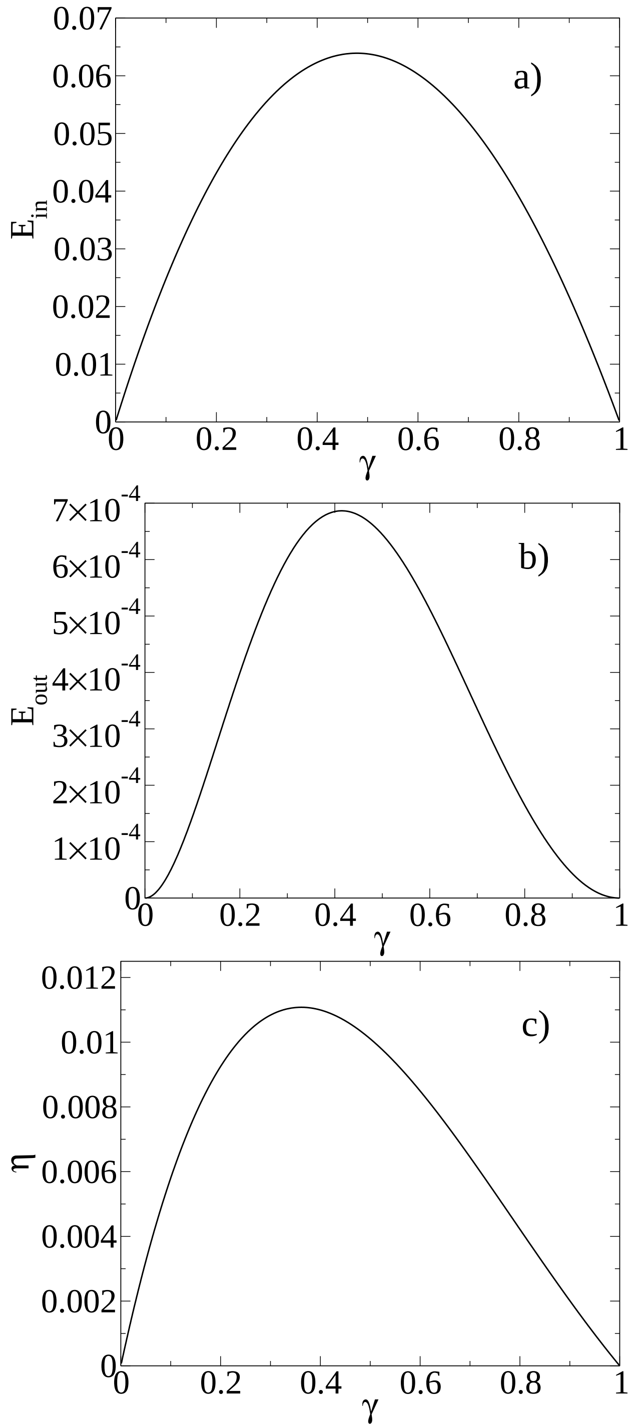



**Figure 6.3.** In these figures, we consider the original Parrondo games defined by a period $L = 3$ and the following set of probabilities $p_i^A = \frac{1}{2} - \varepsilon$, $p_0^B = \frac{1}{10} - \varepsilon$, $p_1^B = p_2^B = \frac{3}{4} - \varepsilon$ in the fair case $\epsilon = 0$. Using the analogy explained in the text, we have computed the energy input (a) and energy output (b) as a function of the mixing probability $\gamma$. The maximum energy input is at $\gamma \approx 0.479$, close to the case of maximum alternation between the games, whereas the maximum for the energy output (or the maximum gain $G$) is located at $\gamma \approx 0.415$. The efficiency $\eta = \mathcal{E}_{in}/\mathcal{E}_{out}$ is displayed in panel (c). Its maximum value $\eta = 0.011$ occurs at $\gamma \approx 0.362$.

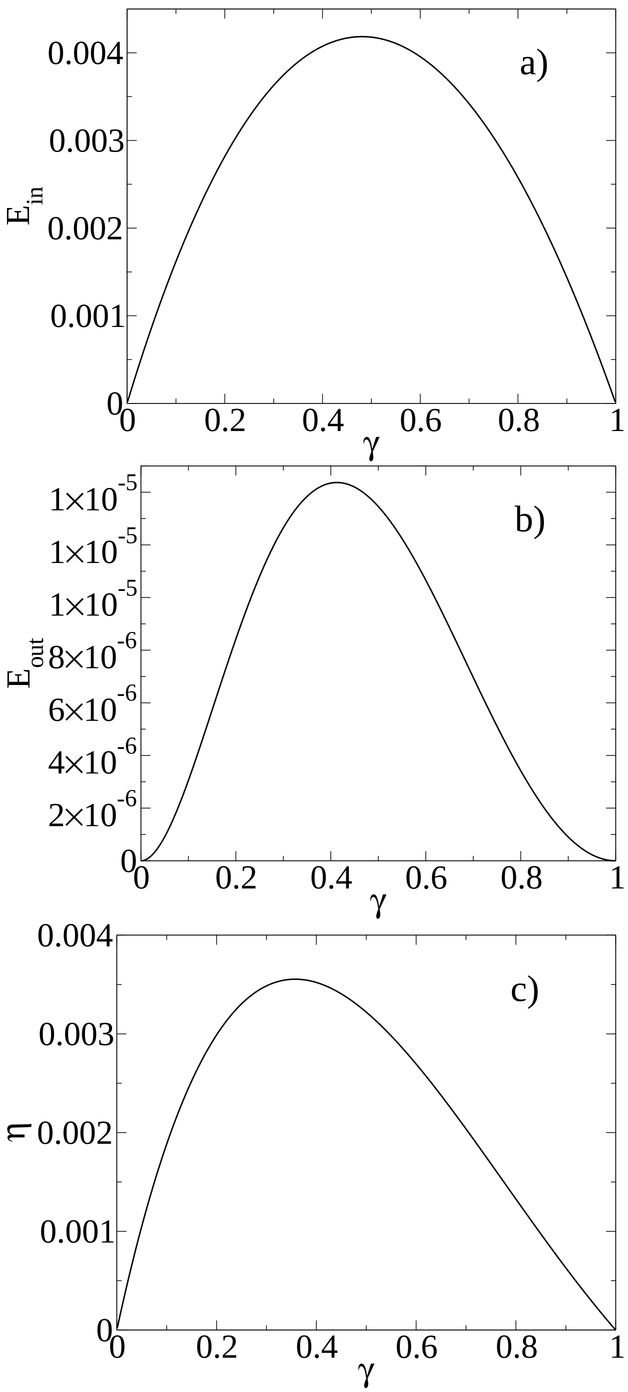



**Figure 6.4.** We plot the energy input (a), energy output (b) and efficiency (c) versus the mixing probability $\gamma$ in the case of fair games whose probabilities have been obtained from ratchet potentials (see the main text for the values of the parameters). The maximum energy input is at $\gamma \approx 0.481$, whereas the maximum for the energy output is at $\gamma \approx 0.413$. The maximum value for the efficiency $\eta = 3.554 \times 10^{-3}$ occurs for $\gamma \approx 0.358$.

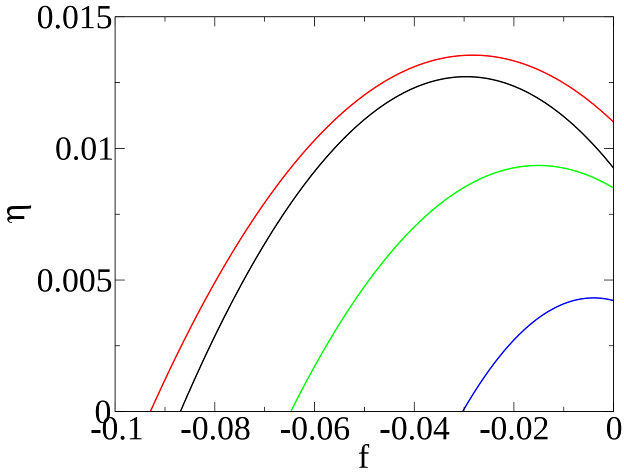



**Figure 6.5.** Plot of the efficiency versus the external load $f$ for different values of the probability $\gamma$ for the original Parrondo's games. From top to bottom $\gamma = 0.4, 0.2, 0.6$ and $0.8$. The highest values for the efficiency are attained for $\gamma = 0.4$, a value close to the $\gamma$ of maximum current.

The results are shown in Fig. 6.5 where we plot the efficiency for the original Parrondo's games as a function of the external forcing $f$ for different values of the mixing probability $\gamma$. Two features can be highlighted. On one hand it can be appreciated that the efficiency attains a maximum value for $f \neq 0$, corresponding to a lower value for the current than in the case of null forcing. This effect has also been found in other models, for example in [81, 82]. On the other hand, we also find a non–monotonic dependence of the position of the maxima for the efficiency depending on the probability $\gamma$ [81, 82].

# Chapter 7

# Collective games

This Chapter will be devoted to the development of a theoretical analysis for a collective game which considers the redistribution of capital between players. These collective games, already described in Sec. 2.3.2, are based on the alternation between a game A and game B. In [38] different games B are used, however, we will restrict ourselves to the case were game B is the original Parrondo game whose probabilities depend on the capital of the player. Game A is basically a mechanism of redistribution of capital between players. Two versions are used in [38]: the first one considers a redistribution of capital to a randomly selected player; the second considers a redistribution of capital to a neighboring player with probabilities that do depend explicitly on the capital of the players.

In Sec. 7.1 we present the analysis when alternating between the original game B and the new game $A'$, consisting on a redistribution of capital to a randomly selected player; Sec. 7.2 considers the alternation of the capital dependent game B with another game $A''$ with constant probabilities. Finally, we analyze the alternation of game B with a version of game $A''$ where the probabilities depend on the capital of the players in Sec.7.3, although this dependence will be slightly different in order to facilitate the analysis.

## 7.1 Distribution of capital to a randomly selected player

Let us denote by $P(c_1,c_2,\ldots,c_N;\tau)$ the probability that at a given time $\tau$ player 1 has a capital $c_1$, player 2 capital $c_2$,... and so on. This probability density function must fullfill the normalization condition

$$\sum_{c_1,c_2,\ldots,c_N} P(c_1,c_2,\ldots,c_N;\tau) = 1. \tag{7.1}$$

The marginal probability for a single player $j$ is obtained simply by carrying out the summation over all players but $j$, i.e.,

$$P_{(c_j;\tau)} = \sum_{c_1}\sum_{c_2}\dots\sum_{c_{j-1}}\sum_{c_{j+1}}\dots\sum_{c_N} P_{(c_1,c_2,\dots,c_N;\tau)} \tag{7.2}$$

We can write down an evolution equation for the probability density function $P_{(c_1,c_2,\dots,c_N;\tau)}$ for a set of $N$ players alternating between game $A'$ with probability $\gamma$ and game B with probability $1-\gamma$. The equation is given by

$$P_{(c_1,c_2,\dots,c_N;\tau+1)} = \frac{\gamma}{N}\sum_{j=1}^{N}\sum_{\substack{j'=1\\ j'\neq j}}^{N}\frac{1}{(N-1)}P_{(c_1,\dots,c_j+1,\dots,c_{j'}-1,\dots,c_N;\tau)}+$$
$$+\left(\frac{1-\gamma}{N}\right)\sum_{j=1}^{N}[a_{-1}^{c_j}P_{(c_1,\dots,c_j-1,c_N;\tau)} + a_0^{c_j}P_{(c_1,\dots,c_N;\tau)} + a_1^{c_j}P_{(c_1,\dots,c_j+1,c_N;\tau)}]. \tag{7.3}$$

The *rhs* of Eq. (7.3) is composed of the following terms:

- The first term accounts for the evolution of the capital when game $A'$ (capital redistribution) is played; with the term $\gamma$ denoting the probability of playing game $A'$, and $\frac{1}{N}$ being the probability of choosing player $j$. Inside the summation we find the term $\frac{1}{N-1}$ indicating the probability for player $j'$ of being chosen. The term $P_{(c_1,\dots,c_j+1,\dots,c_{j'}-1,\dots,c_N;\tau)}$ inside the summation accounts for the probability at time $\tau$ of finding player $j$ with capital $c_j+1$ and player $j'$ with capital $c_{j'}-1$. Both summations for $j$ and $j'$ are done in order to consider all possible combinations between the players.
- The second term accounts for the evolution when the selected player plays game B instead of game $A'$. The term $\frac{1-\gamma}{N}$ includes the probability of playing game B times the probability of choosing player $j$. The term in brackets corresponds to the master equation when player $j$ plays game B alone (we are following the same notation as the one used in Chapter 4).

By means of property (7.2) and after some algebra, we can derive the master equation corresponding to the evolution of the probability for a single player $j$ with capital $c_j$ at time $\tau$ (all the details are explicitly given in Appendix A.1) as

$$P_{(c_j;\tau)} = \frac{1-\gamma}{N}[a_{-1}^{c_j}P_{(c_j-1;\tau)} + a_0^{c_j}P_{(c_j;\tau)} + a_1^{c_j}P_{(c_j+1;\tau)}]+$$
$$+\frac{\gamma}{N}[P_{(c_j+1;\tau)} + P_{(c_j-1;\tau)}] + \frac{N-(1+\gamma)}{N}P_{(c_j;\tau)}. \tag{7.4}$$

It can easily be checked that the latter equation fulfills the normalization condition for a single player $\sum_{c_j} P_{(c_j,\tau)} = 1$. Rewritting Eq. (7.4) as a continuity equation we obtain the following expression

$$P(c_j,\tau+1) - P(c_j,\tau) = \frac{1-\gamma}{N}[a_{-1}^{c_j} P(c_j-1,\tau) - (a_1^{c_j-1} + a_{-1}^{c_j+1})P(c_j,\tau) + a_1^{c_j} P(c_j+1,\tau)] + \\ + \frac{\gamma}{N}[P(c_j+1,\tau) - 2P(c_j,\tau) + P(c_j-1,\tau)]. \quad (7.5)$$

From now on we drop the index $j$, as we are dealing only with one player, and the capital of player $j$ will be denoted instead by $i$, thus $P(c_j,\tau) \equiv P_i(\tau)$.

As we have seen, we have been able to obtain the equation governing the evolution of the probability $P_i(\tau)$ for a single player. Taking a closer look to Eq. (7.5) we can conclude that the effect of game $A'$ of a diffusion of capital from player $j$ to another randomly chosen player $j'$ is equivalent, from the point of view of a single player $j$, to a diffusion of capital of that player only. Therefore, we may define a current $J_i$ as

$$J_i = \left(\frac{1-\gamma}{N}\right)[a_{-1}^{i} P_{i-1}(\tau) - a_1^{i-1} P_i(\tau)] + \frac{\gamma}{N}[P_{i-1}(\tau) - P_i(\tau)]. \quad (7.6)$$

Assuming that the system eventually attains a stationary state, we can solve Eq. (7.6) for $P_i(\tau)$, assuming a constant current $J_i = J \; \forall i$ and $P_i(\tau) = P_i$, obtaining

$$P_n = \prod_{k=1}^{n} A_k \cdot P_0 - \sum_{j=1}^{n} \frac{J}{(1-\gamma)a_1^{j-1} + \gamma} \prod_{k=j+1}^{n} A_k, \quad (7.7)$$

where $A_k = \frac{(1-\gamma)a_{-1}^{k}+\gamma}{(1-\gamma)a_1^{k-1}+\gamma}$. The constants $P_0$ and $J$ are obtained from the periodicity[1] $P_n = P_{n+L}$ and normalization condition $\sum_{k=0}^{L-1} P_k = 1$. The current $J$ reads

$$J = \frac{P_0 \left[\prod_{k=1}^{L} A_k - 1\right]}{\sum_{j=1}^{L} \frac{\prod_{k=j+1}^{L} A_k}{(1-\gamma)a_1^{k-1}+\gamma}}, \quad (7.8)$$

and

$$P_0 = \frac{1}{\sum_{n=1}^{L} \prod_{k=1}^{n} A_k - \frac{\prod_{k=1}^{L} A_k - 1}{\sum_{j'=1}^{L} \frac{\prod_{k=j'+1}^{L} A_k}{(1-\gamma)a_1^{j'-1}+\gamma}} \left(\sum_{n=1}^{L} \sum_{j'=1}^{n} \frac{\prod_{k=j'+1}^{L} A_k}{(1-\gamma)a_1^{j'-1}+\gamma}\right)}. \quad (7.9)$$

In Fig. 7.1 we plot the current $J$ of a single player in terms of the mixing probability $\gamma$ between games $A'$ and B. We check this result with numerical values obtained through simulation with $N = 1000$ players.

[1] As players alternate between the original Parrondo game B and the new game A, we can consider, as in the analysis of the original games, that the system is periodic with periodicity $L$ (where $L$ is given by periodicity of game B).

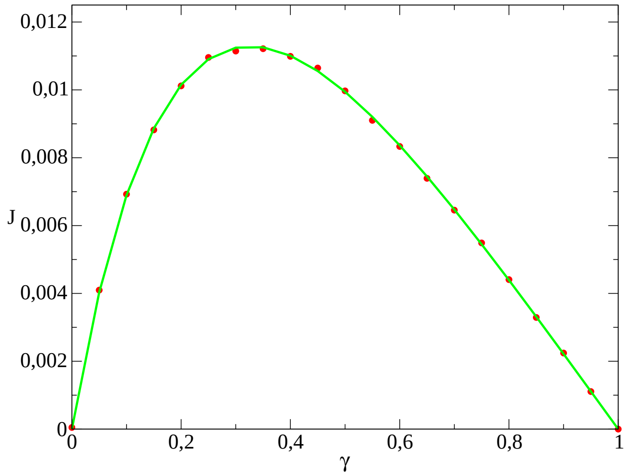



**Figure 7.1.** Plot of the theoretical – continuous line – and numerical current – circles – $J$ versus the mixing probability $\gamma$ for a single player. The probabilities used for game B are that of the original Parrondo game B: $p_0 = \frac{1}{10}$, $p_1 = p_2 = \frac{3}{4}$.

## 7.2 Redistribution of capital to a nearest neighbor with constant probabilities

In this section we present a collective Parrondo game obtained from the alternation of the original Parrondo game B with a new diffusing game $A''$. However, in this case the diffusion of capital of game $A''$ takes place only to nearest neighbors. We consider a general case where with probability $p_r$ player $j$ will give a coin to its neighbor $j+1$ located on the right, and with probability $p_l$ the coin will be given to the neighbor $j-1$ on the left. Then, the general master equation describing the time evolution of the probability density function $P(c_1,c_2,\ldots,c_N;\tau+1)$ when a set of $N$ players alternate between game $A''$ with probability $\gamma$ and game B with probability $1-\gamma$ is given by

$$P(c_1,\ldots,c_N;\tau+1) = \frac{\gamma}{N}\sum_{j'=1}^{N}\left[p_l P(c_1,..c_{j'-1}-1,c_{j'}+1,..c_N;\tau) + p_r P(c_1,..c_{j'}+1,c_{j'+1}-1,..c_N;\tau)\right] +$$

$$+\frac{1-\gamma}{N}\sum_{j=1}^{N}[a_{-1}^{c_j}P(c_1,..,c_j-1,..,c_N;\tau) + a_0^{c_j}P(c_1,\ldots,c_N;\tau) + a_1^{c_j}P(c_1,..,c_j+1,..,c_N;\tau)]. \quad (7.10)$$

All details of the calculation can be found in Appendix A.2. As a result we obtain the same equation for a single player as the one obtained previously, *c.f.* Eq. (7.5).

An interesting case appears when $p_l = p_r = \frac{1}{2}$. It corresponds to a random distribution of capital amongst nearest neighbors. For this case the master equation obtained is

$$P_{(c_1,..,c_N;\tau+1)} = \frac{1-\gamma}{N}\sum_{j=1}^{N}\Big[a_{-1}^{c_j}P_{(c_1,..,c_j-1,..,c_N;\tau)} + a_0^{c_j}P_{(c_1,..,c_N;\tau)}+$$
$$+a_1^{c_j}P_{(c_1,..,c_j+1,..,c_N;\tau)}\Big]+\frac{\gamma}{2N}\sum_{j=1}^{N}\Big[P_{(c_1,..,c_{j-1}-1,c_j+1,..,c_N;\tau)}+P_{(c_1,..,c_j+1,c_{j+1}-1,..,c_N;\tau)}\Big]. \tag{7.11}$$

Which, after some manipulation, can be written in a continuity form as

$$P_{(c_1,..,c_N;\tau+1)} - P_{(c_1,..,c_N;\tau)} =$$
$$= \frac{1-\gamma}{N}\sum_{j=1}^{N}\Big[a_{-1}^{c_j}P_{(c_1,..,c_j-1,..,c_N;\tau)} - (a_1^{c_j-1} + a_{-1}^{c_j+1})P_{(c_1,..,c_j,..,c_N;\tau)}+$$
$$+ a_1^{c_j}P_{(c_1,..,c_j+1,..,c_N;\tau)}\Big] + \frac{\gamma}{2N}\sum_{j=1}^{N}\Big[P_{(c_1,..,c_{j-1}-1,c_j+1,..,c_N;\tau)} - 2P_{(c_1,..,c_N;\tau)}+$$
$$+ P_{(c_1,..,c_j+1,c_{j+1}-1,..,c_N;\tau)}\Big]. \tag{7.12}$$

We already know from a previous chapter –*c.f.* Chapter 4– that the term corresponding to game B is equivalent, in the continuous form, to a ratchet potential acting on the Brownian particle. Therefore, we next proceed to find the equivalent model in the continuous form to that of game $A''$.

Let us consider only game $A''$, thus we may set $\gamma = 1$ in Eq. (7.12) which leads to the following equation

$$P_{(c_1,..,c_N;\tau+1)} - P_{(c_1,..,c_N;\tau)} = \frac{1}{2N}\sum_{j=1}^{N}\Big[P_{(c_1,..,c_{j-1}-1,c_j+1,..,c_N;\tau)} - 2P_{(c_1,..,c_N;\tau)}+$$
$$+ P_{(c_1,..,c_j+1,c_{j+1}-1,..,c_N;\tau)}\Big] \tag{7.13}$$

We may introduce the step-operators [83] $\mathbb{E}$ and $\mathbb{E}^{-1}$, which are defined by its effect on an arbitrary function $f(n)$

$$\mathbb{E}f(n) = f(n+1), \qquad \mathbb{E}^{-1}f(n) = f(n-1), \tag{7.14}$$

and that can be expanded in a Taylor series as

$$\begin{aligned}\mathbb{E} &= 1+\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{3!}\frac{\partial^3}{\partial x^3}+\ldots,\\ \mathbb{E}^{-1} &= 1-\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{1}{3!}\frac{\partial^3}{\partial x^3}+\ldots.\end{aligned} \tag{7.15}$$

Then, rewritting Eq. (7.13) using the previous operators we obtain

$$P(c_1,..,c_N;\tau+1)-P(c_1,..,c_N;\tau)=\frac{1}{2N}\sum_{j=1}^{N}\left[(\mathbb{E}_{j-1}^{-1}\mathbb{E}_j+\mathbb{E}_j\mathbb{E}_{j+1}^{-1}-2)P(c_1,..,c_N;\tau)\right]=$$

$$=\frac{-1}{2N}\sum_{j=1}^{N}\left[\nabla_{j+1}-2\Delta_j+\nabla_{j-1}+\Delta_j(\nabla_{j-1}+\nabla_{j+1})\right]P(c_1,..,c_N;\tau)=$$

$$=\frac{1}{2N}\sum_{j=1}^{N}\left[2(\Delta_j-\nabla_j)-\Delta_j(\nabla_{j-1}+\nabla_{j+1})\right]P(c_1,..,c_N;\tau). \tag{7.16}$$

Where we have defined the terms $\Delta_j$ and $\nabla_j$ so that they can be directly related to an expansion with partial derivatives as $\Delta_j=\mathbb{E}_j-1=\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\ldots$, and $\nabla_j=1-\mathbb{E}_j^{-1}=\frac{\partial}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\ldots$.

Regarding the *l.h.s* of the previous equation as the discretization of a time derivative $\frac{\partial P(c_1,\ldots,c_N;\tau)}{\partial\tau}$, and substituting on the *r.h.s* the terms $\Delta_j$ and $\nabla_j$ by their partial derivatives expansions, to a first approximation, we obtain

$$\frac{\partial P(c_1,..,c_N;\tau)}{\partial\tau}=\frac{-1}{2N}\sum_{j=1}^{N}\left\{\frac{\partial^2 P(c_1,..,c_N;\tau)}{\partial c_{j-1}\partial c_j}-2\frac{\partial^2 P(c_1,\ldots,c_N;\tau)}{\partial c_j^2}+\frac{\partial^2 P(c_1,..,c_N;\tau)}{\partial c_j\partial c_{j+1}}\right\}. \tag{7.17}$$

This equation can be compared to the general Fokker–Planck equation for more than one dimension [83]

$$\frac{\partial P(c_1,..,c_N;\tau)}{\partial\tau}=-\sum_{j=1}^{N}\frac{\partial F(c_1,..,c_N;\tau)P(c_1,..,c_N;\tau)}{\partial c_i}+\frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2 B_{ij}(c_1,..,c_N;\tau)P(c_1,..,c_N;\tau)}{\partial c_i\partial c_j} \tag{7.18}$$

With the result that for game $A''$ there is no drift, i.e., the term $F(c_1,\ldots,c_N;\tau)=0$, and the diffusion matrix $B_{ij}(c_1,\ldots,c_N;\tau)$ is given by

$$\mathbf{B}=\frac{1}{N}\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & -1\\ -1 & 2 & -1 & 0 & \ldots & & 0\\ 0 & -1 & 2 & -1 & 0 & \ldots & 0\\ & \vdots & & & \ddots & & \\ -1 & 0 & & \ldots & 0 & -1 & 2\end{pmatrix} \tag{7.19}$$

The diffusion matrix is related to the diffusion coefficients $d_{ij}$ of the Langevin equation $\dot{x}_i = f_i(\{\mathbf{x}\}) + d_{ij}(\{\mathbf{x}\})\xi_i$ through $B_{ij} = \mathbf{d}\mathbf{d}^T = \sum_k d_{ik}d_{jk}$. This set of equations has an infinite set of solutions due to the symmetry property of $B_{ij} = B_{ji}$. Therefore, we must choose the appropriate solution for this system, which to our consideration might be

$$\mathbf{d} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 \\ & \cdot & & & \ddots & & \\ & \cdot & & & & & \\ -1 & 0 & & \dots & 0 & 0 & 1 \end{pmatrix} \tag{7.20}$$

Then, the equivalent set of Langevin equations would be given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\sqrt{N}}(\xi_1 - \xi_2), \\ \dot{x}_2 &= \frac{1}{\sqrt{N}}(\xi_2 - \xi_3), \\ &\cdot \\ &\cdot \\ \dot{x}_N &= \frac{1}{\sqrt{N}}(\xi_N - \xi_1). \end{aligned} \tag{7.21}$$

This set of equations clearly preserves normalization as $\langle \sum_i \dot{x}_i \rangle = 0$. They could also be rewritten in the form $\dot{x}_1 = \frac{\eta_1}{\sqrt{N}}$, $\dot{x}_2 = \frac{\eta_2}{\sqrt{N}}$, ... $\dot{x}_N = \frac{\eta_N}{\sqrt{N}}$, with the properties: $\langle \eta_i \eta_{i+1} \rangle = -1$ and $\langle \eta_i^2 \rangle = 2$.

Finally, the complete solution would consider the inclusion of a drift term coming from game B, which as stated previously it consisted on a ratchet–like potential. In Fig. 7.2 we plot the average current for a set of $N = 40$ Brownian particles alternating between a state characterized by Eqs. (7.21) and a state with a ratchet–like potential. The ratchet effect is obtained as expected, and the curve presents (as in the single particle case) an optimum flip-rate value for which the system attains a maximum current.

## 7.3 Distribution of capital with capital dependent probabilities

In this section we derive the equation when the probabilities for game $A''$ depend explicitely on the actual value of the capital of the players. However, as stated previously, we will make use of a set of probabilities slightly different from those defined in [38]. In order to facilitate our analysis, the following probabilities for game $A''$ will be used

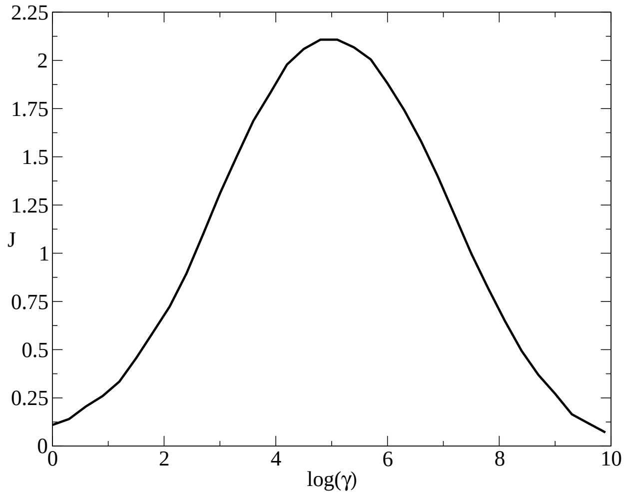



**Figure 7.2.** Plot of the average current per particle in terms of the natural logarithm of the flip rate when they are subjected to a state where a ratchet–like potential –*c.f.* Fig. (2.3)– is acting, and another state characterized by Eqs. (7.21). The number of particles is $N = 40$, and the results have been obtained averaging over $1000$ realizations.

$$p_{j,j+1} = \frac{c_{j-1}}{c_{j+1} + c_{j-1}} \qquad p_{j,j-1} = \frac{c_{j+1}}{c_{j+1} + c_{j-1}} \tag{7.22}$$

where $p_{j,j+1}$ denotes the probability that player $j$ gives away one unit of capital to player $j + 1$, and $p_{j,j-1}$ is the probability that player $j - 1$ receives the coin instead. Clearly, these probabilities fulfill the normalization condition $p_{j,j+1} + p_{j,j-1} = 1$. and the way they are defined – i.e., the probability of player $j + 1$ receiving a coin from player $j$ being proportional to the capital of player $j - 1$ – accomplishes the same task as those defined in [38], that is, those players with less capital possess a higher probability of receiving the coin than those with higher amounts of capital. The only inconvenient is that the capital of the players must remain positive in order to avoid negative values for the probabilities.

The master equation for this game is given by

$$\begin{aligned} P(c_1,..,c_N;\tau+1) &= \frac{\gamma}{N} \sum_{j'=1}^{N} \sum_{j''=1}^{N} p_{j',j''} P(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau) + \\ &+ \frac{(1-\gamma)}{N} \sum_{j'=1}^{N} [a_{-1}^{c'_j} P(c_1,..,c_{j'}-1,..,c_N;\tau) + a_0^{c_{j'}} P(c_1,..,c_N;\tau) + a_1^{c_{j'}} P(c_1,..,c_{j'}+1,..,c_N;\tau)] \end{aligned} \tag{7.23}$$

where the term $p_{j',j''}$ denotes the probability that player $j'$ gives a unit of capital to player $j''$. We are interested, as in previous cases, in obtaining the stationary probability distribution for a single player. Therefore, we must perform the sum (7.2) in Eq. (7.23) in order to obtain the single player distribution $P(c_j; \tau)$. A comparison between Eqs. (7.23) and (7.3) yields that the sum $\sum_{c_1,..,c_{j-1},c_{j+1},..,c_N}$ of the term corresponding to game B gives as a result Eq. (A.6).

The most difficult part comes from game $A''$. The second term on the *rhs* of Eq. (7.23) must be developed in terms of $j'$ and then perform the sum $\sum_{c_1,..,c_{j-1},c_{j+1},..,c_N}$. The calculations are shown in Appendix A.3. Nevertheless, the main result is that again, even though the probabilities of diffusing capital depend explicitly on the capital of the players, the equation we obtain for the probability density function for a single player agrees with the previous results, that is, Eq. (7.4).

## Chapter 8

# Reversals of chance in collective games

Cooperative versions of the games, played by a set of $N$ players, have been studied previously. As already explained in Sec. 2.3.2, ref. [67] considers a set of $N$ players arranged in a ring such that at each round a player is chosen randomly to play either game A or B. The original game A is combined with a new game B, for which the winning probability depends on the state (winner/loser) of the nearest neighbors of the selected player. A player is said to be a winner (loser) if he has won (lost) his last game. In [38], Toral considers again a set of $N$ players, but game A is replaced by another game based on a redistribution of capital. When combining this new game with the original game B, the paradox is reproduced.

In this Chapter we present a new version of collective games with new paradoxical features when they are combined. Besides reproducing the Parrondo effect, where a winning game is obtained from the alternation of two fair games, another feature appears: the games show under certain circumstances a current inversion when varying $\gamma$. In other words, the value of the mixing probability $\gamma$ determines whether you end up with a winning or a losing game AB. As shown in [15], it is not possible to obtain a current inversion in a single player set–up using the standard rules of the original games when game A is state independent. For the collective games considered here, we are able to obtain a current inversion even if one of the games used (game A) uses no information at all about the present state of the system. And so this current inversion is a collective genuine effect, without a corresponding analog in the single player game.

The chapter is organized as follows: in Sec. 8.1 we present the games in detail as well as a theoretical analysis by means of discrete–time Markov chain theory, obtaining analytical expressions for the stationary probabilities for a finite number of players; we also provide some qualitative insight into this new current inversion effect. Finally, in Sec. 8.2 we offer a qualitative picture of the impossibility of a current inversion using the original games.

## 8.1 The games

The games will be played by a set of $N$ players. In each round, a player is selected randomly for playing. Then, with probabilities $\gamma$ and $1-\gamma$ respectively game A or B is played. Game A is the original game in which the selected player wins or loses one coin with probability $p^A$ and $1-p^A$ respectively. The winning probabilities in game B depend on the collective state of all players. Again, as in [67], a player is said to be a winner or a loser when he has won or lost respectively his last game. More precisely, the winning probability can have three possible values, determined by the actual number of winners $i$ within the total number of players $N$, in the following way

$$p_i^B \equiv \text{probability to win in game B} = \begin{cases} p_B^1 & \text{if} \quad i > \frac{2N}{3}, \\ p_B^2 & \text{if} \quad \frac{N}{3} \le i \le \frac{2N}{3}, \\ p_B^3 & \text{if} \quad i < \frac{N}{3}. \end{cases} \tag{8.1}$$

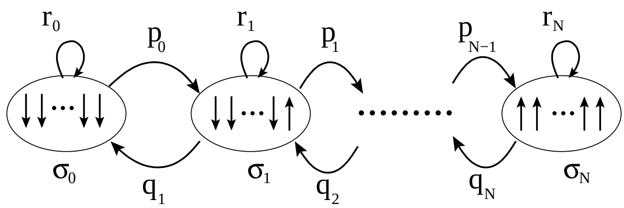



**Figure 8.1.** Different states and allowed transitions for $N$ players. The arrows indicate the state of each player being a winner (*arrow up*) or a loser (*arrow down*).

### 8.1.1 Analysis of the games

The main quantity of interest is the average gain of the collection of $N$ players when playing the stochastic game AB. Since the winning probability of game B only depend on the total number of winners, it is sufficient to describe the games using a set of $N+1$ different states $\{\sigma_0, \sigma_1, \ldots, \sigma_N\}$. A state $\sigma_i$ is the configuration where $i$ players are labeled as winner and $N-i$ as loser. Transitions between the states will be determined by the forward transition probability $p_i$, the backward transition probability $q_i$, and the probability for remaining in the same state $r_i$, see Fig. 8.1.

Denoting as $P_i(t)$ the probability of finding the system in state $\sigma_i$ at the t–*th* round played, we can write the equation governing its time evolution as

$$P_i(t+1) = p_{i-1}P_{i-1}(t) + r_i P_i(t) + q_{i+1}P_{i+1}(t), \tag{8.2}$$

with $0 \le i \le N$ and where the transition probabilities are given by

$$p_i = \frac{N-i}{N}\left[\gamma\, p^A + (1-\gamma)\, p_i^B\right], \tag{8.3}$$

$$r_i = \frac{2i-N}{N}\left[\gamma\, p^A + (1-\gamma)\, p_i^B\right] + \frac{N-i}{N}, \tag{8.4}$$

$$q_i = \frac{i}{N}\left[\gamma\,(1-p^A) + (1-\gamma)\,(1-p_i^B)\right]. \tag{8.5}$$

These transition probabilities have been obtained through the following reasoning: if we recall that in state $i$ there are $N-i$ losers and $i$ winners, the only way that we can go forward to state $i+1$ is by choosing a player labelled as a loser – with probability $\frac{N-i}{N}$ – and that player winning the game. So if there is a probability $\gamma$ of playing game $A$ and a probability $1-\gamma$ of playing game $B$, the combined winning probability will be given by $\gamma\, p^A + (1-\gamma)\, p_i^B$. Considering these two contributions, the forward transition (8.3) from state $i$ to state $i+1$ is obtained. The transition probabilities $r_i$ and $q_i$ follow from the same reasoning.

The set of transition probabilities $(p_i, q_i, r_i)$ must satisfy the normalization condition $p_i + r_i + q_i = 1$, which implies for the probabilities $P_i(t)$ that $\sum_{i=0}^{N} P_i(t) = 1$, as long as $\sum_{i=0}^{N} P_i(t=0) = 1$.

This system of $N+1$ equations can be solved in the stationary state, where the probabilities no longer depend on time $P_i(t) = P_i^{\text{st}}$. In this case Eq. (8.2) can be rewritten as

$$(p_i + q_i)P_i^{st} = p_{i-1}P_{i-1}^{st} + q_{i+1}P_{i+1}^{st}. \tag{8.6}$$

Considering that the system is bounded by states $0$ and $N$ we have

$$\begin{aligned}
&p_0 P_0^{st} = q_1 P_1^{st},\\
&(p_1 + q_1)P_1^{st} = p_0 P_0^{st} + q_2 P_2^{st},\\
&(p_2 + q_2)P_2^{st} = p_1 P_1^{st} + q_3 P_3^{st},\\
&\ldots\\
&(p_i + q_i)P_i^{st} = p_{i-1}P_{i-1}^{st} + q_{i+1}P_{i+1}^{st},\\
&\ldots\\
&(p_{N-1} + q_{N-1})P_{N-1}^{st} = p_{N-2}P_{N-2}^{st} + q_N P_N^{st},\\
&q_N P_N^{st} = p_{N-1}P_{N-1}^{st}.
\end{aligned} \tag{8.7}$$

Writing the previous set of equations in terms of the stationary probability at the origin $P_0$ we get

$$
\begin{aligned}
P_1^{st} &= \frac{p_0}{q_1} P_0^{st}, \\
P_2^{st} &= \frac{p_1}{q_2} P_1^{st} = \frac{p_0 p_1}{q_1 q_2} P_0^{st}, \\
P_3^{st} &= \frac{p_2}{q_3} P_2^{st} = \frac{p_0 p_1 p_2}{q_1 q_2 q_3} P_0^{st}, \\
\ldots \\
P_N^{st} &= \frac{p_{N-1}}{q_N} P_{N-1}^{st} = \frac{p_0 p_1 p_2 \ldots p_{N-1}}{q_1 q_2 q_3 \ldots q_N} P_0^{st}.
\end{aligned} \tag{8.8}
$$

Note that these solutions entail the detailed balance property between two neighboring states $p_i P_i^{st} = q_{i+1} P_{i+1}^{st}$. This is due to the reflecting boundary conditions at $i = 0$ and $i = N$. Through the normalization condition $\sum_{i=0}^{N} P_i^{st} = 1$ we may obtain $P_0^{st}$. Thus, the general solution can be written as

$$
P_i^{\text{st}} \;=\; \frac{1}{Z} p_0\, p_1 \cdots p_{i-1}\, q_{i+1}\, q_{i+2} \cdots q_N, \tag{8.9}
$$

or equivalently,

$$
\begin{aligned}
P_0 &= \frac{1}{Z}\, q_1 q_2 q_3 \ldots q_N \\
P_1 &= \frac{1}{Z} p_0\, q_2 q_3 \ldots q_N \\
P_2 &= \frac{1}{Z} p_0 p_1\, q_3 q_4 \ldots q_N \\
P_3 &= \frac{1}{Z} p_0 p_1 p_2\, q_4 q_5 \ldots q_N \\
\ldots \\
P_N &= \frac{1}{Z} p_0 p_1 p_2 \ldots p_{N-1}
\end{aligned} \tag{8.10}
$$

where $Z$ is the normalization factor. Once the stationary probabilities are calculated, we can obtain the average winning probability over all states for the stochastic combination AB (mixing probability $\gamma$) from

$$
p_{\text{win}}^{AB} = \sum_{i=0}^{N} \left[ \gamma\, p^A + (1-\gamma)\, p_i^B \right] P_i^{\text{st}}. \tag{8.11}
$$

The average gain can then easily be evaluated through the expression $J^{AB} = 2p_{\text{win}}^{AB} - 1$.

The properties of the separate games A and B can be obtained by replacing in the previous expressions $\gamma$ by 1 or 0 respectively.

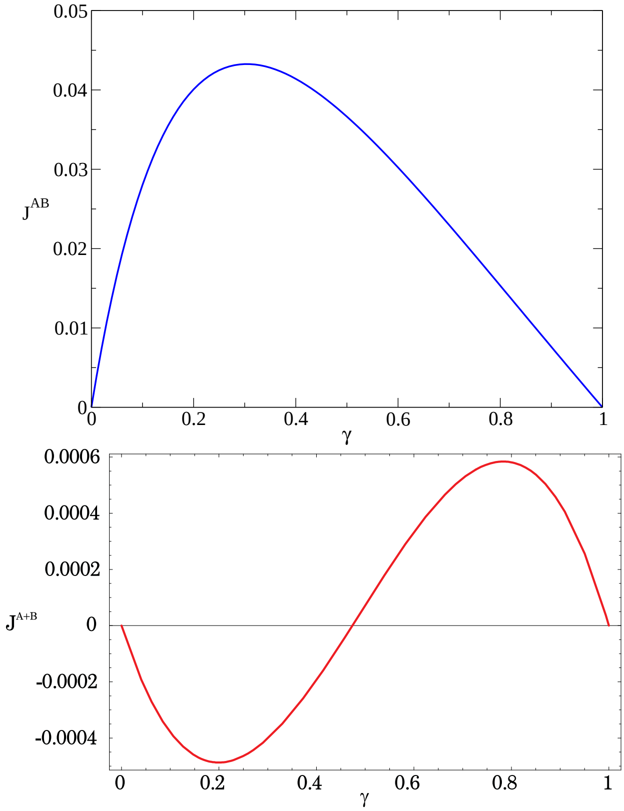



**Figure 8.2.** ***a)***Plot of the current versus the mixing probability $\gamma$ between games A and B for $N = 4$ with probabilities $p^A = \frac{1}{2}$, $p_B^1 = 0.79$, $p_B^2 = 0.65$ and $p_B^3 = 0.15$. ***b)*** Plot of the current versus the mixing probability $\gamma$ between games A and B for $N = 3$ with probabilities $p^A = \frac{1}{2}$, $p_B^1 = 0.686$, $p_B^2 = 0.423$ and $p_B^3 = 0.8$.

| $N$ | $p_B^2$ |
|---|---|
| 2 | $\frac{p_B^1-1}{p_B^1-p_B^3-1}$. |
| 3 | $\frac{(p_B^1-1)(p_B^3+1)+\sqrt{(p_B^1-2)(p_B^1-1)p_B^3(p_B^3+1)}}{(p_B^1+p_B^3-1)}$ |
| 4 | $\frac{(p_B^1-1)^2(p_B^3+1)}{1+p_B^3+(p_B^1-2)(p_B^1+p_B^1p_B^3-(p_B^3)^2)}$ |
| 5 | $\left[1-\frac{p_B^3}{p_B^1-1}\sqrt{\frac{5+2p_B^1(p_B^1-3)}{1+2p_B^3(1+p_B^3)}}\right]^{-1}$ |

**Table 8.1:** Condition on $p_B^2$ in order that game B is fair for $N = 2, \ldots, 5$.

#### *8.1.1.a The Parrondo effect*

We know that the Parrondo effect appears when from the combination of two fair games, we obtain a winning game. Clearly, game A is fair for $p^A = 1/2$. For game B the set of values $\{p_B^1, p_B^2, p_B^3\}$ giving a fair game is more difficult to determine because it depends on the total number of players $N$. The conditions on $p_B^2$ for a fair game B have been found analytically by a symbolic manipulation program up to $N < 13$. In Table 8.1 we find listed the conditions of fairness for $p_B^2$ up to $N = 5$. When playing only game B ($\gamma = 0$), the following symmetry in the stationary distribution can be deduced from Eq.(8.9)

$$P_i^{\text{st},\{p_B^1,p_B^2,p_B^3\}} = P_{N-i}^{\text{st},\{1-p_B^3,1-p_B^2,1-p_B^1\}}. \tag{8.12}$$

This property implies that $p_{\text{win}}^{AB}$ is unaffected by the parameter transformation: $\{p_B^1, p_B^2, p_B^3\} \rightarrow \{1-p_B^3, 1-p_B^2, 1-p_B^1\}$. It also means that for the parameter set $\{p_B^1, p_B^2 = 1/2, 1-p_B^1\}$, the stationary probability distribution is symmetric over the states, i.e. $P_i^{\text{st}} = P_{N-i}^{\text{st}}$. Therefore, when combining this with game A, i.e., alternating two games with symmetric probability distributions, always yields a fair game, independent of the values of $\gamma$, $N$ and $p_B^1$. To see the Parrondo effect, we need another, non-trivial, parameter set which yields a fair game B. For example, for $N = 4$ we obtain a fair game B when $p_B^1 = 0.79$, $p_B^2 = 0.65$ and $p_B^3 = 0.15$. The stochastic combination with game A reproduces the desired Parrondo effect, see Fig. 8.2.a .

### 8.1.2 Results

#### *8.1.2.a Two players*

For $N = 2$ players, there are 3 different states. Fig. 8.3.a shows the regions in parameter space $\{\gamma, p_B^1, p_B^3\}$ where the mixing $(0 < \gamma < 1)$ between games A and B results in a fair, winning or losing game. Note that $p_B^2$ is fixed by the condition to have a fair game B, see Table 8.1. Besides the case $p_B^1 = 1 - p_B^3$, valid for any number of players, also $p_B^1 = p_B^3$ results in a fair game for $N = 2$, independent of the alternation probability $\gamma$. From Eq. (8.9), one can deduce that $p_B^1 = p_B^3$ and $p_B^1 = 1 - p_B^3$ imply a symmetric distribution $P_i^{\text{st}}$ over the states, i.e. $P_0^{\text{st}} = P_2^{\text{st}}$. As mentioned before, this property prohibits any net current in the system. For all other values of $p_B^1$ and $p_B^3$ the Parrondo effect appears, that is, game AB is either a winning or a losing game, cf. Fig. 8.3.a.

#### *8.1.2.b Three players*

Fig. 8.3.b shows for $N = 3$ the surfaces in parameter space $\{\gamma, p_B^1, p_B^3\}$ where AB is a fair game. Besides the plane $p_B^1 = 1 - p_B^3$, there is a second, curved surface with values of $\gamma$ different from 0 and 1 which results in $J^{AB} = 0$. This curved surface is not uniform in $\gamma$ and is therefore the collection of points of flux reversal between a winning and losing

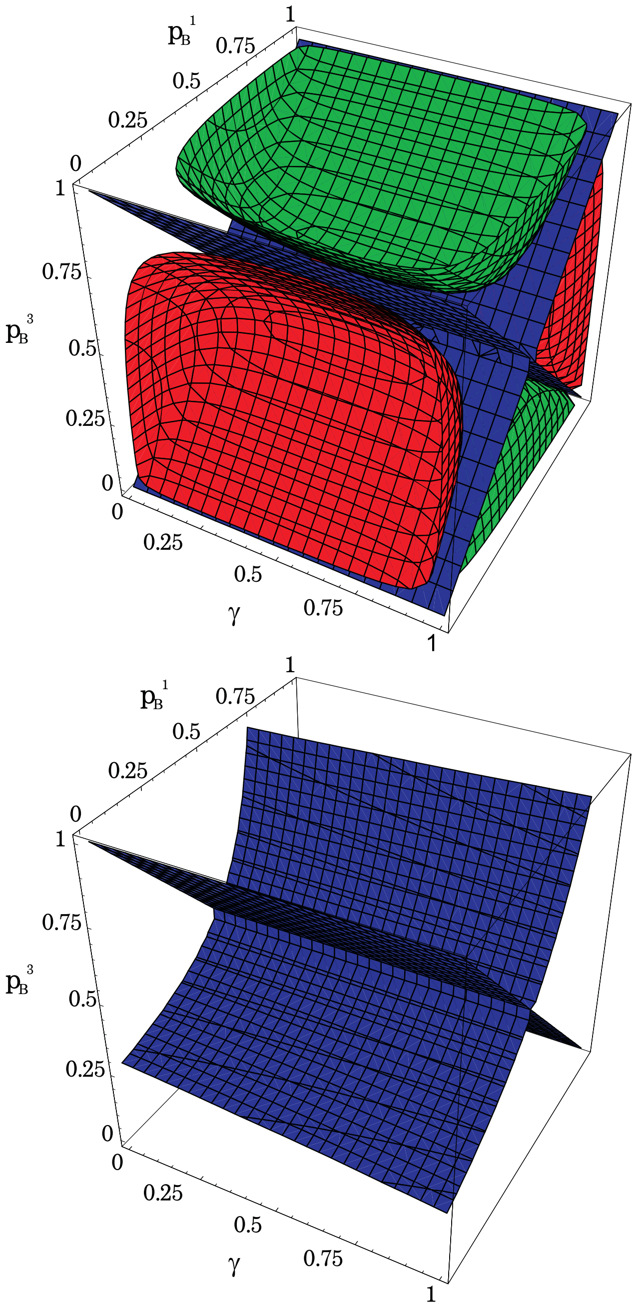



**Figure 8.3.** ***a)*** $N = 2$. The regions in parameter space for for which $p_{\text{win}}^{AB} = 0.5$, $0.499$ and $0.501$, indicating the regions where AB is fair (*blue*), losing (*red*) and winning (*green*) respectively. The blue diagonal planes show the situations $p_B^1 = 1 - p_B^3$ and $p_B^1 = p_B^3$, for which AB is fair, independent of $\gamma$. ***b)*** $N = 3$. The regions in parameter space for which the mixing $(0 < \gamma < 1)$ between game A and B results in a fair game. Besides the trivial diagonal plane, there is a curved plane – not uniform in $\gamma$ – for which $J^{AB} = 0$.

game AB. This implies that, depending on the value of $\gamma$ we can either have a winning game or a losing game by alternating between two fair games. For example, in Fig. 8.2.b we have plotted the current $J^{AB}$ vs. $\gamma$ for the set of probabilities $p^A = \frac{1}{2}$, $p_B^1 = 0.686$, $p_B^2 = 0.423$ and $p_B^3 = 0.8$. For low values of $\gamma$ the resulting game is a losing game, whereas for high values of $\gamma$ the game turns to be a winning game, cf. Fig 8.2.b. In both regions there exists an optimal value for $\gamma$ giving a maximum current. We can provide a qualitative picture that may help understanding the mechanism by which the current inversion phenomenon takes place.

When playing exclusively game B ($\gamma = 0$), the stationary distribution $P_i^{\rm st}$ is not homogeneous. This is reflected by the fact that the central states $\{\sigma_1, \sigma_2\}$ have a higher occupancy probability ($P_i^{st}$) than the boundary states $\{\sigma_0, \sigma_3\}$. On the other hand, if we look to the winning probability, it is higher in the latter set of states rather than in the former one ($p_B^1, p_B^3 > p_B^2$).

Indeed, the central states can be labelled as *losing* states, as when combining game B with game A for any $0 \leq \gamma < 1$, the average losing probability $p_i^l = \gamma(1 - p^A) + (1 - \gamma)(1 - p_i^B) < \frac{1}{2}$, i.e., it is more likely on average for a player to lose money rather than to win when being in one of these states. On the other hand, for the boundary states the contrary is true: it is more likely to win money rather than to lose for any $0 \leq \gamma < 1$, so we can refer to them as *winning* sites, i.e., $p_i^w = \gamma p^A + (1 - \gamma)p_i^B > \frac{1}{2}$.

When combining game B with A, the resulting game will be fair, losing or winning depending on the net balance between the occupancy probabilities and the average winning probability on each set of central and boundary states. For low $\gamma$ values (playing game B more often), the high occupancy probability of $\{\sigma_1, \sigma_2\}$ is the dominant part, and due to the low winning probability on these sites the resulting game is a losing game. On the contrary, for higher $\gamma$ values (playing game A more often), the winning probability on the boundary sites $\{\sigma_0, \sigma_3\}$ is high enough to compensate their low occupancy, resulting in a winning game.

#### *8.1.2.c N players*

For a general number of players, we have not been able to find the analytical expressions for a fair game B. Nevertheless, we will show numerically that the results for $N = 3$ are representative for any $N$. This is illustrated by Fig. 8.4, where the parameter space $\{p_B^2, p_B^3\}$ giving a fair game B is shown, corresponding to a fixed $p_B^1 = 0.4$ and different values of $N$. As shown, the different curves seem to converge to a limiting curve as $N$ increases. Note that all curves intersect at the trivial point $\{p_B^1 = 0.4, p_B^2 = 0.5, p_B^3 = 0.6\}$.

We can also obtain the parameter space where the current inversion takes place, for different values of $N$. For clarity reasons we show in Fig. 8.5 only a vertical slice corresponding to a fixed $\gamma = 0.4$, and different values of $N$. Again, the regions for which a flux inversion exists, doesn't seem to depend much on $N$. The only exception is $N = 4$, for which the curve bends in the other direction. This is a consequence of the fact that

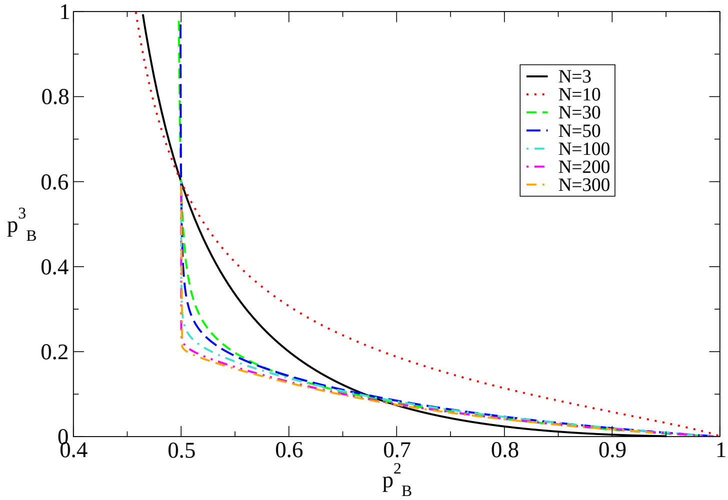



**Figure 8.4.** Plot of the parameter space $\{p_B^2, p_B^3\}$ for a fixed $p_B^1 = 0.4$ that gives a fair game B for different values of $N = 3$, 10, 30, 50, 100, 200 and 300. As it can be seen, the curves seem to converge to a limiting curve as $N$ increases.

for $N = 4$ there exists only one state (namely $\sigma_2$) where the probability $p_B^2$ is used. This is confirmed by our findings when we modify the definition of game B such that there is for any $N$ only one state where $p_B^2$ is used. The fact that all curves of inversion points are symmetric upon reflection about the plane $p_B^1 = 1 - p_B^3$ is a consequence of the property of Eq. (8.12).

## 8.2 Parrondo's games and the current inversion

As stated previously and shown in [15], the effect of a current inversion when varying the mixing probability $\gamma$ is not possible when combining the original game B with a state independent type game A. One way of understanding the reason is through the quantitative relation established in Chapter 4 between the Brownian ratchet and Parrondo's games. It was shown that a fair or unfair paradoxical game corresponds to a periodic or tilted potential respectively in the model of a Brownian ratchet. Thus, the question now reduces to explain why there is no current inversion in the flashing ratchet model when varying the rate of alternation between the potentials.

In the flashing ratchet model, the appearance of a flux when alternating between a flat and an asymmetric potential is due to a rectification process. From Fig. 2.3 we see that the asymmetry present in the ratchet potential will always favor a rightward movement of the Brownian particles. Thus, whatever the rate of alternation between states *ON* (asymmetric potential) and *OFF* (diffusive state), the induced current will always be unidirectional. It is clear then that no current inversion may take place under this scheme unless some other parameters rather than the flip rate are varied.

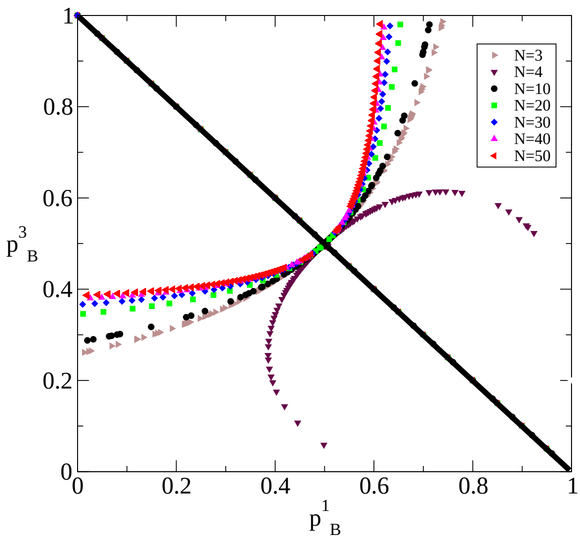



**Figure 8.5.** Plot of the points in parameter space $\{p_B^1, p_B^3\}$ where (for $\gamma = 0.4$ fixed) AB is a fair game. Results for different values of the total number of players $N = 3, 4, 10, 20, 30, 40$ and $50$ are shown. The diagonal line shows the common plane $p_B^1 = 1 - p_B^3$, that corresponds to a fair game B for any number of players $N$.

# Chapter 9

# Truels and N–uels

In this Chapter we present a detailed analysis using Markov chain theory of some versions of truel games in which three players try to eliminate each other in a series of one-to-one competitions, using the rules of the game. These games were first studied by Kilgour [42] from the point of view of game theory. Our treatment reproduces the expressions for the winning probability of each player, including the equilibrium points. Furthermore we give expressions for the distribution of winners in a truel competition. In Section 9.1 we introduce some basic concepts on game theory and the main notions of truel games in Sec 9.2. In Sec. 9.3, and in order to introduce the general methods in an simpler context, we present a detailed analysis for the case of duels. Afterwards, Sec. 9.4 is devoted to the analysis of the strategies –9.4.1– in the random –Sec. 9.4.2– and sequential –Sec. 9.4.3– versions of truels, together with an analysis of the opinion model in Sec. 9.4.4. In Sec. 9.4.5 we present the distribution of winners when playing the truel games as well as the opinion model. We study the effect of introducing spatial dependence in these models in Sec. 9.4.6, and finally truels are generalized to more than three players in Sec. 9.5. Most of the details of the calculations are left for Appendices B and C, showing here only the main results.

## 9.1 Introduction

Making a decision is not an easy task, and it turns to be even more difficult when more than one person is involved, with the result depending of all decisions taken. Besides, in everyday life we encounter many situations in which we are posed with dilemmas appearing from the confrontation of our own interests with that of other individuals or the society surrounding us. Thus we are frequently required to take decisions, with outcomes that not necessarily are those one expected *a priori*. Does exist a rational way of behaving in those situations?

A formal answer to this question was not found until the mid 40's, when the mathematician J. von Neumann (1903-1957) published in collaboration with the Princeton economist Oskar Morgenstern the book *Theory of games and economic behavior* [84]. In this book von Neumann establishes the foundations of what was later coined as *game theory*. He realized that saloon games (like poker,...) raised simple dilemmas that could encounter analogous conflicts in economy, politics, everyday life or even war situations. Following the words of the authors, a game is *a conflictive situation where one has to take a decision knowing that others also take decisions, and the outcome of the conflict is determined, in some way, from all decisions taken*.

Strictly speaking, game theory can be considered as a formal study of conflict and cooperation, a branch of mathematical analysis developed to study decision making in conflict situations. They appear when two or more decision makers having different objectives act on the same system or share the same resources. The main purpose of game theory is to consider situations where instead of agents making decisions as reactions to exogenous prices, their decisions are strategic reactions to other agents actions. The goal for all agents is always trying to obtain the maximum payoff, which can be understood as a quantity reflecting the desirability of an outcome to a player, for whatever reason. The expected payoff incorporates the player's attitude towards risk. These agents (or decision makers) can either be individuals, groups, firms, or any combination of these. In game theory, *games* have always been a metaphor for more serious interactions in human society.

We may distinguish between *cooperative game theory* and *non-cooperative game theory*. The former case investigates coalitional games, characterized by a high-level description, specifying only what payoffs each potential group, or coalition, can obtain by the cooperation of its members. The latter case is concerned with the analysis of strategic choices. The details of the ordering and timing of players' choices are essential to establish the outcome of the non-cooperative games.

von Neumann solved non-cooperative games in the case of *pure rivalries*, i.e., two person zero-sum games, in which one person's gain is another's loss, so the payoffs always sum to zero. In 1950, John Forbes Nash [85] demonstrated that finite games have always an equilibrium point, at which all players choose actions which are best for them given the opponents' choices. This proposal applied to a much wider class of games without restrictions on the payoff structure or the number of players [86,87]. The idea of *Nash equilibrium* [1] is that a set of strategies, one for each player, would be stable if nobody had a unilateral incentive to deviate from the strategy they have adopted. This equilibrium notion supposed a key concept of non–cooperative game theory, revolutioning the use of game theory in economics, and has been object of analysis since then. It was later developed by Harsanyi [88], who extended the Nash equilibrium to a larger class of games of *incomplete information*, where a player making a decision cannot always observe all previous decisions neither know other players' preferences.

Since the pioneering work of von Neumann and Morgenstern, game theory has de-

[1] We will return later in this Chapter to the concept of *Nash equilibria*, explaining it in more detail in Sec. 9.4.2.

veloped considerably and has found many applications in numerous fields such as economics, social science, political science and evolutionary biology. In the following Sections we will present a detailed study of a non-cooperative game known as *truel*, offering an alternative analysis more adequate to the physics community to that conducted by Kilgour [42] in the field of game theory.

## 9.2 Introduction: truel games

A truel game can be considered as the extension of a duel played by three individuals. These players, which will be named as A, B and C, possess different markmanships, that is, the probability of hitting a chosen target. Markmanships will be denoted as $a$, $b$ and $c$ for players A, B and C respectively. Without loss of generality we will assume throughout this Chapter that the players are labeled such that $a > b > c$. In this game all players share the same goal: to eliminate all the opponents. The game ends when there is only one survivor left, the winner of the game. The mechanics of the truel can be described by the following steps:

1. Each round – or time-step –, one of the truelists is chosen for playing.
2. He then decides who will be his target and, with a certain probability – the markmanship – he does achieve the goal of eliminating that opponent from the game.
3. Whatever the result obtained by the player, steps one and two are repeated again until there is only one survivor.

Based on the rules used for selecting the players, we may distinguish between three main types of truels:

- **Random truel.** Each round one of the remaining players is chosen randomly with equal probability.
- **Sequential truel.** In this case there exists an established firing order, which will be followed throughout the whole game. We allow players with worst markmanship to shoot firstly, followed then by players with better markmanship. According to the notation introduced earlier, the firing order in the sequential truel is C–B–A.
- **Simultaneous truel.** In this truel all players shoot at the same time.

A paradoxical or counter–intuitive result appears in this game, as the “truelist” with the highest markmanship does not necessarily possess the highest survival probability. This paradoxical result was already mentioned in the early literature on truels [42]. These games were formally introduced for the first time by Kinnaird in 1946 [89], although the name *truel* was coined later by Shubik [90] in the 1960s.

We find in the literature other models similar to the truel game that present also counterintuitive results, like for instance the *rock–scissors–paper* game. This game has been applied to some convective instabilities in rotating fluids [91], as well as to population dynamics [92, 93]. It consists on a system with three species interacting with each other in such a way that they create a competitive loop (recall that in the *rock–scissors–paper* game a rock beats a pair of scissors, scissors beat a sheet of paper and paper beats a rock). The paradoxical effect in this model is that the least competitive species might be the one with the largest population and, when there are oscillations in a finite population, to be the least likely to die out. This game has also been applied to a voter model [94, 95] obtaining again a paradoxical result, namely, an initial damage and suppression of one candidate may later lead to an enhancement of the same candidate.

Different versions of the truels vary on the number of tries (or "bullets") available to each player, on whether they are allowed to "pass", i.e. missing the shoot on purpose ("shooting into the air"), on the number of rounds being finite or infinite, etc. All these modifications lead to games with different outcomes [39–41]. Besides, they can be further extended through the introduction of coalitions between the truelists, that is, the appearance of cooperations between different players so that they can set a common target (these games are known as *cooperative truels* [96]), in such a way that they can obtain greater benefits from that coalition improving their own survival probability. We will restrict ourselves to the case of unlimited ammunition, and the game will continue until there is only one player left (so that there is no upper limit in the number of rounds); besides, players are also allowed to lose their turn by shooting into the air, a possibility that turns out to be useful in some particular cases.

The strategy of each player consists in choosing the appropriate target when it is his turn to shoot. Rational players will use the strategy that maximizes their own probability of winning (considered as the payoff) and hence the ensemble of players will chose the strategy given by the Nash equilibrium point. In a series of seminal papers [39–41], Kilgour has analyzed the games and determined the equilibrium points under a variety of conditions.

In this Chapter, we analyze the games from the point of view of Markov chain theory. Besides being able to reproduce some of the results by Kilgour, we obtain the probability distribution for the winners of the games. We restrict our study to the case in which there is an infinite number of bullets and consider two different versions of the truel: random and fixed sequential choosing of the shooting player.

Furthermore, we consider a variation of the game in which instead of eliminating the competitors from the game, the objective is to convince them on a topic, making the truel suitable for a model of opinion formation.

## 9.3 The duels

In this simpler game we consider two players, A and B, with markmanships $a$ and $b$ respectively, such that $a > b$. We will consider the random duel in which the person to shoot next is randomly selected with equal probability between the two players, as well as the sequential version in which the bad player, B, starts shooting and then they alternate fires. In any case, the game continues until there is only one survivor. If we take the model as an opinion model, the game continues until one player has convinced the other and hence both share the same opinion. Clearly, in a duel it makes no sense for a player to lose his opportunity to eliminate the opponent by shooting into the air and the only meaningful strategy is to shoot into the other player.

An analytical study done with Markov chains for both the random duel and the opinion model shows that both models can be described through the same Markov chain with three states (see Appendix B.1 for further details). If we denote the survival (or convincing) probabilities of players A and B as $\pi_A$ and $\pi_B$ respectively we have

$$\pi_A = \frac{a}{a+b}\,, \qquad \pi_B = \frac{b}{a+b}\,, \tag{9.1}$$

a result that indicates that the higher the markmanship of a given player, the higher the survival (convincing) probability in the random duel (opinion model).

Turning to the case of the sequential duel, this game can be described with a Markov chain with four states. The analytical expressions obtained for the survival probabilities are

$$\pi_A = \frac{a}{1-(1-a)(1-b)}\,, \qquad \pi_B = \frac{b(1-a)}{1-(1-a)(1-b)}\,, \tag{9.2}$$

A closer study of Eqs. (9.2) shows that even though the worst player B starts shooting first, he achieves a higher survival probability than A only when $b > \frac{a}{1+a}$. Thus, in the sequential duel the unfavorable situation of player B having a lower markmanship than A is partially compensated by being the one shooting in first place.

## 9.4 The truels

### 9.4.1 Strategies in truels

If a third individual comes into play, the previous situation of a duel is no longer simple. Now every player in the truel must consider all possible actions that other opponents may take and their corresponding outcomes. In this case, we must consider strategies and make use of some concepts of game theory. For concreteness, and without loss of generality, we consider that the third player C has the lowest marksmanship, $c$, such that $a > b > c$.

It turns out that strategies followed by the players are a key point in determining the winner of the truel. As explained previously, all players in the truel share the same goal: to be the only one surviving the truel.This can be explicitly imposed through the inclusion of a “payoff”, a concept introduced in game theory and that corresponds to some sort of reward the player receives for achieving the goal. In order to maximize their payoff, players have to chose strategies that maximize their survival probability. When the three players are still in the game, a player has three possible strategies: two correspond to choosing one of the two opponents and the third strategy is to shoot into the air (or missing the shot on purpose). If one of the three players has been removed from the game, we are in a duel situation and, as discussed before, the only strategy is to aim at the remaining opponent. We also assume that strategies adopted by the players are non–cooperative, in the sense that alliances or pacts between them are not allowed.

### 9.4.2 Random firing

Let us first fix the notation. We denote by $P_{AB}$, $P_{AC}$ and $P_{A\emptyset}$ the probability of player A shooting into player B, C, or into the air, respectively, with equivalent definitions for players B and C. These probabilities verify $P_{AB} + P_{AC} + P_{A\emptyset} = 1$. We will consider only "pure" strategies, namely, only one of these three probabilities is taken equal to $1$ and the other two equal to $0$[2]. Finally, we denote by $\pi(a; b, c)$ the probability that player with marksmanship $a$ wins the game when playing against other two players with marksmanships $b$ and $c$. This definition implies $\pi(a; b, c) = \pi(a; c, b)$ and $\pi(a; b, c) + \pi(b; a, c) + \pi(c; a, b) = 1$. Recall that we use the convention $a > b > c$.

The corresponding Markov chain for this game is composed of 7 different states labeled as ABC, AB, AC, BC, A, B, C according to the players remaining in the game. Three of these states, A, B and C are absorbent states. The details of the calculation for the winning probabilities as well as a diagram of the allowed transitions between states are shown in Appendix B.2. We now discuss the results in different cases.

Let us first imagine that players do not adopt any thought strategy and each one shoots randomly to any of the other two players. Clearly, this is equivalent to setting $P_{AB} = P_{AC} = P_{BA} = P_{BC} = P_{CA} = P_{CB} = 1/2$. The winning probabilities in this case are:

$$\pi(a; b, c) = \frac{a}{a+b+c}, \quad \pi(b; a, c) = \frac{b}{a+b+c}, \quad \pi(c; a, b) = \frac{c}{a+b+c}, \tag{9.3}$$

a result indicating that the player with the higher marksmanship possesses the higher probability of winning. Identical result is obtained if players include shooting in the air as one of their equally likely possibilities.

It is conceivable, though, that players will not decide the targets randomly, but will use some strategy in order to maximize their winning probability. As explained previously, completely rational players will choose strategies that are best responses (i.e. strategies that are utility–maximizing) to the strategies used by the other players. This defines an equilibrium point when all players are better off keeping their actual strategy than changing to another one. Accordingly, this equilibrium point can be defined

[2] Another possibility that we do not consider in this game is the"mixed" strategy, which consists on taking two or more of the probabilities strictly greater than 0.

as the set of probabilities $P_{\alpha\beta}$ (with $\alpha$ =A,B,C and $\beta$ =A,B,C,$\emptyset$) such that the winning probabilities have a local maximum. This idea, that is nothing but the concept of *Nash equilibria* introduced earlier, is clarified with the following example: in Table 9.1 we present the different survival –or winning– probabilities $\pi_A$, $\pi_B$ and $\pi_C$ of players A, B and C respectively for different strategies adopted by the players when they play the random truel. These values are calculated considering that player A has $100\%$ of effectiveness ($a = 1$), player B has $80\%$ ($b = 0.8$) and player C $50\%$ ($c = 0.5$).

Let us start by looking in Table 9.1 at the set of strategies given by $\{C, C, B\}$, which consists on player A aiming at player C, player B aiming at player C and player C aiming at player B. In this case we can see how the player with the highest survival probability is A with a $58\%$ percentage of winning, followed by player B with $34.8\%$ percentage and finally player C with a very low percentage of $7.2\%$. If player C analyzes this situation, he concludes that if players A and B adopt these strategies in the game, it is better for him to change his own strategy and instead of aiming to B, set as a new target player A. Reasoning in this way, he increases his survival probability up to a $8.5\%$.

| **A** | **B** | **C** | $\pi_{\mathbf{A}}$ | $\pi_{\mathbf{B}}$ | $\pi_{\mathbf{C}}$ |
|---|---|---|---|---|---|
| *C* | *C* | *B* | 0.58 | 0.348 | 0.072 |
| *C* | *C* | *A* | 0.434 | 0.481 | 0.085 |
| *C* | *A* | *B* | 0.386 | 0.407 | 0.207 |
| *C* | *A* | *A* | 0.2415 | 0.541 | 0.2175 |
| *B* | *C* | *B* | 0.628 | 0.155 | 0.217 |
| *B* | *C* | *A* | 0.483 | 0.288 | 0.229 |
| *B* | *A* | *B* | 0.4348 | 0.214 | 0.3512 |
| *B* | *A* | *A* | 0.29 | 0.348 | 0.362 |

**Table 9.1.** Table corresponding to the survival probabilities $\pi_A$, $\pi_B$ and $\pi_C$ of players A, B and C respectively, for the different set of strategies adopted in the case of the random truel. Player A has $100\%$ of effectiveness, player B an $80\%$ and player C a $50\%$.

Once we are found in the set $\{C, C, A\}$, we can follow the same reasoning but for player B, and see that it is better for him to change his strategy –aiming at player C– setting as a new target player A (increasing $\pi_B$ from $48.1\%$ to $54.1\%$). This leads us to the set $\{C, A, A\}$. Now it is the turn of player A who decides to change strategy and set B as a new target thus leading the the set BAA where $\pi_A$ has indeed increased from $24.2\%$ to $29.0\%$.

Executing the same procedure for the rest of strategies, we see that all lead to the same strategy set: $\{B, A, A\}$. This is the unique *Nash equilibrium point* of the random truel, meaning that no player improves his survival probability by changing his strategy, as long as the rest of players keep theirs. Therefore, this set corresponds to a local maximum of all survival probabilities of the players. Besides, when all players use their 'best' strategy $\{B, A, A\}$ we are lead to the paradoxical result that the player with the worst marksmanship can become the player with the highest winning probability. This some-what surprising result can be easily understood if one realizes that players set as primary target either player A or B, leaving player C as the last option and therefore increasing his winning expectation.

The strategy $\{B, A, A\}$ is known [40, 42] as the *strongest opponent strategy*, as all players aim at the opponent with the highest markmanship. For the random truel it is the equilibrium point whatever markmanships $a$, $b$ and $c$, as long as the condition $a > b > c$ is fulfilled (in Appendix C.1 is shown the demonstration for arbitrary values $a$, $b$ and $c$).

Using this strategy, the winning probabilities for the random truel are

$$\begin{aligned} \pi(a;b,c) &= \frac{a^2}{(a+c)(a+b+c)}, \\ \pi(b;a,c) &= \frac{b}{a+b+c}, \\ \pi(c;a,b) &= \frac{c(c+2a)}{(a+c)(a+b+c)}. \end{aligned} \tag{9.4}$$

This set can be obtained from Eqs. (C.1) from Appendix C with $P_{AB} = P_{CA} = P_{BA} = 1$ and $P_{AC} = P_{A\emptyset} = P_{BC} = P_{B\emptyset} = P_{CB} = P_{C\emptyset} = 0$.

In Fig. 9.1 we plot by colour code the region in parameter space in which each player possesses the highest survival probability when playing the random truel, varying markmanships $b$ and $c$ and keeping $a$ fixed and equal to 1. It can be appreciated that the region of player A is larger than the ones for B and C. In this figure, markmanship $a$ has been set to its highest possible value 1, because other values $a \neq 1$ can be related through the scaling relations $\pi(a;b,c) = \pi(1;b/a,c/a)$, $\pi(b;a,c) = \pi(b/a;1,c/a)$, $\pi(c;a,b) = \pi(c/a;1,b/a)$.

### 9.4.3 Sequential firing

In this version of the truel there is an established order of firing. The players will shoot in increasing value of their marksmanship, i.e., if $a > b > c$ the first player to shoot will be player C, followed by player B and the last to shoot is player A. The sequence repeats until only one player remains. Again, we have left for Appendix B.3 the details of the calculation of the winning probabilities. In Appendix C.2 we reproduce the analysis of the optimal strategies which agrees with that obtained by Kilgour [40]. The main result is that there are two equilibrium points depending on the value of the function $g(a,b,c) = a^2(1-b)^2(1-c) - b^2c - a\,b\,(1-b\,c)$: if $g(a,b,c) > 0$ the equilibrium point is the strongest opponent strategy $P_{AB} = P_{BA} = P_{CA} = 1$, while for $g(a,b,c) < 0$ it turns out that the equilibrium point strategy is $P_{AB} = P_{BA} = P_{C\emptyset} = 1$, where the worst player C is better off by shooting into the air and hoping that the second best player B succeeds in eliminating the best player A from the game. Player C would use the next turn to try to eliminate the remaining player, becoming the winner of the truel.

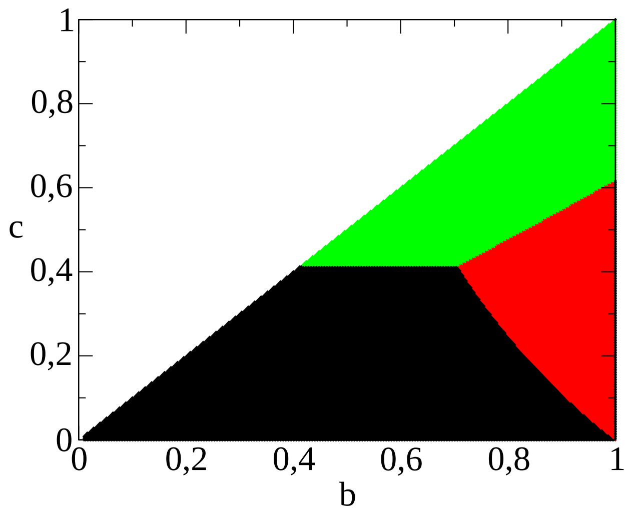



**Figure 9.1.** Diagram $b$ vs $c$ setting $a = 1$ where it is plotted with color codes which is the player with the highest survival probability for the case of the random truel and using the optimal strategy, as given by Eq. (9.4). Black color corresponds to the region where player A has the highest winning probability, red color corresponds to player B having the highest winning probability and finally the green color corresponds to player C being the player with the highest survival probability.

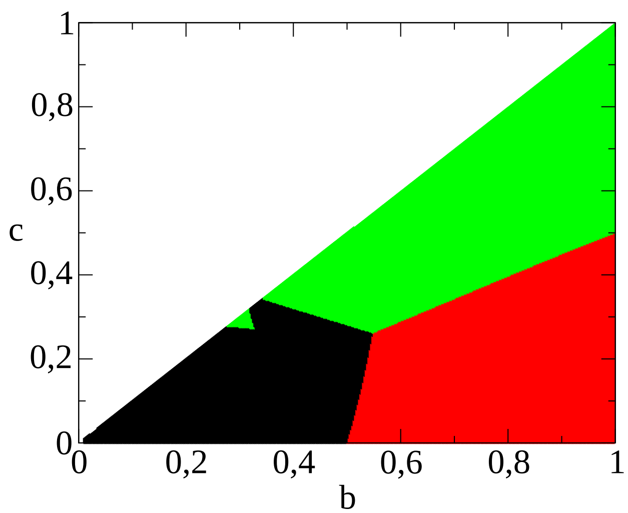



**Figure 9.2.** Same as Fig. 9.1 in the case that players play sequentially in increasing order of their marksmanship.

The winning probabilities for this case, assuming $a > b > c$, are given by

$$
\begin{aligned}
\pi(a;b,c) &= \frac{(1-c)(1-b)a^2}{[c(1-a)+a][b(1-a)+a]},\\
\pi(b;a,c) &= \frac{(1-c)b^2}{(c(1-b)+b)(b(1-a)+a)},\\
\pi(c;a,b) &= \frac{c[bc+a[b(2+b(-1+c)-3c)+c]]}{[c+a(1-c)][b+a(1-b)][a+b(1-a)]},
\end{aligned} \tag{9.5}
$$

if $g(a,b,c) > 0$, and

$$
\begin{aligned}
\pi(a;b,c) &= \frac{a^2(1-b)(1-c)^2}{[a+(1-a)c][a+b(1-a)+c(1-a)(1-b)]},\\
\pi(b;a,c) &= \frac{b\left(b(1-c)^2+c\right)}{[b+(1-b)c][a+b(1-a)+c(1-a)(1-b)]},\\
\pi(c;a,b) &= \frac{\frac{ac(1-b)(1-c)}{a+c(1-a)}+\frac{c(b+c(1-2b))}{b+c(1-b)}}{[a+b(1-a)+c(1-a)(1-b)]},
\end{aligned} \tag{9.6}
$$

if $g(a,b,c) < 0$. Again, as in the case of random firing, the paradoxical result appears that the player with the smallest marksmanship has the largest probability to win the game.

Due to the imposed firing order (C-B-A), player A is the last one to shoot. Therefore, the a priori advantageous situation given by a high marksmanship is partially lost. This is reflected in Fig. 9.2, since the region where player A is the favorite has decreased considerably compared to that of Fig. 9.1. In fact, the a priori worst player C is the favorite in a larger number of occasions. We explained previously that there were two equilibrium points in the sequential truel, $\{B, A, A\}$ and $\{B, A, \emptyset\}$. The last one is the relevant in the small green region located in the black region seen in Fig. 9.2.

### 9.4.4 Opinion model

We reinterpret the truel as a game in which three people holding different opinions, A, B and C, on a topic, aim to convince each other in a series of one-to-one discussions. The marksmanship $a$ (resp. $b$, $c$) are now interpreted as the probabilities that player holding opinion A (resp. B or C) have of convincing another player of adopting this opinion. The main difference with the previous games is that the number of players present is always constant and equal to three, a fact that will strongly conditionate the results.

The states belonging to the Markov chain for this model are ABC, AAB, ABB, AAC, ACC, BBC, BCC, AAA, BBB and CCC. As in previous cases, we have left the analysis of the convincing probabilities for Appendix B.4. We consider only the random case in which the person that tries to convince another one is chosen randomly amongst the three players.

The study of equilibrium points (*c.f.* App. C.3) reveals the existence of a unique equilibrium point corresponding again to the strongest opponent strategy, in which each player tries to convince the opponent with the highest marksmanship. The probabilities of a final consensus opinion being A, B or C, assuming $a > b > c$ are given by

$$\begin{aligned}
\pi(a;b,c) &= \frac{a^2\left[2cb^2 + a\left((a+b)^2 + 2(a+2b)c\right)\right]}{(a+b)^2(a+c)^2(a+b+c)}, \\
\pi(b;a,c) &= \frac{b^2(b+3c)}{(b+c)^2(a+b+c)}, \\
\pi(c;a,b) &= \frac{c^2\left[c^3 + 3(a+b)c^2 + a(a+8b)c + ab(3a+b)\right]}{(a+c)^2(b+c)^2(a+b+c)},
\end{aligned} \tag{9.7}$$

respectively. Notice that, as before, they satisfy the scaling relations $\pi(a;b,c) = \pi(1;b/a,c/a)$, $\pi(b;a,c) = \pi(b/a;1,c/a)$, $\pi(c;a,b) = \pi(c/a;1,b/a)$. As in previous cases, we have plotted in Fig. 9.3 in colour code the opinion with the highest probability of becoming majority. In this case opinion A becomes majority nearly for all values of $b$ and $c$. Only for a small region opinion C can become the majority opinion. This overwhelming dominion of A can be understood if we recall that the total number of players always remains the same throughout the game. Only the opinions held by the players change. So, once opinion A convinces either a player with opinion B or a player with opinion C, it is very likely that it will eventually become the majority opinion due to its high *convincing* probability.

### 9.4.5 Distribution of winners

Imagine that we set up a league scheme: everybody plays against everybody else. Sets of three players are chosen randomly amongst a population whose marksmanship are uniformly distributed in the interval $(0,1)$. The distribution of winners is characterized by a probability density function, $f(x)$, such that $f(x)dx$ is the proportion of winners whose marksmanship lies in the interval $(x, x+dx)$. This distribution is obtained as:

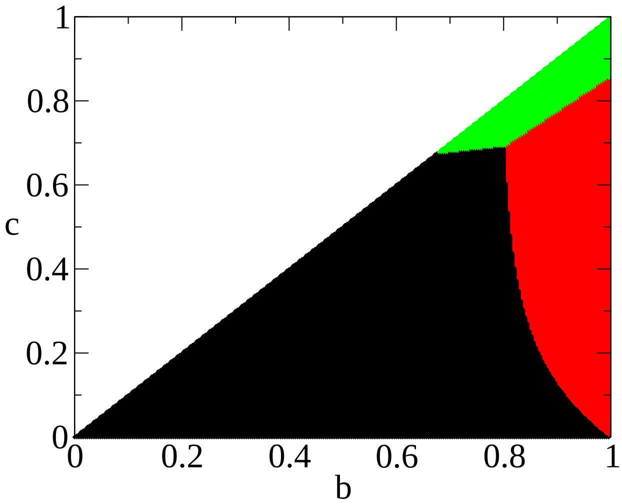



**Figure 9.3:** Same as Fig.9.1 for the convincing opinion model.

$$f(x) = \int da\, db\, dc\, [\pi(a;b,c)\,\delta_a + \pi(b;a,c)\,\delta_b + \pi(c;a,b)\,\delta_c] =$$
$$= \int db\, dc\, \pi(x;b,c) + \int da\, dc\, \pi(x;a,c) + \int da\, db\, \pi(c;a,b) =$$
$$= 3 \int_0^1 db \int_0^1 dc\, \pi(x;b,c). \quad (9.8)$$

where $\delta_i$ accounts for the Dirac delta $\delta(x-i)$.

We may also consider a variation of the competition in which the winner of one game keeps on playing against other two randomly chosen players. The resulting distribution of players, $\bar{f}(x)$, can be computed as the steady state solution of the recursion equation:

$$\bar{f}(x,t+1) = \int da\, db\, dc\, [\pi(a;b,c)\,\delta_a + \pi(b;a,c)\,\delta_b + \pi(c;a,b)\,\delta_c]\, \bar{f}(a,t)$$
$$= \bar{f}(x,t) \int db\, dc\, \pi(x;b,c) + \int da\, dc\, \pi(x;a,c)\bar{f}(a,t) + \int da\, db\, \pi(c;a,b)\bar{f}(a,t), \quad (9.9)$$

performing the variable change $a \longrightarrow b$ in the second integral, and $\left\{ \begin{array}{l} a \longrightarrow b \\ b \longrightarrow c \end{array} \right.$ in the third one we obtain

$$\bar{f}(x) = \frac{1}{3}\bar{f}(x)f(x) + 2\int_0^1 db \int_0^1 dc\,\pi(x;b,c)\bar{f}(b) \tag{9.10}$$

For the case of the random truel, with players using the random strategy whose winning probabilities are given by Eq. (9.3), the distribution of winners is $f(x) = 3x\,[x\ln x - 2(1+x)\ln(1+x) + (2+x)\ln(2+x)]$. In Fig. 9.4 we observe that the function $f(x)$ attains its maximum at $x = 1$ indicating that the best marksmanship players are the ones which win in more occasions. For the same strategy set, the distribution of winners if the winner keeps on playing is [3] $\bar{f}(x) = 2x$.

If, on the other hand, players adopt the equilibrium point strategy, Eq. (9.4), the resulting $f(x)$ has been plotted in Fig. 9.5. Notice that, despite the paradoxical result mentioned before, the distribution of winners still has it maximum at $x = 1$, indicating that the best marksmanship players are nevertheless the ones who win in more occasions. In the same figure, we have also plotted the distribution $\bar{f}(x)$ of the competition in which the winner of a game keeps on playing. In this case, the integral relation Eq. (9.10) has been solved numerically.

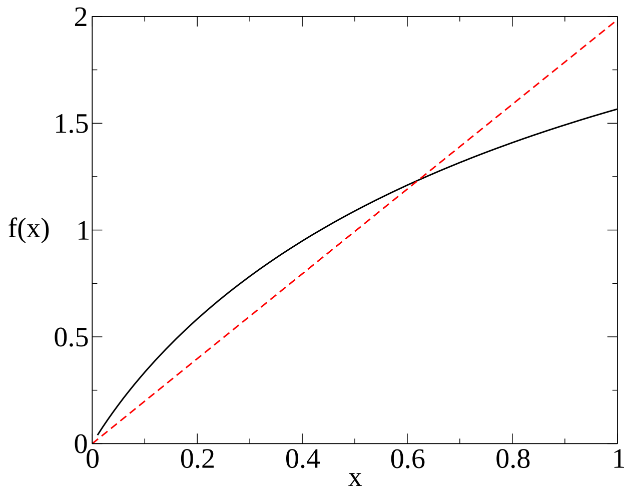



**Figure 9.4.** Distribution function $f(x)$ for the winners of truels of randomly chosen triplets (solid line) in the case of players using random strategies, Eq. (9.3); distribution $\bar{f}(x)$ of winners in the case where the winner of a truel remains in the competition (dashed line).

In Fig. 9.6 we plot the distribution of winners $f(x)$ and $\bar{f}(x)$ in a competition where players play the sequential truel. As before, the solid line corresponds to the former truel competition and the discontinuous line corresponds to the competition where the winner of the truel goes on playing. Notice that now the distribution of winners $f(x)$ has a

[3] The result is more general: if $\pi(a;b,c) = G(a)/[G(a)+G(b)+G(c)]$, for an arbitrary function $G(x)$, the solution is $\bar{f}(x) = G(x)/\int_0^1 G(y)dy$.

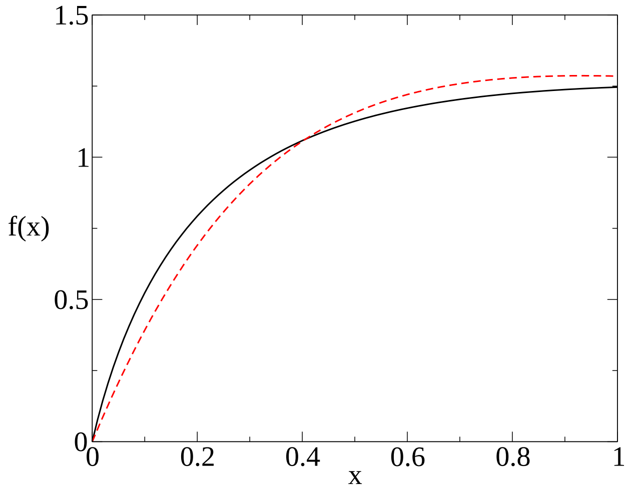



**Figure 9.5.** Similar to Fig.(9.4) in the case of the competition where players use the rational strategy of the equilibrium point given by Eq.(9.4).

maximum at $x \approx 0.57$. This result reflects the counter–intuitive result obtained earlier, and is that players who perform better on average are *not* those with higher markmanship, instead, are those with *intermediate* values.

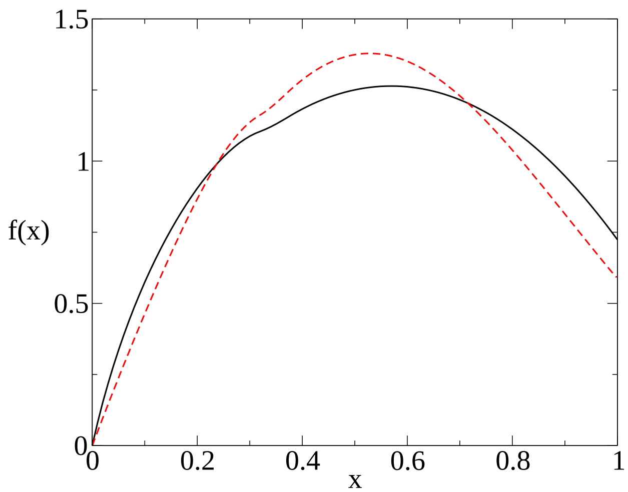



**Figure 9.6.** Same as Fig.9.4 in the case that players play sequentially in increasing order of their marksmanship. Notice that now both distributions of winners present maxima for $x < 1$ indicating that the best a priori players do not win the game in the majority of the cases.

Similarly to other versions, we plot in Fig. 9.7 the distribution of winning opinions, $f(x)$ and $\bar{f}(x)$. As in the case of the random truel, we can observe how the player most favored on average is the one with the highest markmanship available.

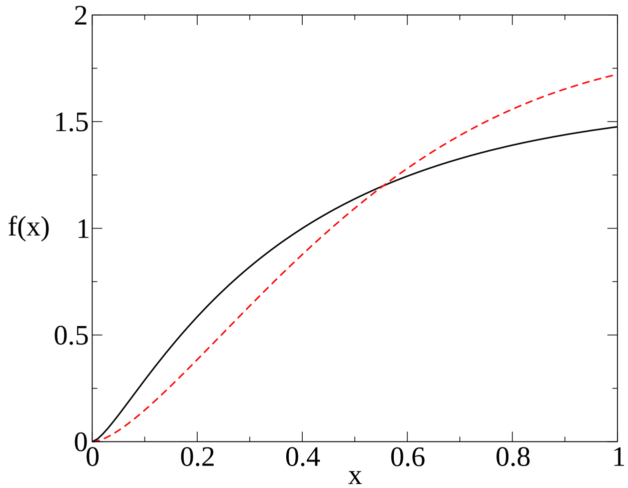



**Figure 9.7**: Same as Fig.9.4 for the convincing opinion model.

### 9.4.6 Truels with spatial dependence

A natural step forward in the truels would involve the introduction of a spatial structure in the system. This reflects the fact that players do not interact with any other player, but only with those which are closer in some sense. Although one could devise some sort of social network of interaction [97, 98], we consider here a simple two dimensional lattice. In this case we have a set of $N$ individuals arranged in a grid, each surrounded by four nearest neighbor links. The lattice is initialized by putting randomly on each site one player of groups A, B or C in the respective proportions $x_A$, $x_B$ and $x_C$, ($x_A + x_B + x_C = 1$) and respective marksmanships $a$, $b$ and $c$. An important ingredient of this generalization is that players never shoot to a person of the same group.

The rules of the random *collective truel* are as follows:

1. One of the remaining players is chosen at random.
2. The chosen player selects randomly two players amongst the occupied neighbors sites and the three of them play a random truel. The losers of the truel are eliminated from the system. If the chosen player has only one neighbor, the two of them will play a duel with the loser being removed from the system. If no neighbors are left, the player will walk to a randomly chosen neighbor site.
3. Steps 1 and 2 are repeated until all the survivors belong to the same group.

In step 2, it is possible that some of the chosen players belong to the same group. In this case, they observe strictly the rule of no shooting between members of the same group. Accordingly, it could happen that there is more than one survivor of that game. In any event, players use the strongest–opponent strategy. If, for example, the three players in a truel belong to groups A, A and B, the two A players will aim at B, while B will aim to one of the two A (again chosen at random). The outcome of that particular situation could be either player B eliminating both A players or player B being eliminated by the two players A. Since the analytical treatment appears rather difficult, we present now the results coming from a direct numerical simulation of the aforementioned rules. We use throughout this section the values $a = 1$, $b = 0.8$, $c = 0.5$ for the marksmanships.

In Fig. 9.8 we show some snapshots concerning different stages of a simulation carried out for the random truel. The initial population proportion was $x_A = 0.3$, $x_B = 0.3$ and $x_C = 0.4$. We can see how in early stages of the run, populations B and C diminish considerably whereas group A resists and eventually becomes the winner of the *collective truel*.

In this collective truel, the group that will survive at the end depends, for a fixed values of the marksmanships, on the initial proportions of players. This dependence is summarized in Fig. 9.9, where we plot in a color code the group that has the highest winning probability as a functions of the initial proportions.

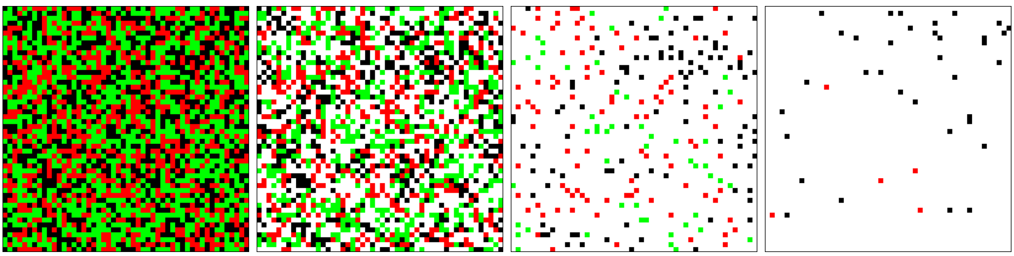


**Figure 9.8.** Snapshots corresponding to different stages of a simulation carried out for the random truel with initial proportions $x_A = 0.3$ (black colour), $x_B = 0.3$ (red colour) and $x_C = 0.4$ (green colour). The total number of players is $N = 2500$ arranged in a two–dimensional grid.

It is easy to modify step 2 by considering the rule of the sequential truel by which players shoot in inverse order to their marksmanship. A typical realization is shown in Fig. 9.10. In this occasion the winning group is the weakest one, group C. This *survival of the weakest* effect is also present in the diagram of Fig. 9.11, as now groups B and C have increased the region in parameter space where they win the truel, compared to the diagram of the random truel in Fig.9.9.

It is possible to distinguish two different regimes in the dynamics. Almost all truel competitions take place during the first steps where a large fraction of the population is removed. At the end of this first regime, the largest remaining population is the one that possesses the higher survival probability when playing a single truel and the system presents many empty sites. Later, in a second regime, players start to diffuse to neighboring sites increasing the appearance of duel encounters. Consequently, the evolution will result from a balance between the population favored by the existence of *duels* (the one with the highest marksmanship), and the one favored by possessing a high proportion of the remaining population.

Finally, in Fig. 9.12 we show some snapshots of a simulation carried out for the

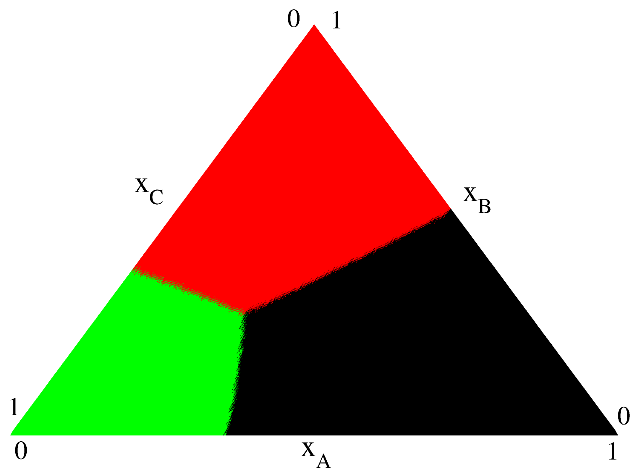



**Figure 9.9.** Diagram where it is shown the winning group in colour code (black colour corresponds to group A, red to group B and green to group C) in terms of initial proportions $x_A$, $x_B$ and $x_C$, for a set of $N = 400$ players arranged in a two–dimensional grid and playing the random truel. The results are obtained after averaging over 10000 realizations.

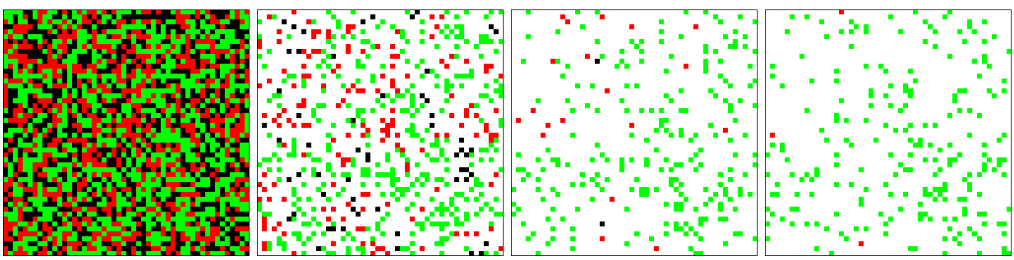


**Figure 9.10.** Snapshots corresponding to different stages of a simulation carried out for the sequential truel with an initial population of $x_A = 0.3$ (black colour), $x_B = 0.3$ (red colour) and $x_C = 0.4$ (green colour) for a set of $N = 2500$ players arranged in a spatial two–dimensional grid.

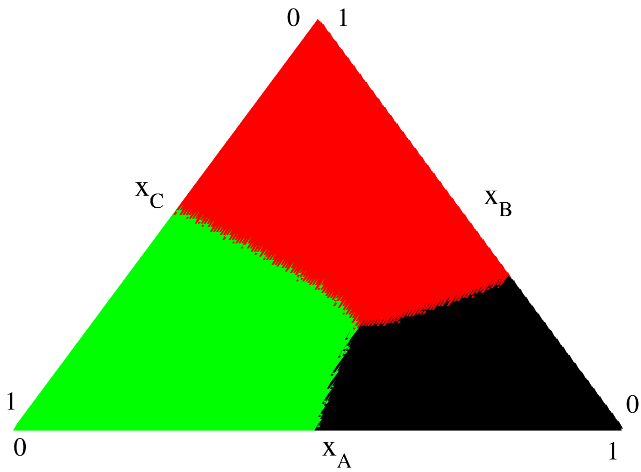



**Figure 9.11.** Diagram where it is shown the winning group in colour code (black colour corresponds to group A, red to group B and green to group c) in terms of the initial proportions $x_A$, $x_B$ and $x_C$, for a set of $N = 400$ players arranged in a two–dimensional grid that play the sequential truel. The probabilities have been obtained averaging over 100000 realizations.

case of the opinion model. As it happened in the three players case, the total number of players remains constant troughout the simulation, only the opinions held by the players may vary. For the set of marksmanships chosen $a = 1$, $b = 0.8$ and $c = 0.5$ we find that the opinion most likely to become majority opinion is always the one with highest marksmanship, A. This occurs even for very small initial proportion $x_A$ and it is a reflection of the large region in parameter space where A becomes the favorite opinion, as it was shown in Fig. 9.3.

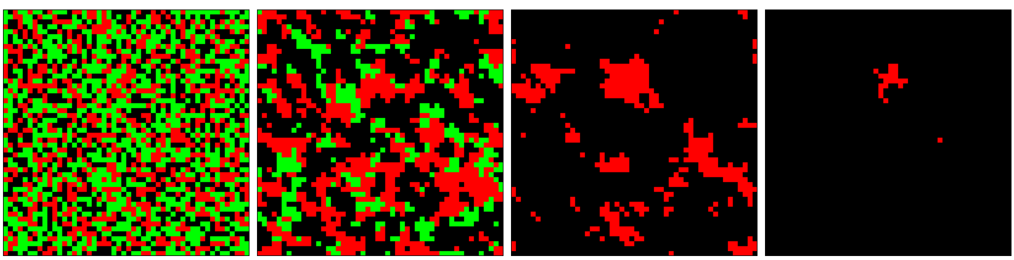


**Figure 9.12.** Snapshots corresponding to different stages of a simulation of the opinion model, carried out with an initial population of $x_A = 0.3$ (black colour) , $x_B = 0.3$ (red colour) and $x_C = 0.4$ (green colour) for a set of $N = 2500$ players arranged in a spatial two–dimensional grid.

## 9.5 Generalization to N players : N–uels

We have shown for three players the existence of an interesting and *a priori* counter–intuitive result where the player with the highest markmanship does not win the truel in all cases. But, what happens if there are more than three players? For a general case of $N$ players, it is rather difficult to obtain exact analytic expressions. Already for a low number of individuals the expressions obtained increase very rapidly in complexity. However, we can make use of numerical simulations in order to obtain the distribution of winners for a number of players $N > 3$. We will also restrict our analysis to the random case.

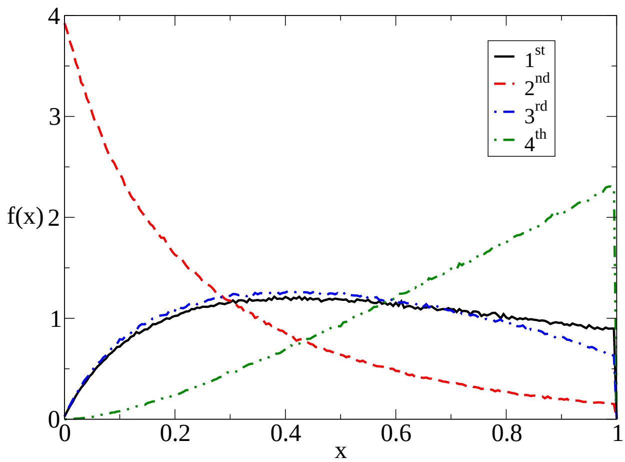



**Figure 9.13.** Histogram of the classified corresponding to the random truel for $N = 4$ players.

In Fig. 9.13 we show a histogram corresponding to the classification obtained when the game is played by $4$ players. The fourth classified would correspond to the distribution of players eliminated from the game in first place, the third classified would be the one eliminated in second place and so on. The distribution of the fourth classified shows that individuals eliminated firstly in the game are those with higher markmanships. Indeed, the maximum is located at $x = 1$, indicating then that the better you are the higher the probability of being eliminated first. Another aspect we can extract from this figure has to deal with the distribution of first and second classifieds: these curves correspond to the case where there are only two players left in the game, i.e., to a duel. Therefore, it is more likely in this situation that players with lower markmanships are eliminated firstly rather than those with higher markmanships (that is the reason why the curve for the second classified presents a maximum in the origin). It is worth mentioning that already for $4$ players the histogram associated to the first classified – i.e., the winner of the $4$–uel – presents a maximum for a value of $x < 1$. This result implies that the best performing player does not correspond anymore to the player with the highest markmanship, as it happened when $N = 3$. Indeed, the optimum value is located in $x \sim 0.49$.

For greater values of $N$, we can develop a simple theory that helps us to understand the distribution. The mechanics of this collective game is quite simple: we start from a

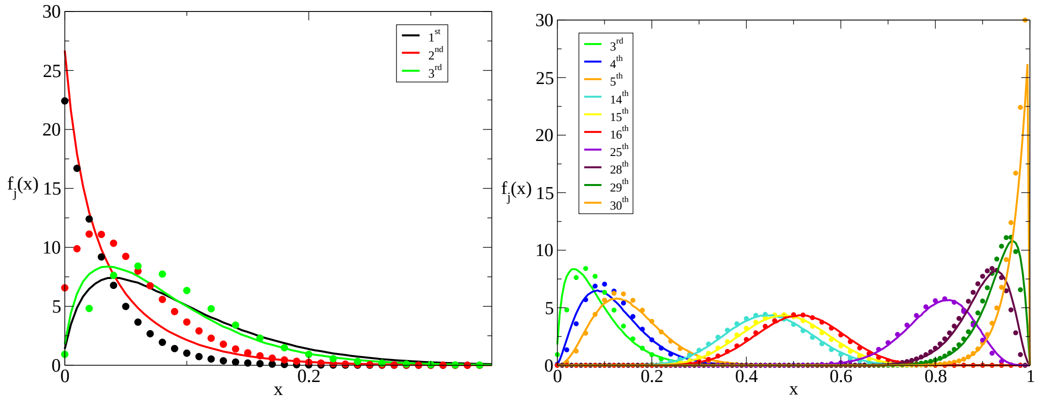



**Figure 9.14.** Left panel: Distribution of the first, second and third classifieds corresponding to the random truel for $N = 30$ players. The solid line corresponds to the numerical values, and circles correspond to the theoretical calculation. Right panel: Distribution ranging from the $3^{rd}$ classified (left side) to the $30^{th}$ (right side).

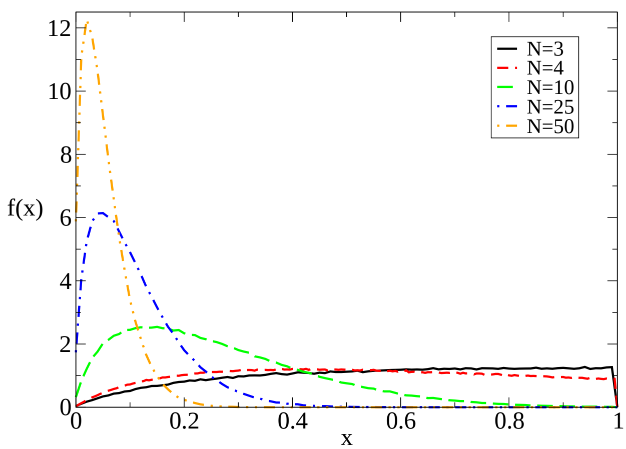



**Figure 9.15.** Different histograms of the first classifieds when playing the random N–uel corresponding to different values of $N = 3, 4, 10, 25$ and $50$.

set of $N$ players whose markmanship is uniformly distributed between $(0,1)$. Then, each time step one player is chosen randomly, and then he aims to the remaining player with the highest markmanship. This process continues until there is only one survivor left.

A similar distribution can be obtained if we consider a set of $N$ numbers $a_1,\ldots,a_N$ uniformly distributed in the interval $(0,1)$. As the probability density function describing each number $a_j$ is equal to 1 we have $f(a_1,\ldots,a_N)=1$. If we classify them in increasing value such that $a_1<a_2<\ldots<a_N$, we need to consider all different ways of ordering these terms through the inclusion of a factorial term in the probability density function and consequently $f(a_1,\ldots,a_N)=N!$. Thus, if we consider that these numbers are being *suppressed* in decreasing order, that is, greater numbers are eliminated first, we can calculate the distribution of terms $a_j$ occupying the j-$th$ place in the classification as

$$\begin{aligned} f_j(a) &= \int_0^a da_1 \int_{a_1}^a da_2 \cdots \int_{a_{j-2}}^a da_{j-1} \int_a^1 da_{j+1} \int_{a_{j+1}}^1 da_{j+2} \cdots \int_{a_{N-1}}^1 da_N f(a_1,..,a_N) \\ &= N! \int_0^a da_1 \int_{a_1}^a da_2 \cdots \int_{a_{j-2}}^a da_{j-1} \int_a^1 da_{j+1} \int_{a_{j+1}}^1 da_{j+2} \cdots \int_{a_{N-1}}^1 da_N. \end{aligned} \quad (9.11)$$

The first set of integrals $\int_0^a da_1 \ldots \int_{a_{j-2}}^a da_{j-1}$ gives as a result $\frac{a^{j-1}}{(j-1)!}$; on the other hand, the second set $\int_a^1 da_{j+1} \ldots \int_{a_{N-1}}^1 da_N$ gives $\frac{(1-a)^{N-j}}{(N-j)!}$. Joining both results Eq. (9.11) yields

$$f_j(a) = N! \frac{a^{j-1}(1-a)^{N-j}}{(j-1)!(N-j)!} = \frac{a^{j-1}(1-a)^{N-j}}{B(j,N-j+1)} \quad (9.12)$$

where $B(j,N-j+1)$ accounts for the binomial coefficient. In Fig. 9.14 we have plotted the distributions corresponding to different classifieds, obtained for a set of $N=30$ players. We compare the results obtained through numerical simulations –solid line– with the theoretical description explained above –circles–. We can deduce from the right panel in Fig. 9.14 that the theoretical description works rather well with the classifieds ranging from the third up to the last one, the thirtieth. However, we can see from the right panel from Fig. 9.14 that it does not work quite well for the first and second classifieds. This is so because our approach considers that players are *eliminated* according to their markmanship: the higher is the markmanship of a player, the higher the probability of being suppressed from the game. But when there are two players left in the game, we know from duel analysis carried out in Sec. 9.3 that the opposite is true for this case: players with low markmanship are those with higher probability of being eliminated. This is the reason why our approach does not provide a good description of the first and second classifieds.

Our next step would be then the survey for different values of $N$. Fig. 9.15 shows the histogram of the winners of a N–uel when varying $N$. It can be clearly seen that for values of $N \geq 4$ the optimum/maximum value of the distribution is indeed progressively enhanced and shifted towards zero when $N$ is increased.

# Chapter 10

# Conclusions

This thesis has considered two kinds of paradoxical games: Parrondo's games and truels. We now summarize the main original results as well as outlining some of the perspectives for future work.

We have introduced in Chapter 3 a new version of Parrondo's games including the *self–transition* probability. The original Parrondo games are then a special case with self-transition probabilities set to zero. Discrete–time Markov chain analysis have been performed for these new games, showing that Parrondo's paradox still occurs if the appropriate conditions are fulfilled. New expressions for the rates of winning have been obtained, with the result that under certain conditions a higher rate of winning than in the original games can be obtained. We have also studied the region of parameter space where the paradox exists with the self–transition variables, concluding that the parameter space of the original games is a limiting case of maximum volume – as the self-transition probabilities increase in value the volume shrinks to zero. However, despite this decrease in volume, the rates of winning that can be obtained are higher than in the original games.

One of the main results of the thesis concerns the quantitative relation established between Parrondo's games and the Brownian ratchet in Chapter 4. We have been able to write the master equation describing the Parrondo's games as a consistent discretization of the formalism of the Fokker–Planck equation for an overdamped Brownian particle. In this way we can relate the probabilities of the games $\{p_0, \ldots, p_{L-1}\}$ to the dynamical potential $V(x)$. Our approach yields a periodic potential for a fair game and a tilted potential for an unfair game, with positive slope for losing games and negative for winning games. The resulting expressions, in the limit $\Delta x \to 0$ could be used to obtain the effective potential for a flashing ratchet as well as its current. This relation also works in two ways: we can obtain the physical potential corresponding to a set of probabilities defining a Parrondo game, as well as the current and its stationary probability distribution. Inversely, the probabilities corresponding to a given physical potential can also be obtained. Our relations work both in cases of additive and multiplicative noise, showing that the former case is equivalent to the original Parrondo's games, whereas the latter corresponds to Parrondo's games with self–transition probability already introduced in Chapter 3.

With the relations introduced for the cases of additive and multiplicative noise, we have now a precise and of general validity connection between individual Brownian ratchets and single Parrondo's games. This work confirms Parrondo's original intuition based on a flashing ratchet is correct with rigour.

Besides, the similarity between the original Parrondo's games and the flashing ratchet is further extended to the field of information theory. In Chapter 5 we have quantified the amount of transfer of information (negentropy) for the original Parrondo's games as well as other versions. The relation between the gain in the games and the entropy difference follows a similar behavior for every version of the games analyzed, showing its robustness, and it is the equivalent of the result obtained in the case of the Brownian ratchets. In the case of the original Parrondo's paradox mixing two games, A and B, we have obtained analytically an estimation of the entropy considering that the capital originates from a combination of two ergodic sources, reflecting the different winning probabilities when the capital is a multiple of three or not. We have shown that the entropy behaves very differently for low and high values of the delay parameter $\delta_t$: while for $\delta_t = 1$ there is a monotonic dependence on the switching parameter $\gamma$, the relation between the gain and the current is only apparent for large values of $\delta_t$.

In Chapter 6 we have rewritten the master equations describing the alternation between two Parrondo games A and B with different transition probabilities $\gamma_{AB}$, $\gamma_{BA}$ as a conveniently discretized set of Fokker–Planck equations for a Brownian particle. In the particular case $\gamma_{AB} + \gamma_{BA} = 1$, we have obtained analytical expressions for the stationary probabilities in terms of the potential function already developed in Chapter 4. Using this analogy we have been able to provide suitable definitions for the energy input, energy output, average gain and efficiency of the Parrondo games. The efficiency quantifies the relationship between the gain of the games (the energy output $E_{out}$ is directly related to the current $J$) and a convenient measure of the difference between the probabilities defining the games (given by their difference in the potentials). We have evaluated the efficiency for biased and unbiased games and studied its dependence on the mixing probability $\gamma$, showing that it shares many qualitative features with the continuous model of a flashing Brownian ratchet. Our results provide a framework for comparing different Parrondian or discrete-time ratchets, and should provide a basis for the search of higher efficiency discrete systems.

Once a quantitative connection between Parrondo's games and a Brownian particle has been established in the case of a single player, we turn our attention to the case of collective games.

Chapter 7 has been devoted to a theoretical analysis of the collective games introduced in ref. [38]. We have analyzed the alternation of the original capital dependent game B with different versions of game A, in which a redistribution of capital takes place amongst the players. It has been shown that for all cases it is possible to find an equation describing the evolution (on average) of capital for a single player, and surprisingly, this equation turns to be the same for all cases studied. Besides, for the case of random diffusion to nearest neighbors it has been possible to find, in a first approximation, a direct relation with a set of $N$ coupled Brownian particles. This coupling was present in the noise terms, conserving on average the mean value of the position, as it occurs with the discrete model.

In Chapter 8 we have presented a new type of collective Parrondo games. They present, besides the Parrondo effect, a current inversion when varying the alternation probability $\gamma$ between the two games A and B. The novelty introduced in these games lies on the fact that the current inversion appears from the combination of a collective game – i.e., game B – and a totally unbiased, state independent, game A. Analytical expressions for the games have been obtained for a finite number of players using discrete–time Markov chain techniques. We have also been able to explain qualitatively the reason of this current inversion.

In the last Chapter 9 we have performed a detailed analysis of the truels, using the methods of Markov chain theory. Hence, we have been able to reproduce in a language which is more familiar to the Physics community most of the results of the original analysis by Kilgour [40]. In particular, we have obtained the survival probabilities for every truel game and for arbitrary values of markmanships $a$, $b$ and $c$, as well as their equilibrium points. Besides computing the optimal rational strategy, we have focused on computing the distribution of winners in a truel competition. We have shown that in the random case, the distribution of winners still has its maximum at the highest possible marksmanship $x = 1$, despite the fact that in some cases players with a lower marksmanship have a higher probability of winning the game. In the sequential firing case, a player performs better on average if he has intermediate values of the markmanship. This is reflected in the fact that the distribution of winners has a maximum at $x < 1$.

We have reinterpreted the random truel as an opinion model, obtaining its equilibrium points and the distribution of winners. As it happened in the random truel, the distribution of winners presents a maximum at $x = 1$, indicating that on average the opinion most likely to become majority is that with the highest convincing probability.

We have also analyzed the effect of including a spatial dependence in the random and sequential truels, as well as the opinion model. We distinguish two regimes in the dynamics: one being characterized by truel competitions, and a second characterized by duel competitions due to the diffusion of players to neighboring sites in the grid. The winning population will result from a balance between these two regimes.

Finally, we have shown the effect of generalizing the random truel to more than three players. In this case, already for $4$ players we highlight the appearance of an optimum value for the markmanship which is lower than one, a similar effect as in the sequential truel but in this case it appears due to an increase of the number of players. Furthermore, as $N$ is increased, this optimal value shifts towards lower and lower values of the markmanship.

## 10.1 Perspectives and future work

Once a complete relation has been established between the physical model of the flashing ratchet and Parrondo games for a single player, we should focus our future work on the establishment of a similar relation between collective games and collective models of Brownian particles. As in the single case, it would be desirable to obtain a relation between the probabilities defining the collective game and the drift and diffusion matrices defining a multivariate Fokker-Planck equation. Furthermore, we could also obtain an *effective* potential yielding an unbiased potential for fair games and biased for unfair games, as it occurs for the single player case. This connection should be as general as possible, so that it can be applied to a wider range of collective games.

Concerning the collective games introduced in Chapter 8, it remains as an open question the possible implications of these findings in the field of the Brownian ratchet, as well as the possibility of finding a physical model equivalent to this collective game.

Regarding truel games, our next step would involve a deeper study of the dynamics of these games in terms of the spatial grid used. Small-world or even scale-free networks could be introduced in the model so as to analyze the effect of different topologies on the final population comparing the results obtained for the two-dimensional grid.

Furthermore, it would be worth studying a generalization of the sequential truel to a number of players greater than three. It seems reasonable to consider either for the random truel and the opinion model that the unique equilibrium point is given by the *strongest opponent strategy*. However, for the sequential truel the situation turns to be far different. This case entails a greater complexity in determining its equilibrium points, as the number of strategies feasible is quite large.

An interesting extension for truel games would be that of including a dynamics based on selection and evolution. We could allow strategies to evolve, in the sense that players would modify their own strategy if they contemplate the possibility of improving their own payoff. Hence we could study the evolution of the strategies adopted by the players and check whether they tend to a fixed set. On the other hand, we could also allow the fitness –markmanship– of the players to change/evolve over time, therefore studying the dynamics of the system under this scheme.

# Appendix A

# Collective Parrondo games with redistribution of capital

## A.1 Distribution to a randomly selected player

Our starting point is the general master equation (7.3) which we reproduce here

$$P_{(c_1,..,c_N;\tau+1)} = \sum_{j=1}^{N} \sum_{\substack{j'=1 \\ j'\neq j}}^{N} \frac{\gamma}{N(N-1)} P_{(c_1,..,c_j+1,..,c_{j'}-1,..,c_N;\tau)} +$$

$$+ \frac{1-\gamma}{N} \sum_{j=1}^{N} \left[ a_{-1}^{c_j} P_{(c_1,..,c_j-1,..,c_N;\tau)} + a_0^{c_j} P_{(c_1,..,c_N;\tau)} + a_1^{c_j} P_{(c_1,..,c_j+1,..,c_N;\tau)} \right]. \quad \text{(A.1)}$$

From Eq. (A.1) we can obtain the probability density function for a single player $j$, i.e., $P_{(c_j;\tau)}$, performing the following sum

$$P_{(c_j;\tau+1)} = \sum_{c_1,..c_{j-1},c_{j+1},..c_N} P_{(c_1,..,c_N;\tau+1)}. \quad \text{(A.2)}$$

For simplicity we will calculate separately the two contributions of Eq. (A.1) to this sum, the first one being that of game $A'$, and the second being that of game $B$. Let us first calculate the sum for game $A'$

$$\frac{\gamma}{N(N-1)} \sum_{c_1..c_{j-1},c_{j+1}..c_N} \left[ \sum_{j'=1}^{N} \sum_{\substack{j''=1 \\ j''\neq j'}}^{N} P(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau) \right] =$$

$$= \frac{\gamma}{N(N-1)} \sum_{c_1..c_{j-1},c_{j+1}..c_N} \left[ \sum_{\substack{j''=1 \\ j''\neq j}}^{N} P(c_1,..,c_j+1,..,c_{j''}-1,..,c_N;\tau) + \right.$$

$$\left. + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} P(c_1,..,c_{j'}+1,..,c_j-1,..,c_N;\tau) + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} \sum_{\substack{j''=1 \\ j''\neq j,j'}}^{N} P(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau) \right]. \quad \text{(A.3)}$$

Carrying out the sum $\sum_{c_1,\ldots,c_{j-1},c_{j+1},\ldots c_N}$ we obtain

$$\frac{\gamma}{N(N-1)} \left[ \sum_{\substack{j''=1 \\ j''\neq j}}^{N} P(c_j+1;\tau) + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} P(c_j-1;\tau) + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} \sum_{\substack{j''=1 \\ j''\neq j,j'}}^{N} P(c_j;\tau) \right] =$$

$$= \frac{\gamma}{N(N-1)} \Big[ (N-1)P(c_j+1;\tau) + (N-1)P(c_j-1;\tau) + (N-2)(N-1)P(c_j;\tau) \Big] =$$

$$= \frac{\gamma}{N} \left[ P(c_j+1;\tau) + (N-2)P(c_j;\tau) + P(c_j-1;\tau) \right]. \quad \text{(A.4)}$$

We now proceed with the second term of Eq. (A.1), that of game $B$ [1]

$$\sum_{c_1..,c_{j-1},c_{j+1},..c_N} \sum_{j'=1}^{N} \Big[ a_{-1}^{c_{j'}} P(c_1,..,c_{j'}-1,..,c_N;\tau) + a_0^{c_{j'}} P(c_1,..,c_N;\tau) +$$

$$+ a_1^{c_{j'}} P(c_1,..,c_{j'}+1,..,c_N;\tau) \Big] = \sum_{c_1..,c_{j-1},c_{j+1},..c_N} \Big[ a_{-1}^{c_j} P(c_1,..,c_j-1,..,c_N;\tau) + a_0^{c_j} P(c_1,..,c_N;\tau) +$$

$$+ a_1^{c_j} P(c_1,..,c_j+1,..,c_N;\tau) + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} ( a_{-1}^{c_{j'}} P(c_1,..,c_{j'}-1,..,c_N;\tau) + a_0^{c_{j'}} P(c_1,..,c_N;\tau) +$$

[1] For simplicity we will omit the coefficient $\frac{1-\gamma}{N}$ until the final result for game $B$ is obtained.

$$+ a_1^{c_{j'}} P(c_1,..,c_{j'}+1,..,c_N;\tau))\Bigg] = \Bigg[a_{-1}^{c_j} P(c_j-1;\tau) + a_0^{c_j} P(c_j;\tau) + a_1^{c_j} P(c_j+1;\tau) +$$

$$+ \sum_{\substack{j'=1 \\ j'\neq j}}^{N} \sum_{c_{j'}} (a_{-1}^{c_{j'}} P(c_j,c_{j'}-1;\tau) + a_0^{c_{j'}} P(c_j,c_{j'};\tau) + a_1^{c_{j'}} P(c_j,c_{j'}+1;\tau))\Bigg]. \quad \text{(A.5)}$$

By means of normalization condition $a_{-1}^{c_{j'}+1} + a_0^{c_{j'}} + a_1^{c_{j'}-1} = 1$, the previous expression can be simplified obtaining

$$\left[a_{-1}^{c_j} P(c_j-1;\tau) + a_0^{c_j} P(c_j;\tau) + a_1^{c_j} P(c_j+1;\tau) + \sum_{\substack{j'=1 \\ j'\neq j}}^{N} \sum_{c_{j'}} P(c_j,c_{j'};\tau)\right] =$$

$$= \frac{1-\gamma}{N}\Bigg[a_{-1}^{c_j} P(c_j-1;\tau) + a_0^{c_j} P(c_j;\tau) + a_1^{c_j} P(c_j+1;\tau) + (N-1)P(c_j;\tau)\Bigg]. \quad \text{(A.6)}$$

Finally, adding both results (A.4) and (A.6) we obtain the final expression for the evolution of the probability for a single player $j$ with capital $c_j$

$$P(c_j;\tau+1) = \frac{1-\gamma}{N}\Bigg[a_{-1}^{c_j} P(c_j-1;\tau) + a_0^{c_j} P(c_j;\tau) + a_1^{c_j} P(c_j+1;\tau)\Bigg] +$$

$$+ \frac{\gamma}{N}\Bigg[P(c_j+1;\tau) + P(c_j-1;\tau)\Bigg] + \frac{N-(1+\gamma)}{N} P(c_j;\tau) \quad \text{(A.7)}$$

## A.2 Distribution to nearest neighbor with constant probabilities

In this section we calculate the equation for a single player $j$ when alternating between the original Parrondo game $B$ and another version of the redistributing game $A''$, in which there are different probabilities $p_r$ and $p_l$ of giving a coin to neighbor $j+1$ on the right and to $j-1$ on the left respectively. The master equation describing the evolution of $P(c_1,c_2,..,c_N;\tau+1)$ of all $N$ players is given by Eq. (7.10), that is

$$P(c_1,..,c_N;\tau+1) = \frac{\gamma}{N} \sum_{j'=1}^{N} \Bigg[p_l\, P(c_1,..,c_{j'-1}-1,c_{j'}+1,..,c_N;\tau) + p_r\, P(c_1,..,c_{j'}+1,c_{j'+1}-1,..,c_N;\tau)\Bigg] +$$

$$+ \frac{1-\gamma}{N} \sum_{j=1}^{N} \Bigg[a_{-1}^{c_j} P(c_1,..,c_j-1,c_N;\tau) + a_0^{c_j} P(c_1,..,c_j,c_N;\tau) + a_1^{c_j} P(c_1,..,c_j+1,c_N;\tau)\Bigg]. \quad \text{(A.8)}$$

Again, the sum $\sum_{c_1..c_{j-1},c_{j+1},..,c_N}$ must be performed on the previous equation in order to obtain the single probability density function. As already calculated in the previous section, the result of the sum for the term corresponding to game $B$ is given by Eq. (A.6); therefore, we need only to calculate that of game $A''$, thus we have [2]

$$\sum_{c_1,..,c_{j-1},c_{j+1},..c_N} \sum_{j'=1}^{N} \Big[p_l\, P(c_1,..,c_{j'-1}-1,c_{j'}+1,..,c_N;\tau) + p_r\, P(c_1,..,c_{j'}+1,c_{j'+1}-1,..,c_N;\tau)\Big] =$$

$$= \sum_{c_1,..,c_{j-1},c_{j+1},..,c_N} \Bigg[p_l\, P(c_1,..,c_{j-1}-1,c_j+1,..,c_N;\tau) + p_r\, P(c_1,..,c_j+1,c_{j+1}-1,..,c_N;\tau)+$$
$$+ p_l\, P(c_1,..,c_j-1,c_{j+1}+1,..,c_N;\tau) + p_r\, P(c_1,..,c_{j+1}+1,c_{j+2}-1,..,c_N;\tau)+$$
$$+ p_l\, P(c_1,..,c_{j-2}-1,c_{j-1}+1,..,c_N;\tau) + p_r\, P(c_1,..,c_{j-1}+1,c_j-1,..,c_N;\tau)+$$
$$\sum_{\substack{j'=1\\ j'\neq j,j\pm1}}^{N} \big(p_l\, P(c_1,..,c_{j'-1}-1,c_{j'}+1,..,c_N;\tau) + p_r\, P(c_1,..,c_{j'}+1,c_{j'+1}-1,..,c_N;\tau)\big)\Bigg] =$$

$$= \left( P(c_j+1;\tau) + P(c_j-1;\tau) + P(c_j;\tau) + \sum_{\substack{j'=1\\ j'\neq j,j\pm1}}^{N} [p_l\, P(c_j;\tau) + p_r\, P(c_j;\tau)] \right) =$$

$$= \frac{\gamma}{N}\Big[P(c_j+1;\tau) + P(c_j-1;\tau) + (N-2)P(c_j;\tau)\Big]. \quad \text{(A.9)}$$

Again, the result obtained for game $A''$ when the capital is redistributed to nearest neighbors agrees with that of random distribution of capital between players. It is remarkable though that the final result does not depend on the actual probabilities $p_l$ and $p_r$. Therefore, joining results from Eqs. (A.6) and (A.9) we obtain the same equation governing the evolution of $P(c_j;\tau)$ for a single player $j$, namely, Eq. (A.7).

## A.3 Distribution to nearest neighbor with capital dependent probabilities

This section will be dedicated to the derivation of the equation for the evolution of the probability for a single player $j$, when alternating between a new version of game $A''$ in which the probabilities do depend on the capital of the neighbors, and the original game $B$. Our starting point is the master equation for the total probability $P(c_1,\cdots,c_N;\tau+1)$, i.e., Eq. (7.23),

[2]Again, we omit the coefficient $\frac{\gamma}{N}$ until the final result for game $A''$ is obtained.

$$P_{(c_1,..,c_N;\tau+1)} = \frac{\gamma}{N}\sum_{j',j''=1}^{N}\Big[p_{j',j''}P_{(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau)}\Big]+$$
$$+\frac{1-\gamma}{N}\sum_{j'=1}^{N}\left[a_{-1}^{c'_j}P_{(c_1,..,c_{j'}-1,..,c_N;\tau)} + a_0^{c_{j'}}P_{(c_1,..,c_N;\tau)} + a_1^{c_{j'}}P_{(c_1,..,c_{j'}+1,..,c_N;\tau)}\right], \tag{A.10}$$

where, as in previous cases, game $A''$ is played with probability $\gamma$ and game $B$ with probability $1-\gamma$.

We must perform the sum $\sum_{c_1..,c_{j-1},c_{j+1},..c_N}$ in Eq. (A.10) in order to obtain the single distribution $P_{(c_j;\tau)}$. However, from a previous calculation we already know the result corresponding to the term of game $B$, *c.f.* Eq. (A.6). Therefore, we need only to calculate the remaining sum, that corresponding to game $A''$, [3]

$$\sum_{c_1..c_{j-1},c_{j+1}..c_N}\sum_{j',j''=1}^{N}p_{j',j''}P_{(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau)} =$$
$$= \sum_{c_1..c_{j-1},c_{j+1}..c_N}\sum_{j'=1}^{N}\big[p_{j',j'-1}P_{(c_1,..,c_{j'-1}-1,c_{j'}+1,..,c_N;\tau)}+$$
$$+p_{j',j'+1}P_{(c_1,..,c_{j'}+1,c_{j'+1}-1,..,c_N;\tau)}\big] = \sum_{c_1..c_{j-1},c_{j+1}..c_N}\big[p_{j,j-1}P_{(c_1,..c_{j-1}-1,c_j+1..,c_N;\tau)}+$$
$$+\,p_{j,j+1}P_{(c_1,..c_j+1,c_{j+1}-1..,c_N;\tau)} + p_{j+1,j}\,P_{(c_1,..,c_j-1,c_{j+1}+1,..,c_N;\tau)}+$$
$$+\,p_{j+1,j+2}\,P_{(c_1,..,c_{j+1}-1,c_{j+2}-1,..,c_N;\tau)} + p_{j-1,j-2}\,P_{(c_1,..,c_{j-2}-1,c_{j-1}+1,..,c_N;\tau)}+$$
$$+\,p_{j-1,j}\,P_{(c_1,..,c_{j-1}+1,c_j-1,..,c_N;\tau)} + \sum_{\substack{j'=1\\ j'\neq j\\ j'\neq j-1,j+1}}^{N}\big[p_{j',j'-1}\,P_{(c_1,..,c_{j'-1}-1,c_{j'}+1,..,c_N;\tau)}+$$
$$+\,p_{j',j'+1}\,P_{(c_1,..,c_{j'}+1,c_{j'+1}-1,..,c_N;\tau)}\big] =$$
$$=\Bigg\{\sum_{\substack{c_{j-1}\\ c_{j+1}}}\big[p_{j,j-1}P_{(c_{j-1}-1,c_j+1,c_{j+1};\tau)} + p_{j,j+1}P_{(c_{j-1},c_j+1,c_{j+1}-1;\tau)}\big]+$$
$$+\sum_{c_{j+1},c_{j+2}}\big[p_{j+1,j}P_{(c_j-1,c_{j+1}+1,c_{j+2};\tau)} + p_{j+1,j+2}P_{(c_j,c_{j+1}+1,c_{j+2}-1;\tau)}\big]+$$

[3] As in previous calculations, we omit $\frac{\gamma}{N}$ until the final result for $A''$ is obtained

$$+\sum_{c_{j-2},c_{j-1}}\left[p_{j-1,j-2}P(c_{j-2}-1,c_{j-1}+1,c_j;\tau)+p_{j-1,j}P(c_{j-2},c_{j-1}+1,c_j-1;\tau)\right]+$$

$$\left.+\sum_{\substack{j'=1\\ j'\neq j,\\ j'\neq j-1,j+1}}^{N}\sum_{\substack{c_{j'-1}\\ c_{j'+1}\\ c_{j'}}}\left[p_{j',j'-1}P(c_{j'-1}-1,c_{j'}+1,c_{j'+1};\tau)+p_{j',j'+1}P(c_{j'-1},c_{j'}+1,c_{j'+1}-1;\tau)\right]\right\} \tag{A.11}$$

In the last part of Eq. (A.11) we have splitted the sum over $j'$ into four terms. The three first factors correspond to those in which the probabilities $p_{i,i'}$ depend on the capital $c_j$ of player $j$; the last factor is simply the sum over the rest of terms where $c_j$ is not present. We can perform the sum in the latter factor substituting the expressions for $p_{i,i'}$ obtaining

$$\sum_{\substack{j'=1\\ j'\neq j\\ j'\neq j-1,j+1}}^{N}\sum_{\substack{c_{j'-1}\\ c_{j'+1}\\ c_{j'}}}\frac{1}{c_{j'+1}+c_{j'-1}-1}\left[c_{j'+1}P(c_{j'-1}-1,c_{j'}+1,c_{j'+1},c_j;\tau)+\right.$$

$$\left.+\,c_{j'-1}P(c_{j'-1},c_{j'}+1,c_{j'+1}-1,c_j;\tau)\right]=$$

$$=\sum_{\substack{j'=1\\ j'\neq j\\ j'\neq j-1,j+1}}^{N}\sum_{c_{j'-1},c_{j'+1}}\frac{1}{c_{j'+1}+c_{j'-1}-1}\left[c_{j'+1}P(c_{j'-1}-1,c_{j'+1},c_j;\tau)+c_{j'-1}P(c_{j'-1},c_{j'+1}-1,c_j;\tau)\right]$$

$$=\left\{\begin{matrix}c_{j'-1}-1\longrightarrow c_{j'-1}\\ c_{j'+1}-1\longrightarrow c_{j'+1}\end{matrix}\right\}=\sum_{\substack{j'=1\\ j'\neq j\\ j'\neq j-1,j+1}}^{N}P(c_j;\tau)=(N-3)P(c_j;\tau). \tag{A.12}$$

Due to the summations carried out for $\sum_{\substack{c_{j'-1}\\ c_{j'+1}}}$, we have changed the values for $c_{j'+1}$ and $c_{j'-1}$, increasing their values in one unit. We can now proceed with the remaining three sums in Eq. (A.11). The sum with terms $c_{j+1}, c_{j-1}$ results in

$$\sum_{c_{j-1},c_{j+1}}\left[p_{j,j-1}P(c_{j-1}-1,c_j+1,c_{j+1};\tau)+p_{j,j+1}P(c_{j-1},c_j+1,c_{j+1}-1;\tau)\right]=$$

$$\sum_{c_{j-1},c_{j+1}}\left[\frac{c_{j+1}}{c_{j+1}+c_{j-1}}P(c_{j-1},c_j+1,c_{j+1};\tau)+\frac{c_{j-1}}{c_{j+1}+c_{j-1}}P(c_{j-1},c_j+1,c_{j+1};\tau)\right]=$$

$$= \sum_{c_{j-1},c_{j+1}} P(c_{j-1},c_j+1,c_{j+1};\tau) = P(c_j+1;\tau). \quad \text{(A.13)}$$

We are left now with the terms corresponding to the sums for $c_{j+1}, c_{j+2}$ and that for $c_{j-1}, c_{j-2}$. In order to solve them, we must assume the following set of hypothesis,

$$\left.\begin{array}{l} P(c_j-1,c_{j+2};\tau)\to P(c_j,c_{j+2}-1;\tau) \\ P(c_{j-2}-1,c_j;\tau)\to P(c_{j-2},c_j-1;\tau) \end{array}\right\} \tau \to \infty \quad \text{(A.14)}$$

In some sense, this hypothesis might imply that for large times $\tau$ two individuals become indistinguishable. Therefore, by means of hypothesis (A.14), we are able to perform the remaining sums as

$$\sum_{\substack{c_{j+1}\\c_{j+2}}} \left[\tfrac{c_{j+2}}{c_{j+2}+c_{j}-1} P(c_j-1,c_{j+1}+1,c_{j+2};\tau) + \tfrac{c_j}{c_{j+2}+c_j-1} P(c_j,c_{j+1}+1,c_{j+2}-1;\tau)\right] +$$

$$+ \sum_{\substack{c_{j-2}\\c_{j-1}}} \left[\tfrac{c_j}{c_{j-2}+c_j-1} P(c_{j-2}-1,c_{j-1}+1,c_j;\tau) + \tfrac{c_{j-2}}{c_{j-2}+c_j-1} P(c_{j-2},c_{j-1}+1,c_j-1;\tau)\right] =$$

$$= \sum_{c_{j+2}} \left[\tfrac{c_{j+2}}{c_{j+2}+c_j-1} + \tfrac{c_j}{c_{j+2}+c_j-1}\right] P(c_j-1,c_{j+2};\tau) +$$

$$+ \sum_{c_{j-2}} \left[\tfrac{c_j}{c_j+c_{j-2}-1} + \tfrac{c_{j-2}}{c_{j-2}+c_j-1}\right] P(c_{j-2},c_j;\tau) = P(c_j-1;\tau) + P(c_j;\tau). \quad \text{(A.15)}$$

Finally, joining results from Eqs. (A.12),(A.13),(A.15) we obtain the desired result for the term corresponding to game $A''$

$$\sum_{c_1..c_N} \frac{\gamma}{N} \sum_{j',j''=1}^{N} \left[p_{j',j''} P(c_1,..,c_{j'}+1,..,c_{j''}-1,..,c_N;\tau)\right] =$$

$$= \frac{\gamma}{N} \left[P(c_{j-1};\tau) + (N-2)P(c_j;\tau) + P(c_{j+1};\tau)\right] \quad \text{(A.16)}$$

Finally, joining both Eqs. (A.6) and (A.16) what we obtain is exactly the same equation as (A.7) for the evolution of the probability $P(c_j;\tau)$ of a single player $j$.

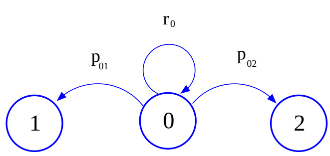
# Appendix B

# Survival probabilities for duels and truels

In this Appendix we will deduce the expressions corresponding to the survival probabilities when playing either a duel or a truel. Both games can be described with discrete–time Markov chains with a finite number of states. Besides, they are characterized by the existence of a certain number of *absorbing states*, which means that once the system reaches this state, it never leaves it (they correspond to those states where there is only one survivor in the game). As we are dealing with finite Markov chains, it is certain [64] that this system will eventually end up in one of its absorbent states. We will first calculate the survival probabilities for the simplest case of duels in Sec. B.1, followed then by their analogous in truels in Secs. B.2, B.2, B.3 and B.4.

## B.1 Duels

In Fig. B.1 we show a Markov chain with three states $0, 1, 2$ corresponding to the random duel and also the opinion model. The Table in Fig. B.1 shows the correspondence between the players remaining on the game and their corresponding state for both the random duel and the opinion model.




| | *Random Duel* | *Opinion Duel* |
|---|---|---|
| *States* | *Players* | *Opinions* |
| 0 | A B | A B |
| 1 | A | A A |
| 2 | B | B B |

**Figure B.1.** Table: description of the different states for the random duel and opinion model. Diagram: Markov chain corresponding to both the random duel and opinion model with two opinions.

From Markov chain theory[41] we can calculate the probability $u_i^j$ that starting from state $i$ we eventually end up in state $j$ after a sufficiently large number of steps. We are interested in calculating the probability that starting from state 0 we end up either in state 1 or state 2. The set of equations to be solved are

$$u_0^1 = p_{01}u_1^1 + r_0u_1^1 \tag{B.1}$$
$$u_0^2 = p_{02}u_2^2 + r_0u_0^2 \tag{B.2}$$

where the transition probabilities $p_{ij}$ between states are given by :

$$r_0 = \frac{1}{2}[2-a-b]\,,\quad p_{01} = \frac{1}{2}a\,,\quad p_{02} = \frac{1}{2}b \tag{B.3}$$

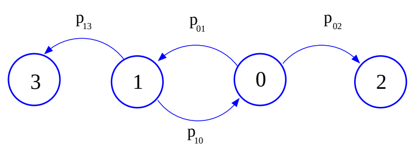
Recalling that by definition $u_j^j = 1$ we may solve Eqs. (B.1), (B.2) obtaining

$$u_0^1 = \frac{p_{01}}{1-r_0}\,,\qquad u_0^2 = \frac{p02}{1-r_0}\,, \tag{B.4}$$

Substituting the transition probabilities in the previous set of equations we obtain the survival probabilities for player A ($u_0^1$) and player B ($u_0^2$)

$$\pi_A = \frac{a}{a+b}\,,\qquad \pi_B = \frac{b}{a+b}\,, \tag{B.5}$$

We may now consider the Markov chain describing the sequential duel. It is composed of four states $0,1,2,3$ and is depicted in Fig. B.2. The table from Fig. B.2 shows the relation between the states and the players that are still on the game.




| *States* | *Remaining players* |
|---|---|
| 0 | **A** B |
| 1 | A **B** |
| 2 | A |
| 3 | B |

**Figure B.2.** Table: description of the different states for the sequential duel. Diagram: Markov chain with four states corresponding to the sequential duel.

The set of equations to be solved are

$$u_0^2 = p_{02}u_2^2 + p_{01}u_1^2 \tag{B.6}$$
$$u_0^3 = p_{01}u_1^3 \tag{B.7}$$
$$u_1^2 = p_{10}u_0^2 \tag{B.8}$$
$$u_1^3 = p_{13}u_3^3 + p_{10}u_0^3 \tag{B.9}$$

where

$$p_{01} = 1 - a\,, \quad p_{02} = a\,, \quad p_{10} = 1 - b\,, \quad p_{13} = b \tag{B.10}$$

The general solutions for Eqs. (B.6)–(B.9) are

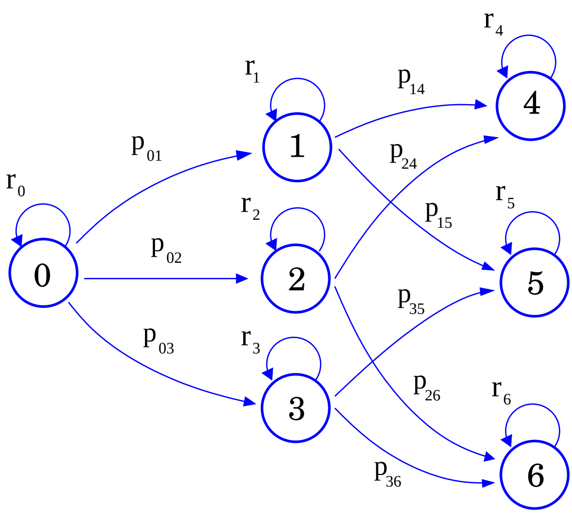
$$u_0^2 = \frac{p_{02}}{1 - p_{01}p_{10}}\,, \qquad u_0^3 = \frac{p_{01}p_{13}}{1 - p_{01}p_{10}}\,, \tag{B.11}$$

which, after substituting the transition probabilities give as a result

$$\pi_A = u_0^2 = \frac{a}{1-(1-a)(1-b)}\,, \qquad \pi_B = u_0^3 = \frac{b(1-a)}{1-(1-a)(1-b)}\,, \tag{B.12}$$

## B.2 Random firing

For this game there are seven possible states according to the remaining players. These are labeled as $0, 1, \ldots, 6$. The allowed transitions between states are shown in the diagram in Fig. B.3, where $p_{ij}$ denotes the transition probability from state $i$ to state $j$ (the self–transition probability $p_{ii}$ is denoted by $r_i$).




| *States* | *Remaining players* |
|---|---|
| 0 | ABC |
| 1 | AB |
| 2 | AC |
| 3 | BC |
| 4 | A |
| 5 | B |
| 6 | C |

**Figure B.3.** Table with the description of all possible states for the random firing game, and diagram representing the allowed transitions between the states shown in the table.

From Markov chain theory [64] we can evaluate the probability $u_i^j$ that starting from state $i$ we eventually end up in state $j$ after a sufficiently large number of steps. In particular, if we start from state $0$ (with the three players active), the nature of the game is such that the only non-vanishing probabilities are $u_0^4$, $u_0^5$ and $u_0^6$ corresponding to the winning of the game by player A, B and C respectively. The relevant set of equations is

$$
\begin{array}{ll}
u_0^4 = p_{01}\, u_1^4 + p_{02}\, u_2^4 + p_{03}\, u_3^4 + r_0\, u_0^4, & u_0^5 = p_{01}\, u_1^5 + p_{02}\, u_2^5 + p_{03}\, u_3^5 + r_0\, u_0^5,\\
u_0^6 = p_{01}\, u_1^6 + p_{02}\, u_2^6 + p_{03}\, u_3^6 + r_0\, u_0^6, & u_1^4 = p_{14}\, u_4^4 + r_1\, u_1^4,\\
u_1^5 = p_{15}\, u_5^5 + r_1\, u_1^5, & u_1^6 = r_1\, u_1^6,\\
u_2^4 = p_{24}\, u_4^4 + r_2\, u_2^4, & u_2^5 = r_2\, u_2^5,\\
u_2^6 = r_2\, u_2^6 + p_{26} u_6^6, & u_3^4 = r_3\, u_3^4,\\
u_3^5 = r_3\, u_3^5 + p_{35}\, u_5^5, & u_3^6 = r_3\, u_3^6 + p_{36}\, u_6^6.
\end{array}
\tag{B.13}
$$

We can solve the previous set of equations for $u_0^4$, $u_0^5$ and $u_0^6$, considering that by definition $u_j^j = 1\ \forall j$. The solutions are

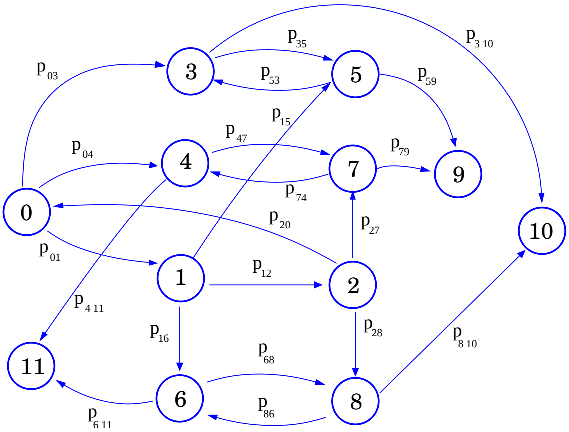
$$
\begin{aligned}
u_0^4 &= \frac{p_{01}\, p_{14}}{(1-r_0)(1-r_1)} + \frac{p_{02}\, p_{24}}{(1-r_0)(1-r_2)},\\
u_0^5 &= \frac{p_{01}\, p_{15}}{(1-r_0)(1-r_1)} + \frac{p_{03}\, p_{35}}{(1-r_0)(1-r_3)},\\
u_0^6 &= \frac{p_{02}\, p_{26}}{(1-r_0)(1-r_2)} + \frac{p_{03}\, p_{36}}{(1-r_0)(1-r_3)}.
\end{aligned}
\tag{B.14}
$$

We can now derive the expressions for the transition probabilities $p_{ij}$. Remember that we denote by $a$ the probability that player A eliminates from the game the player he has aimed at (and similarly for $b$ and $c$), and that $P_{\alpha\beta}$ ($\alpha =$A,C,B and $\beta =$ A,B,C,0) the probability of player $\alpha$ choosing player $\beta$ (or into the air if $\beta = 0$) as a target when it is his turn to play (a situation that only appears when the three players are still active). We have then:

$$
\begin{array}{ll}
r_0 = 1 - \frac{1}{3}(a(1-P_{A0}) + b(1-P_{B0}) + c(1-P_{C0})), & p_{01} = \frac{1}{3}(aP_{AC} + bP_{BC}),\\
p_{02} = \frac{1}{3}(aP_{AB} + cP_{CB}), & p_{03} = \frac{1}{3}(bP_{BA} + cP_{CA}),\\
p_{14} = p_{24} = \frac{1}{2}a, & p_{15} = p_{35} = \frac{1}{2}b,\\
p_{26} = p_{36} = \frac{1}{2}c, & r_1 = 1 - \frac{1}{2}(a+b),\\
r_2 = 1 - \frac{1}{2}(a+c), & r_3 = 1 - \frac{1}{2}(b+c).
\end{array}
\tag{B.15}
$$

## B.3 Sequential firing

As in the random firing case, we describe this game as a Markov chain composed of 11 different states, also with three absorbent states: 9 , 10 and 11. In Fig. B.4 we show the corresponding diagram for this game, together with a table describing all possible states. Based on this diagram, we can write down the relevant set of equations for the transition probabilities $u_i^j$:




| *States* | *Remaining players* |
|---|---|
| 0 | A B **C** |
| 1 | A **B** C |
| 2 | **A** B C |
| 3 | **B** C |
| 4 | **A** C |
| 5 | B **C** |
| 6 | **A** B |
| 7 | A **C** |
| 8 | A **B** |
| 9 | C |
| 10 | B |
| 11 | A |

**Figure B.4.** Table: Description of the different states of the game for the case of sequential firing. The highlighted player is the one chosen for shooting in that state. Diagram: scheme representing all the allowed transitions between the states shown in the table for the case of a truel with sequential firing in the order C→ B → A with $a > b > c$.

$$
\begin{array}{ll}
u_0^9 = p_{03}u_3^9 + p_{01}u_1^9 + p_{04}u_4^9, & u_0^{10} = p_{03}u_3^{10} + p_{01}u_1^{10},\\
u_0^{11} = p_{01}u_1^{11} + p_{04}u_4^{11}, & u_1^{10} = p_{12}u_2^{10} + p_{15}u_5^{10} + p_{16}u_6^{10},\\
u_1^9 = p_{12}u_2^9 + p_{15}u_5^9, & u_1^{11} = p_{12}u_2^{11} + p_{16}u_6^{11},\\
u_2^{11} = p_{28}u_8^{11} + p_{27}u_7^{11} + p_{20}u_0^{11}, & u_2^9 = p_{27}u_7^9 + p_{20}u_0^9,\\
u_2^{10} = p_{28}u_8^{10} + p_{20}u_0^{10}, & u_3^9 = p_{35}u_5^9,\\
u_3^{10} = p_{35}u_5^{10} + p_{3\,10}, & u_4^9 = p_{47}u_7^9,\\
u_4^{11} = p_{47}u_7^{11} + p_{4\,11}, & u_5^9 = p_{53}u_3^9 + p_{59},\\
u_5^{10} = p_{53}u_3^{10}, & u_6^{10} = p_{68}u_8^{10},\\
u_6^{11} = p_{68}u_8^{11} + p_{6\,11}, & u_7^9 = p_{74}u_4^9 + p_{79},\\
u_7^{11} = p_{74}u_4^{11}, & u_8^{10} = p_{86}u_6^{10} + p_{8\,10},\\
u_8^{11} = p_{86}u_6^{11}. &
\end{array}
\tag{B.16}
$$

The general solutions for the probabilities $u_0^9$, $u_0^{10}$ and $u_0^{11}$ are given by

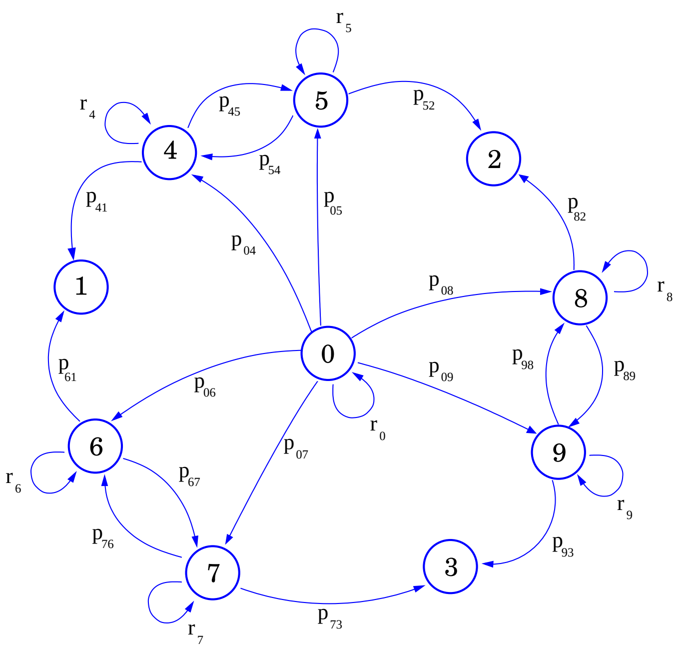
$$
\begin{aligned}
u_0^9 &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{59}(p_{03}p_{35}+p_{01}p_{15})}{1-p_{35}p_{53}} + \frac{p_{79}(p_{04}p_{47}+p_{01}p_{12}p_{27})}{1-p_{47}p_{74}}\right],\\
u_0^{10} &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{3\,10}(p_{03}+p_{01}p_{15}p_{53})}{1-p_{35}p_{53}} + \frac{p_{01}p_{8\,10}(p_{16}p_{68}+p_{12}p_{28})}{1-p_{68}p_{86}}\right],\\
u_0^{11} &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{4\,11}(p_{04}+p_{01}p_{12}p_{27}p_{74})}{1-p_{47}p_{74}} + \frac{p_{01}p_{6\,11}(p_{16}+p_{12}p_{28}p_{86})}{1-p_{68}p_{86}}\right],
\end{aligned}
\tag{B.17}
$$

with transition probabilities given by

$$
\begin{array}{lll}
p_{01} = (1-c) + cP_{C0}, & p_{03} = cP_{CA}, & p_{04} = cP_{CB}, \\
p_{12} = (1-b) + bP_{B0}, & p_{15} = bP_{BA}, & p_{16} = bP_{CA}, \\
p_{20} = (1-a) + aP_{A0}, & p_{27} = aP_{AB}, & p_{28} = aP_{AC}, \\
p_{35} = p_{86} = 1-b, & p_{3\,10} = p_{8\,10} = b, & \\
p_{47} = p_{68} = 1-a, & p_{4\,11} = p_{6\,11} = a, & \\
p_{53} = p_{74} = 1-c, & p_{59} = p_{79} = c. &
\end{array}
$$

## B.4 Convincing opinion

For this model we show in Fig. B.5 the diagram of the allowed states and transitions, together with a table describing the possible states.




| States | Opinions |
|---|---|
| 0 | A B C |
| 1 | C C C |
| 2 | B B B |
| 3 | A A A |
| 4 | C C B |
| 5 | B B C |
| 6 | A C C |
| 7 | A A C |
| 8 | A B B |
| 9 | A A B |

**Figure B.5.** Table: description of the different states of the opinion model. Diagram: scheme representing the allowed transitions between the states.

The corresponding set of equations describing this convincing opinion model, as derived from the diagram, are

$$
\begin{aligned}
u_0^1 &= r_0 u_0^1 + p_{06} u_6^1 + p_{04} u_4^1 + p_{05} u_5^1 + p_{07} u_7^1, \\
u_0^2 &= r_0 u_0^2 + p_{04} u_4^2 + p_{05} u_5^2 + p_{08} u_8^2 + p_{09} u_9^2, \\
u_0^3 &= r_0 u_0^3 + p_{08} u_8^3 + p_{09} u_9^3 + p_{07} u_7^3 + p_{06} u_6^3, \\
u_4^1 &= r_4 u_4^1 + p_{45} u_5^1 + p_{41}, & u_4^2 &= r_4 u_4^2 + p_{45} u_5^2, \\
u_5^1 &= r_5 u_5^1 + p_{54} u_4^1, & u_5^2 &= r_5 u_5^2 + p_{54} u_4^2 + p_{52}, \\
u_6^1 &= r_6 u_6^1 + p_{67} u_7^1 + p_{61}, & u_6^3 &= r_6 u_6^3 + p_{67} u_7^3, \\
u_7^1 &= r_7 u_7^1 + p_{76} u_6^1, & u_7^3 &= r_7 u_7^3 + p_{76} u_6^3 + p_{73}, \\
u_8^2 &= r_8 u_8^2 + p_{89} u_9^2 + p_{82}, & u_8^3 &= r_8 u_8^3 + p_{89} u_9^3, \\
u_9^2 &= r_9 u_9^2 + p_{98} u_8^2, & u_9^3 &= r_9 u_9^3 + p_{98} u_8^3 + p_{93}.
\end{aligned}
\tag{B.18}
$$

And the general solution for the probabilities $u_0^1$, $u_0^2$ and $u_0^3$ is

$$
\begin{aligned}
u_0^1 &= \frac{1}{1-r_0}\left[\frac{p_{61}(p_{06}(1-r_7)+p_{07}p_{76})}{(1-r_6)(1-r_7)-p_{67}p_{76}} + \frac{p_{41}(p_{04}(1-r_5)+p_{05}p_{54})}{(1-r_4)(1-r_5)-p_{45}p_{54}}\right], \\
u_0^2 &= \frac{1}{1-r_0}\left[\frac{p_{52}(p_{04}p_{45}+p_{05}(1-r_4))}{(1-r_4)(1-r_5)-p_{45}p_{54}} + \frac{p_{82}(p_{08}(1-r_9)+p_{09}p_{98})}{(1-r_8)(1-r_9)-p_{89}p_{98}}\right], \\
u_0^3 &= \frac{1}{1-r_0}\left[\frac{p_{73}(p_{06}p_{67}+p_{07}(1-r_6))}{(1-r_6)(1-r_7)-p_{67}p_{76}} + \frac{p_{93}(p_{09}(1-r_8)+p_{08}p_{89})}{(1-r_8)(1-r_9)-p_{89}p_{98}}\right],
\end{aligned}
\tag{B.19}
$$

where the transition probabilities are given by

$$
\begin{aligned}
&p_{04} = \tfrac{1}{3} c P_{CA}, && p_{06} = \tfrac{1}{3} c P_{CB}, && p_{08} = \tfrac{1}{3} b P_{BC}, \\
&p_{05} = \tfrac{1}{3} b P_{BA}, && p_{07} = \tfrac{1}{3} a P_{AB}, && p_{09} = \tfrac{1}{3} a P_{AC}, \\
&p_{41} = p_{61} = \tfrac{2}{3} c && p_{45} = p_{98} = \tfrac{1}{3} b, && p_{54} = p_{76} = \tfrac{1}{3} c, \\
&p_{52} = p_{82} = \tfrac{2}{3} b, && p_{67} = p_{89} = \tfrac{1}{3} a, && p_{73} = p_{93} = \tfrac{2}{3} a, \\
&r_0 = \tfrac{1}{3}[3-a-b-c], && r_4 = \tfrac{2}{3}(1-c) + \tfrac{1}{3}(1-b), \\
&r_5 = \tfrac{1}{3}(1-c) + \tfrac{2}{3}(1-b), && r_6 = \tfrac{2}{3}(1-c) + \tfrac{1}{3}(1-a), \\
&r_7 = \tfrac{1}{3}(1-c) + \tfrac{2}{3}(1-a), && r_8 = \tfrac{2}{3}(1-b) + \tfrac{1}{3}(1-a), \\
&r_9 = \tfrac{1}{3}(1-b) + \tfrac{2}{3}(1-a).
\end{aligned}
\tag{B.20}
$$

# Appendix C

# Equilibrium points for truels and the opinion model

In this Appendix we will demonstrate the existence of equilibrium points for the truel games. Concretely we will show that either for the random truel and the opinion model there exists a unique equilibrium point which is the so-called *strongest opponent strategy*. For the sequential truel we will show the existence of two equilibrium points depending on the values of the markmanships $a$, $b$, $c$ of the players.

## C.1 Random firing

Let us denote by $\pi_A(P_{A0}, P_{AB}, P_{AC})$ the survival probability for player A given the values of the probability set $\{P_{A0}, P_{AB}, P_{AC}\}$ defining the strategy followed by player A (the same notation follows for players B and C).

The general expressions for $\pi_A(P_{A0}, P_{AB}, P_{AC})$, $\pi_B(P_{B0}, P_{BA}, P_{BC})$ and $\pi_C(P_{C0}, P_{CA}, P_{CB})$ with arbitrary values for the probabilities defining the strategies and the markmanships $a$, $b$ and $c$ is too lengthy to present here. Instead, we will show the following terms:

$$\pi_A(1,0,0) = \frac{a\,b(a+c)P_{BC} + a\,(a+b)c\,P_{CB}}{(a+b)(a+c)(b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$

$$\pi_A(0,1,0) = \frac{a\left(a^2+(P_{BC}b+b+cP_{CB})a+b\,c(P_{BC}+P_{CB})\right)}{(a+b)(a+c)(a+b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$

$$\pi_A(0,0,1) = \frac{a(a+c)(a+b\,P_{BC})+a(a+b)c\,P_{CB}}{(a+b)(a+c)(a+b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$

$$\pi_B(1,0,0) = \frac{a\,b(b+c)P_{AC} + b(a+b)cP_{CA}}{(a+b)(b+c)(a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))},$$

$$\pi_B(0,1,0) = \frac{b(b(b+cP_{CA}) + a(b(P_{AC}+1) + c(P_{AC}+P_{CA})))}{(a+b)(b+c)(b+a(P_{AB}+P_{AC}) + c(P_{CA}+P_{CB}))},$$
$$\pi_B(0,0,1) = \frac{b(b+c)(b+aP_{AC}) + b(a+b)c\,P_{CA}}{(a+b)(b+c)(b+a(P_{AB}+P_{AC}) + c(P_{CA}+P_{CB}))},$$
$$\pi_C(1,0,0) = \frac{a\,c(b+c)P_{AB} + b\,c(a+c)P_{BA}}{(a+c)(b+c)(a(P_{AB}+P_{AC}) + b(P_{BA}+P_{BC}))}, \qquad \text{(C.1)}$$
$$\pi_C(0,1,0) = \frac{c(c(c+bP_{BA}) + a(c(P_{AB}+1) + b(P_{AB}+P_{BA})))}{(a+c)(b+c)(c+a(P_{AB}+P_{AC}) + b(P_{BA}+P_{BC}))},$$
$$\pi_C(0,0,1) = \frac{c(b+c)(c+a\,P_{AB}) + b\,c(a+c)P_{BA}}{(a+c)(b+c)(c+a(P_{AB}+P_{AC}) + b(P_{BA}+P_{BC}))}.$$

We are interested in evaluating for all players which term $\pi(1,0,0)$, $\pi(0,1,0)$, $\pi(0,0,1)$ is greater depending on the values for $a$, $b$ and $c$. This will give us the equilibrium point of the system. For that purpose we may define new terms $S_i$ as

$$\begin{aligned}
S_1 &= \pi_A(1,0,0) - \pi_A(0,1,0),\\
S_2 &= \pi_A(1,0,0) - \pi_A(0,0,1),\\
S_3 &= \pi_A(0,1,0) - \pi_A(0,0,1),\\
S_4 &= \pi_B(1,0,0) - \pi_B(0,1,0),\\
S_5 &= \pi_B(1,0,0) - \pi_B(0,0,1),\\
S_6 &= \pi_B(0,1,0) - \pi_B(0,0,1),\\
S_7 &= \pi_C(1,0,0) - \pi_C(0,1,0),\\
S_8 &= \pi_C(1,0,0) - \pi_C(0,0,1),\\
S_9 &= \pi_C(0,1,0) - \pi_C(0,0,1).
\end{aligned} \qquad \text{(C.2)}$$

Thus, substituting the set of probabilities (C.1) in the previous expressions and after some manipulation we obtain

$$S_1 = \frac{a^2(-b(aP_{BA}-cP_{BC}+b(P_{BA}+P_{BC}))-(a+b)cP_{CA})}{(a+b)(a+c)(b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))(a+b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$
$$S_2 = \frac{a^2((b-c)cP_{CB}-(a+c)(bP_{BA}+cP_{CA}))}{(a+b)(a+c)(b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))(a+b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$
$$S_3 = \frac{a^2(b-c)}{(a+b)(a+c)(a+b(P_{BA}+P_{BC})+c(P_{CA}+P_{CB}))},$$
$$S_4 = \frac{b^2(-a(bP_{AB}-cP_{AC}+a(P_{AB}+P_{AC}))-(a+b)cP_{CB})}{(a+b)(b+c)(a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))(b+a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))},$$
$$S_5 = -\frac{b^2(a(b+c)P_{AB}-acP_{CA}+c(bP_{CB}+c(P_{CA}+P_{CB})))}{(a+b)(b+c)(a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))(b+a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))},$$

$$S_6 = \frac{b^2(a-c)}{(a+b)(b+c)(b+a(P_{AB}+P_{AC})+c(P_{CA}+P_{CB}))},$$
$$S_7 = \frac{c^2(-a(-bP_{AB}+cP_{AC}+a(P_{AB}+P_{AC}))-b(a+c)P_{BC})}{(a+c)(b+c)(a(P_{AB}+P_{AC})+b(P_{BA}+P_{BC}))(c+a(P_{AB}+P_{AC})+b(P_{BA}+P_{BC}))},$$
$$S_8 = -\frac{c^2(a(b+c)P_{AC}-abP_{BA}+b(cP_{BC}+b(P_{BA}+P_{BC})))}{(a+c)(b+c)(a(P_{AB}+P_{AC})+b(P_{BA}+P_{BC}))(c+a(P_{AB}+P_{AC})+b(P_{BA}+P_{BC}))},$$
$$S_9 = \frac{(a-b)c^2}{(a+c)(b+c)(c+a(P_{AB}+P_{AC})+b(P_{BA}+P_{BC}))}. \tag{C.3}$$

We can clearly see that all denominators in the previous expressions are strictly positive. Therefore, if we want to evaluate the sign of $S_i$ we need only to analyze the sign of the numerator.

Assuming that $a > b > c$ we already obtain the result that $S_3 > 0$, $S_6 > 0$ and $S_9 > 0$ implying that

$$\pi_A(0,1,0) > \pi_A(0,0,1),$$
$$\pi_B(0,1,0) > \pi_B(0,0,1),$$
$$\pi_C(0,1,0) > \pi_C(0,0,1).$$

Thus, we conclude that aiming at the weakest player it is not a conceivable strategy for any player and hence we may set $P_{AC} = P_{BC} = P_{CB} = 0$. This lead us to the following expressions

$$S_1 = \frac{-a^2(a+b)(bP_{BA}+cP_{CA})}{(a+b)(a+c)(bP_{BA}+cP_{CA})(a+bP_{BA}+cP_{CA})},$$
$$S_2 = \frac{-a^2(a+c)(bP_{BA}+cP_{CA})}{(a+b)(a+c)(bP_{BA}+cP_{CA})(a+bP_{BA}+cP_{CA})},$$
$$S_3 = \frac{a^2(b-c)}{(a+b)(a+c)(a+bP_{BA}+cP_{CA})},$$
$$S_4 = \frac{-ab^2P_{AB}(b+a)}{(a+b)(b+c)(aP_{AB}+cP_{CA})(b+aP_{AB}+cP_{CA})},$$
$$S_5 = -\frac{b^2(a(b+c)P_{AB}-cP_{CA}(a+c)}{(a+b)(b+c)(aP_{AB}+cP_{CA})(b+aP_{AB}+cP_{CA})},$$
$$S_6 = \frac{b^2(a-c)}{(a+b)(b+c)(b+aP_{AB}+cP_{CA})},$$
$$S_7 = \frac{-aP_{AB}c^2(a-b)}{(a+c)(b+c)(aP_{AB}+bP_{BA})(c+aP_{AB}+bP_{BA})},$$
$$S_8 = \frac{c^2bP_{BA}(a-b)}{(a+c)(b+c)(aP_{AB}+bP_{BA})(c+aP_{AB}+bP_{BA})},$$

$$S_9 = \frac{(a-b)c^2}{(a+c)(b+c)(c+aP_{AB}+bP_{BA})}. \tag{C.4}$$

It can be clearly seen that whatever values $a$, $b$, $c$, $P_{BA}$ and $P_{CA}$ the terms $S_1 < 0$ and $S_2 < 0$; recalling that $S_3 > 0$ we obtain for player A: $\pi_A(0,1,0) > \pi_A(0,0,1) > \pi_A(1,0,0)$ and therefore $P_{A0} = P_{AC} = 0$, $\mathbf{P_{AB} = 1}$.

Besides, the fact that $S_4 < 0$ and $S_6 > 0$ imposes $\pi_B(0,1,0) > \pi_B(1,0,0)$ and $\pi_B(0,1,0) > \pi_B(0,0,1)$. Then for player B we obtain $P_{B0} = P_{BC} = 0$ and $\mathbf{P_{BA} = 1}$. Finally, $S_7$ is also negative and together with $S_9 > 0$ we obtain $\pi_C(0,1,0) > \pi_C(1,0,0)$ and $\pi_C(0,1,0) > \pi_C(0,0,1)$. Hence $P_{C0} = P_{CB} = 0$ and $\mathbf{P_{CA} = 1}$.

As a conclusion, we have demonstrated for the random truel the existence of a unique equilibrium point, which is given by the strongest opponent strategy: $\mathbf{P_{AB} = P_{BA} = P_{CA} = 1}$.

## C.2 Sequential firing

For the sequential truel we can proceed as in the previous section, and so we may first present the expressions corresponding to $\pi(1,0,0)$, $\pi(0,1,0)$, $\pi(0,0,1)$ for players A, B and C. The expressions are

$$\begin{aligned}
\pi_A(1,0,0) &= \frac{\frac{acP_{CB}}{-ca+a+c} - \frac{abP_{BC}(c(P_{C0}-1)+1)}{a(b-1)-b}}{1-(b(P_{B0}-1)+1)(c(P_{C0}-1)+1)},\\
\pi_A(0,1,0) &= \frac{\frac{a(cP_{CB}-a(c-1)(b(P_{B0}-1)+1)(c(P_{C0}-1)+1))}{-ca+a+c} - \frac{abP_{BC}(c(P_{C0}-1)+1)}{a(b-1)-b}}{(a-1)(b(P_{B0}-1)+1)(c(P_{C0}-1)+1)+1},\\
\pi_A(0,0,1) &= \frac{\frac{a(a(b-1)(b(P_{B0}-1)+1)-bP_{BC})(c(P_{C0}-1)+1)}{a(b-1)-b} + \frac{acP_{CB}}{-ca+a+c}}{(a-1)(b(P_{B0}-1)+1)(c(P_{C0}-1)+1)+1},\\
\pi_B(1,0,0) &= -\frac{\frac{abP_{AC}(c(P_{C0}-1)+1)}{a(b-1)-b} - \frac{bc(P_{C0}+P_{CB}-1)}{b(c-1)-c}}{1-(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)},\\
\pi_B(0,1,0) &= -\frac{\frac{ab(P_{AC}-bP_{AC})(c(P_{C0}-1)+1)}{a(b-1)-b} - \frac{b(b(c-1)(c(P_{C0}-1)+1)+c(P_{C0}+P_{CB}-1))}{b(c-1)-c}}{(b-1)(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)+1},\\
\pi_B(0,0,1) &= -\frac{\frac{b(aP_{AC}-b(P_{AC}a+a-1))(c(P_{C0}-1)+1)}{a(b-1)-b} - \frac{bc(P_{C0}+P_{CB}-1)}{b(c-1)-c}}{(b-1)(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)+1},\\
\pi_C(1,0,0) &= \frac{\frac{bc(P_{B0}+P_{BC}-1)}{b(c-1)-c} - \frac{ac(P_{A0}+P_{AC}-1)(b(P_{B0}-1)+1)}{-ca+a+c}}{1-(a(P_{A0}-1)+1)(b(P_{B0}-1)+1)}, \qquad \text{(C.5)}\\
\pi_C(0,1,0) &= -\frac{\frac{a(1-c)c(P_{A0}+P_{AC}-1)(b(P_{B0}-1)+1)}{-ca+a+c} + \frac{c(c+b(-P_{B0}-P_{BC}+c(P_{B0}+P_{BC}-2)+1))}{b(c-1)-c}}{(c-1)(a(P_{A0}-1)+1)(b(P_{B0}-1)+1)+1},
\end{aligned}$$

$$\pi_C(0,0,1) = \frac{\frac{c(-ac+c+a(c-1)(P_{A0}+P_{AC}-1)(b(P_{B0}-1)+1))}{-ca+a+c} - \frac{b(c-1)c(P_{B0}+P_{BC}-1)}{b(c-1)-c}}{(c-1)(a(P_{A0}-1)+1)(b(P_{B0}-1)+1)+1}.$$

The next step would be to substitute the previous expressions into the terms $S_i$ from (C.2). However, we will not present them because the expressions obtained are of considerable length. Nevertheless, it can be shown that the terms $S_3$, $S_6$ and $S_9$ are greater than zero. This implies that we can set $P_{AC} = P_{BC} = P_{CB} = 0$, simplifying the expressions for $S_i$ which now read

$$\begin{aligned}
S_1 &= \frac{-a^2(1-c)(1-b(1-P_{B0}))(1-c(1-P_{C0}))}{(a+c(1-a))(1-(1-a)(1-b(1-P_{B0}))(1-c(1-P_{C0})))},\\
S_2 &= -\frac{a^2(1-b)(1-b(1-P_{B0}))(1-c(1-P_{C0}))}{(a(1-b)+b)(1-(1-a)(1-b(1-P_{B0}))(1-c(1-P_{C0})))},\\
S_3 &= \frac{a^2(b-c)(1-b(1-P_{B0}))(1-c(1-P_{C0}))}{(a(b-1)-b)(a(c-1)-c)(-P_{C0}c+c+a(1-b(1-P_{B0}))(1-c(1-P_{C0}))+b(P_{B0}-1)(-P_{C0}c+c-1))},\\
S_4 &= \frac{b\left(\frac{c(P_{C0}-1)}{1-(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)} - \frac{b(c-1)(c(P_{C0}-1)+1)+c(P_{C0}-1)}{(b-1)(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)+1}\right)}{b(c-1)-c},\\
S_5 &= \frac{bc(P_{C0}-1)}{(b(c-1)-c)(1-(a(P_{A0}-1)+1)(c(P_{C0}-1)+1))} + \frac{b\left(\frac{(a-1)b(-P_{C0}c+c-1)}{a(b-1)-b} + \frac{c-cP_{C0}}{b(c-1)-c}\right)}{(b-1)(a(P_{A0}-1)+1)(c(P_{C0}-1)+1)+1},\\
S_6 &= \frac{b^2(a-c)(c(P_{C0}-1)+1)}{(a(b-1)-b)(b(c-1)-c)(-cb+b+c+a(b-1)(P_{A0}-1)(c(P_{C0}-1)+1)+(b-1)cP_{C0})},\\
S_7 &= \frac{c^2\left(-(b-1)(c-1)(P_{A0}-1)(b(P_{B0}-1)+1)a^2+b(P_{A0}-b(c-P_{A0})(P_{B0}-1)-1)a+b^2c(P_{B0}-1)\right)}{D},\\
S_8 &= \frac{c^2\left((b(c-1)-c)(P_{A0}-1)(-P_{B0}b+b-1)a^2-b(b(c-1)-1)(P_{B0}-1)a+b^2(c-1)(P_{B0}-1)\right)}{D},\\
S_9 &= \frac{(a-b)c^2}{(a(c-1)-c)(b(c-1)-c)(c+a(c-1)(P_{A0}-1)(b(P_{B0}-1)+1)+b(c-1)(P_{B0}-1))}.
\end{aligned} \tag{C.6}$$

where $D = (a(c-1)-c)(b(c-1)-c)(a(P_{A0}-1)(b(P_{B0}-1)+1)+b(P_{B0}-1))(c+a(c-1)(P_{A0}-1)(b(P_{B0}-1)+1)+b(c-1)(P_{B0}-1))$.

It can easily be checked that both terms $S_1$ and $S_2$ are negative, which together with the condition $S_3 > 0$ give as a result that $\pi_A(0,1,0) > \pi_A(0,0,1) > \pi_A(1,0,0)$ and hence $P_{AC} = P_{A0} = 0$, $\mathbf{P_{AB} = 1}$. Substituting this result into $S_4$ and $S_5$ we get

$$S_4 = \frac{-b\left(\frac{-c(1-P_{C0})}{a+(1-a)(c-P_{C0})} + \frac{b(1-c)(1-c(1-P_{C0}))+c(1-P_{C0})}{1-(1-a)(1-b)(1-c(1-P_{C0}))}\right)}{c+b(1-c)}, \tag{C.7}$$

$$S_5 = \frac{bc(1-P_{C0})}{(c+b(1-c))(a+c(1-a)(1-P_{C0}))} - \frac{b\left(\frac{(1-a)b(1-c(1-P_{C0})}{b+a(1-b)} + \frac{c(1-P_{C0})}{c+b(1-c)}\right)}{1-(1-a)(1-b)(1-c(1-P_{C0}))}. \tag{C.8}$$

The previous equations for $S_4$ and $S_5$ are both negative either when $P_{C0} = 0$ or $P_{C0} = 1$. Thus, this result together with $S_6 > 0$ results in $\pi_B(0,1,0) > \pi_B(0,0,1) > \pi_B(1,0,0)$, thus $P_{BC} = P_{B0} = 0$ and $\mathbf{P_{BA} = 1}$.

Finally, substituting these results into the equations $S_7$, $S_8$ and $S_9$ we get

$$S_7 = \frac{-c^2\big((1-c)(1-b)^2a^2-ba(1-bc)-b^2c\big)}{(a(1-c)+c)(b(1-c)+c)(a(1-b)+b)(c+a(1-c)(1-b)+b(1-c))}, \tag{C.9}$$

$$S_8 = \frac{c^2\big((c+b(1-c))(1-b)a^2+ba-b^2(1-c)(1-a)\big)}{(a(1-c)+c)(b(1-c)+c)(a(1-b)+b)(c+a(1-c)(1-b)+b(1-c))}, \tag{C.10}$$

$$S_9 = \frac{(a-b)c^2}{(c+a(1-c))(c+b(1-c))(c+a(1-c)(1-b)+b(1-c))}. \tag{C.11}$$

We know that $S_9$ is positive, implying that $\pi_C(0,1,0) > \pi_C(0,0,1)$. Besides, in order to evaluate the sign in Eq. (C.9) we need only to analyze the numerator, as the denominator is always positive. Defining the function $g(a,b,c) = (1-c)(1-b)^2a^2 - ba(1-bc) - b^2c$ we have

- If $g(a,b,c) > 0$: $S_7 < 0$, $S_9 > 0$

$$\left\{\begin{array}{l} \pi_C(0,1,0) > \pi_C(1,0,0), \\ \pi_C(0,1,0) > \pi_C(1,0,0), \end{array}\right. \longrightarrow P_{C0} = P_{CB} = 0,\ \mathbf{P_{CA} = 1}$$

- If $g(a,b,c) < 0$: $S_7 > 0$, $S_9 > 0$

$$\left\{\begin{array}{l} \pi_C(1,0,0) > \pi_C(0,1,0), \\ \pi_C(0,1,0) > \pi_C(1,0,0), \end{array}\right. \longrightarrow P_{CA} = P_{CB} = 0,\ \mathbf{P_{C0} = 1}$$

Hence we see that depending on the sign of $g(a,b,c)$ the equilibrium point will be given by the strongest opponent strategy $\mathbf{P_{AB}} = \mathbf{P_{BA}} = \mathbf{P_{CA}} = \mathbf{1}$ when $g(a,b,c) > 0$ or by $\mathbf{P_{AB}} = \mathbf{P_{BA}} = \mathbf{P_{C0}} = \mathbf{1}$ when $g(a,b,c) < 0$.

## C.3 Convincing opinion

Following the same methodology as in previous sections we can write down the solutions corresponding to the convincing probabilities of opinions A, B and C in terms of the strategies adopted by the players

$$\pi_A(1,0,0) = \frac{a^2\big(a^3-(b(P_{BA}-3)+c(P_{CA}-3))a^2+c(c-2b(P_{BA}+P_{CA}-4))a+b\,c(-P_{CA}b+b-c(P_{BA}-3))\big)}{(a+b)^2(a+c)^2(a+b+c)},$$

$$\pi_A(0,1,0) = \frac{a^2\big(a^3-(b(P_{BA}-3)+c(P_{CA}-3))a^2+b(b-2c(P_{BA}+P_{CA}-4))a+b\,c(-P_{BA}c+c-bP_{CA}-3))\big)}{(a+b)^2(a+c)^2(a+b+c)},$$

$$\pi_A(0,0,1) = \frac{a^2\big(a^3-(b(P_{BA}-3)+c(P_{CA}-3))a^2+c(c-2b(P_{BA}+P_{CA}-4))a+bc(-P_{CA}b+b-c(P_{BA}-3))\big)}{(a+b)^2(a+c)^2(a+b+c)},$$

$$\pi_B(1,0,0) = \frac{b^2\left(\frac{b-a(P_{AB}-3)}{(a+b)^2}+\frac{cP_{CA}}{(b+c)^2}\right)}{a+b+c},$$

$$\pi_B(0,1,0) = \frac{b^2\Big((b+c(P_{CA}+2))a^2-\Big((P_{AB}-3)b^2+2c(P_{AB}-P_{CA}-3)b+c^2(P_{AB}-1)\Big)a+b^2(b+c(P_{CA}+2))\Big)}{(a+b)^2(b+c)^2(a+b+c)},$$

$$\pi_B(0,0,1) = \frac{b^2\left(\frac{b-a(P_{AB}-3)}{(a+b)^2}+\frac{cP_{CA}}{(b+c)^2}\right)}{a+b+c},$$

$$\pi_C(1,0,0) = \frac{c^2\Big(bP_{BA}(a+c)^2+(b+c)^2(c+a(P_{AB}+2))\Big)}{(a+c)^2(b+c)^2(a+b+c)},$$

$$\pi_C(0,1,0) = \frac{c^2\Big((c+b(P_{BA}+2))a^2+\Big(P_{AB}b^2+2c(P_{AB}+P_{BA}+2)b+c^2(P_{AB}+2)\Big)a+c^2(c+b(P_{BA}+2))\Big)}{(a+c)^2(b+c)^2(a+b+c)},$$

$$\pi_C(0,0,1) = \frac{c^2\Big(bP_{BA}(a+c)^2+(b+c)^2(c+a(P_{AB}+2))\Big)}{(a+c)^2(b+c)^2(a+b+c)}. \qquad \text{(C.12)}$$

And the terms $S_i$ read

$$\begin{aligned}
S_1 &= \frac{a^2(c-b)(2bc+a(b+c))}{(a+b)^2(a+c)^2(a+b+c)},\\
S_2 &= 0,\\
S_3 &= \frac{a^2(b-c)(2bc+a(b+c))}{(a+b)^2(a+c)^2(a+b+c)},\\
S_4 &= -\frac{b^2(a-c)(bc+a(b+2c))}{(a+b)^2(b+c)^2(a+b+c)},\\
S_5 &= 0,\\
S_6 &= \frac{b^2(a-c)(bc+a(b+2c))}{(a+b)^2(b+c)^2(a+b+c)},\\
S_7 &= -\frac{(a-b)c^2(2ab+(a+b)c)}{(a+c)^2(b+c)^2(a+b+c)},\\
S_8 &= 0,\\
S_9 &= \frac{(a-b)c^2(2ab+(a+b)c)}{(a+c)^2(b+c)^2(a+b+c)}.
\end{aligned} \qquad \text{(C.13)}$$

By the way markmanships $a$, $b$ and $c$ are defined, we see that $S_1 < 0$, $S_3 > 0$ and thus $\pi_A(0,1,0) > \pi_A(1,0,0) = \pi_A(0,0,1)$; besides, $S_4 < 0$ and $S_6 > 0$ implying that $\pi_B(0,1,0) > \pi_B(1,0,0) = \pi_B(0,0,1)$; and finally $S_7 < 0$ and $S_9 > 0$ and so $\pi_C(0,1,0) > \pi_C(1,0,0) = \pi_C(0,0,1)$. Hence, there is only one equilibrium point in the opinion model that corresponds to the *strongest opponent strategy*: $\mathbf{P_{AB} = P_{BA} = P_{CA} = 1}$.

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# List of Figures

# Publications

## *Journals*


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2. P. Amengual, A. Allison, R. Toral & D. Abbott. “*Discrete–time ratchets, the Fokker–Planck equation and Parrondo’s paradox*”. Proceedings of the Royal Society of London A **460**, 2269-2284 (2004).
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6. P. Amengual & R. Toral. “*Truels, or the survival of the weakest*”. To appear in *Computing in Science and Engineering*.


## *Conference Proceedings*


1. R. Toral, P. Amengual & S. Mangioni. “*A Fokker-Planck description for Parrondos Games*”, *Noise in Complex Systems and Stochastic Dynamics*. Proceedings of SPIE Vol. **5114**, 309–317 (2003).
2. P. Amengual & R. Toral. “*Parrondos games and the zipping algorithm*”. *Noise in Complex Systems and Stochastic Dynamics II*. Proceedings of SPIE Vol. **5471**, 203–212 (2004).
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