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Effective Markovian approximation for non-Gaussian noises: a path integral approach

M.A. Fuentes[a,*,1], Horacio S. Wio[a,1,2], Raúl Toral[b,3]

[a] *Grupo de Física Estadística, Centro Atómico Bariloche (CNEA) and Instituto Balseiro (CNEA and UNC), 8400-San Carlos de Bariloche, Argentina*

[b] *Instituto Mediterráneo de Estudios Avanzados, IMEDEA (CSIC-UIB), Campus UIB, 07071-Palma de Mallorca, Spain*

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Abstract

We have analyzed diffusion in a double well potential driven by a colored non-Gaussian noise. Using a path-integral approach we have obtained a consistent Markovian approximation to the initially non-Markovian problem. Such an approximation allows us to get analytical expressions for the "mean-first-passage-time" that has been tested against extensive numerical simulations.

Keywords: Non-Gaussian processes; Non-Markovian processes; First passage time; Path integrals

1. Introduction

The consideration of noise sources with finite correlation time (i.e. colored noise) has become a subject of current study in the context of realistic models of physical systems [1–3]. Some authors have focused on the obtention of Markovian approximations, trying to capture the essential features of the original non-Markovian problem. One particular case is the "unified colored noise approximation" (UCNA) of Hänggi and collaborators [4,5]. The original formulation of the problem is in terms of a non-Markovian stochastic differential equation in the relevant variable. However, this problem can be transformed into a Markovian one by extending the number of variables and equations.

* Corresponding author.

E-mail addresses: fuentesm@cab.cnea.gov.ar (M.A. Fuentes), wio@cab.cnea.gov.ar (H.S. Wio), raul@imedea.uib.es (R. Toral).

[1] http://cab.cnea.gov.ar/cab/invBasica/fisEstad/estadis.htm.

[2] Member of CONICET, Argentina.

[3] http://www.imedea.uib.es.

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The UCNA consists of an adiabatic elimination-like procedure [6–8], that allows us to reduce this extended problem to an "effective" Markovian one in the original variable space. The ultimate goal of this procedure is achieving a consistent single variable Fokker–Planck approximation for the probability distribution of the original variable. The UCNA approximation has been justified as a reliable Markovian approximation by means of path integral techniques [9,10].

Although, for the sake of mathematical simplicity, a majority of studies treats the white-noise case only, it is expected that, because of their nature, fluctuations coupled *multiplicatively* to the system will show some degree of time correlation or "color" [1–3], and hence give rise to new effects. A few examples are: reentrant behavior as a consequence of color in a noise-induced transition [11], an ordering non-equilibrium phase transition induced in a Ginzburg–Landau model by varying the correlation time of the additive noise [12,13], and a new reentrant phenomenon found in a simple model of noise induced phase transitions when multiplicative colored noise is considered [14,15].

In another context, most studies of the phenomenon of *stochastic resonance* (SR) [16] have been done assuming white noise sources, with a few exceptions that studied the effect of colored noises [17,18]. In all cases the noises are assumed to be Gaussian [6–8]. However, some experimental results in sensory systems, particularly for one kind of crayfish [19] as well as recent results for rat skin [20], offer strong indications that the noise source in these systems could be non-Gaussian. This point of view is supported by the results obtained in a recent contribution [21], where the study of a particular class of Langevin equations having non-Gaussian stationary distribution functions [22,23] was made use of. The work in Refs. [22,23] is based on the generalized thermostatistics proposed by Tsallis [24–26] which has been successfully applied to a wide variety of physical systems [27–35].

Here we present an UCNA-like approximation to the problem of non-Gaussian colored noise. We exploit a scheme, based on a path integral description of the problem for Gaussian colored noise, analogous to that used for the obtention of the UCNA and its generalizations [9,10,36–38]. Such a procedure allows us to obtain an "effective Markovian" approximation to the original non-Markovian problem. Our aim is to obtain analytical expressions for some relevant quantities (particularly the mean first passage time) that could be exploited to predict qualitatively the behavior, due to the presence of non-Gaussian colored noise, of many relevant systems.

We consider the following problem:

$$\dot{x} = f(x) + g(x)\eta(t) , \tag{1}$$

$$\dot{\eta} = -\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) + \frac{1}{\tau}\xi(t) , \tag{2}$$

where $\xi(t)$ is a Gaussian white noise of zero mean and correlation $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$, $V_q(\eta)$ is given by [22]

$$V_q(\eta) = \frac{1}{\beta(q-1)}\ln\left[1 + \beta(q-1)\frac{\eta^2}{2}\right] , \tag{3}$$

where $\beta=\tau/D$. The function $f(x)$ is derived from a double well potential $U(x)$, $f(x)=-\mathrm{d}U(x)/\mathrm{d}x=-U'(x)$. This problem corresponds to the case of diffusion in a potential $U(x)$, induced by η, a colored non-Gaussian noise. Clearly, when $q\to 1$ we recover the limit of η being a Gaussian colored noise.

The outline of the paper is as follows: in Section 2 we briefly analyze the properties of the process η. In Section 3, we show how to obtain the effective Markovian Fokker–Planck equation using a path integral treatment similar to the case of Gaussian colored noises [9,10,36,37] and derive the stationary probability. In Section 4, we derive the expression for the mean first passage time (MFPT). In Section 5, the latter results are compared with exhaustive Monte Carlo simulations. Finally, in Section 6, we draw some conclusions.

2. Process η

In this section in order to determine the properties of the process η and the range of validity of the present study, we briefly analyze the stochastic process characterized by the Langevin equation given in Eq. (2), that is

$$\dot{\eta}=-\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta)+\frac{1}{\tau}\xi(t)\,. \tag{4}$$

This has the following associated Fokker–Planck equation (FPE) for the time-dependent probability density function $P_q(\eta,t)$:

$$\partial_t P_q(\eta,t)=\frac{1}{\tau}\partial_\eta\left(P_q\frac{\mathrm{d}V_q}{\mathrm{d}\eta}\right)+\frac{D}{2\tau^2}\partial^2_\eta P_q\,. \tag{5}$$

It turns out that the stationary distribution $P_q^{st}(\eta)$ is only well-defined for $q\in(-\infty,3)$, whereas for $q\geqslant 3$, $P_q^{st}(\eta)$ is not normalizable and cannot be accepted as a true probability function. The final expression for $P_q^{st}(\eta)$ depends on q:

- For $q\in(1,3)$, we obtain a Tsallis-exponential type form

$$P_q^{st}(\eta)=\frac{1}{Z_q}\left[1+\beta(q-1)\frac{\eta^2}{2}\right]^{-1/(q-1)}\qquad\forall\eta\in(-\infty,\infty) \tag{6}$$

  with the normalization

$$\begin{aligned}Z_q&=\int_{-\infty}^{\infty}\mathrm{d}\eta\left[1+\beta(q-1)\frac{\eta^2}{2}\right]^{-1/(q-1)}\\&=\sqrt{\frac{\pi}{\beta(q-1)}}\frac{\Gamma(1/(q-1)-1/2)}{\Gamma(1/(q-1))}\,.\end{aligned} \tag{7}$$

  (Γ indicates the Gamma function.) The asymptotic behavior is $P_q^{st}(\eta)\sim\eta^{-2/(q-1)}$ for $|\eta|\to\infty$. As anticipated, for $q\geqslant 3$ the normalization factor, Z_q, diverges.

- For $q=1$ we recover the Gaussian distribution

$$P_1^{st}(\eta)=\frac{1}{Z_1}\exp\left(-\beta\frac{\eta^2}{2}\right) \tag{8}$$

with $Z_1=\sqrt{\pi/\beta}$. In this case of $q=1$ the process η is nothing but an Ornstein–Uhlenbeck noise.

- Finally, for $q\in(-\infty,1)$ we obtain a cut-off distribution, namely

$$P_q^{st}(\eta)=\begin{cases}\frac{1}{Z_q}[1-(\frac{\eta}{w})^2]^{1/(1-q)} & \text{if } |\eta|<w\,,\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

with the cut-off value given by $w=[(1-q)\beta/2]^{-1/2}$, and the normalizing factor being

$$Z_q=\int_{-w}^{w}\mathrm{d}\eta\left[1-\left(\frac{\eta}{w}\right)^2\right]^{1/(1-q)}=\sqrt{\frac{\pi}{\beta(1-q)}}\,\frac{\Gamma(1/(1-q)+1)}{\Gamma(1/(1-q)+3/2)}\,. \tag{10}$$

The distribution $P_q^{st}(\eta)$ is an even function and therefore the mean value is $\langle\eta\rangle=0$. It can be easily verified that the second moment $\langle\eta^2\rangle$ is finite for $q<\frac{5}{3}$ and diverges for $q\in\left[\frac{5}{3},3\right)$. More precisely we have

$$\langle\eta^2\rangle=\begin{cases}\frac{2}{\beta(5-3q)} & \text{if } q\in(-\infty,\frac{5}{3})\,,\\ \infty & \text{if } q\in\left[\frac{5}{3},3\right)\,.\end{cases} \tag{11}$$

In order to characterize even further the stochastic process η, we now consider its normalized time correlation function $C(t)=\langle\eta(t+t')\eta(t')\rangle/\langle\eta^2\rangle$, in the stationary regime $t'\to\infty$. It is not possible to obtain exact analytical expressions for $C(t)$. However, given the form of Eq. (4) it is possible to scale out the parameters τ and D. Defining $z=\sqrt{(\tau/D)}\eta$ and $s=t/\tau$, we arrive at an equation independent of τ and D:

$$\frac{\mathrm{d}z}{\mathrm{d}s}=\frac{-z}{1+(q-1)z^2/2}+\hat{\xi}(s) \tag{12}$$

with $\langle\hat{\xi}(s)\hat{\xi}(s')\rangle=2\delta(s-s')$. This leads to $C(t)=C_q(t/\tau)$ where $C_q(s)=\langle z(s+s')z(s')\rangle/\langle z^2\rangle$ is a universal function depending only on the parameter q. In the case $q=1$, process z is an Ornstein–Uhlenbeck noise and the correlation function is easily obtained as $C_1(s)=\exp(-s)$. We have observed, numerically, that this exponential decay of the correlations is still valid for $q<1$ where we can write $C_q(s)=\exp(-s/s_q)$. This exponential behavior fails for $q>1$ where, on the other hand, $C_q(s)$ can be approximated by a Tsallis-like exponential [24–26] $C_q(s)=[1+(q-1)s/s_q]^{1/(1-q)}$. The characteristic correlation time s_q defined, for instance as

$$s_q=\int_0^{\infty}\mathrm{d}s\,C_q(s)\,, \tag{13}$$

is such that it diverges for $q\to\frac{5}{3}$ as $s_q=2/(5-3q)$. As shown in Fig. 1, this phenomenological relation is clearly consistent from the numerical simulations of the

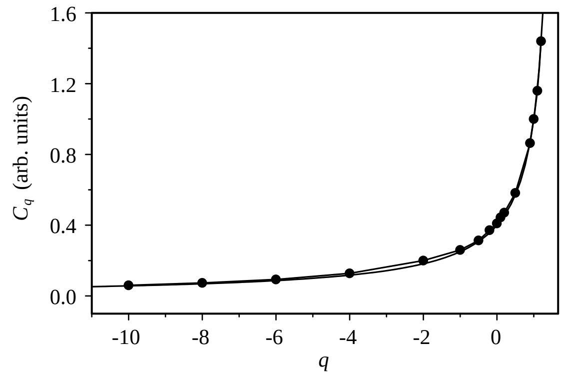


Fig. 1. Correlation time s_q of $C_q(s)$, the correlation function of the process η, as a function of q. Simulations are indicated by dots and theory by a full line.

process z. Notice that in the limit $q \to 1$ it gives the exact result $s_1 = 1$ (corresponding to the Ornstein–Uhlenbeck process). Although we have not been able to derive this result analytically, a very simple calculation is able to predict the divergence of s_q for $q = q_c = \frac{5}{3}$. We have

$$\frac{dC_q(s)}{ds} = -\left\langle \frac{z(s)z(0)}{1+(q-1)z(s)^2/2} \right\rangle \approx -\frac{1}{s_q} C_q(s) , \tag{14}$$

where we have approximated

$$s_q \approx \frac{1}{1+(q-1)\langle z^2 \rangle/2} = \frac{2(2-q)}{5-3q} , \tag{15}$$

which indeed diverges as $s_q \sim (5-3q)^{-1}$ although with a different prefactor from the one observed numerically.

To conclude, in this section we have characterized the process η by computing its stationary probability distribution and correlation function. It turns out that the pdf does not exist for $q \geqslant 3$ while the second moment diverges for $q \geqslant \frac{5}{3}$. The distribution extends to $\pm\infty$ if $q \geqslant 1$ while for $q < 1$ there is a cut-off. Finally, the correlation function can be fitted by an exponential decay for $q \leqslant 1$, while for $q > 1$ it is fitted by a Tsallis exponential. The characteristic time for the decay of the correlations diverges as $q \to q_c = \frac{5}{3}$ as $s_q \sim (q_c - q)^{-1}$.

3. Effective Markovian approximation

As indicated in Section 1, applying the path-integral formalism to the Langevin equations given in Eqs. (1) and (2), and making an adiabatic-like elimination procedure [9,10,36,37] it is possible to arrive to an *effective Markovian approximation*. Here we sketch such a procedure.

The FPE associated to Eqs. (1) and (2) is

$$\frac{\partial P_q(x,\eta,\tau,t)}{\partial t} = -\frac{\partial}{\partial x}([f(x)+g(x)\eta]P_q(x,\eta,\tau,t)) + \frac{\partial}{\partial \eta}\left(\tau^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta)\right]P_q(x,\eta,\tau,t)\right) + \frac{D}{\tau^2}\frac{\partial^2}{\partial \eta^2}P_q(x,\eta,\tau,t)\,. \tag{16}$$

The path-integral representation for the transition probability, corresponding to the Langevin equations given in Eqs. (1) and (2) or to the associated FPE in Eq. (16) is [9,10]

$$P_q(x_b,\eta_b,t_b\,|\,x_a,\eta_a,t_a;\tau) = \int_{x(t_a)=x_a,\,\eta=\eta_b}^{x(t_b)=x_b,\,\eta=\eta_a} \mathscr{D}[x(t)]\mathscr{D}\,[\eta(t)]\mathscr{D}[p_x(t)]\mathscr{D}[p_\eta(t)]\mathrm{e}^{\mathscr{S}_{q,1}}\,, \tag{17}$$

where $p_x(t)$ and $p_\eta(t)$ are the canonically conjugate variables to $x(t)$ and $\eta(t)$, respectively. $\mathscr{S}_{q,1}$ is the stochastic *action* given by

$$\mathscr{S}_{q,1} = \int_{t_a}^{t_b} \mathrm{d}s\Bigg(\mathrm{i}\,p_x(s)[\dot{x}(s) - f(x(s)) - g(x(s))\eta(s)] + \mathrm{i}\,p_\eta(s)\left[\dot{\eta}(s) + \tau^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta(s))\right]\right] + \frac{D}{\tau^2}(\mathrm{i}\,p_\eta(s))^2\Bigg)\,. \tag{18}$$

The Gaussian integration over $p_\eta(s)$ yields

$$P_q(x_b,\eta_b,t_b\,|\,x_a,\eta_a,t_a;\tau) = \int_{x(t_a)=x_a,\,\eta=\eta_b}^{x(t_b)=x_b,\,\eta=\eta_a} \mathscr{D}[x(t)]\mathscr{D}[\eta(t)]\mathscr{D}[p_x(t)]\mathrm{e}^{\mathscr{S}_{q,2}} \tag{19}$$

with

$$\mathscr{S}_{q,2} = \int_{t_a}^{t_b} \mathrm{d}s\Bigg(\mathrm{i}\,p_x(s)[\dot{x}(s) - f(x(s)) - g(x(s))\eta(s)] + \frac{\tau^2}{4D}\int_{t_a}^{t_b}\mathrm{d}s' \times\left[\dot{\eta}(s) + \tau^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta(s))\right]\delta(s-s')\right]\left[\dot{\eta}(s') + \tau^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta(s'))\right]\right]\Bigg)\,. \tag{20}$$

The integration over $p_x(s)$ is also immediate, yielding

$$P_q(x_b,\eta_b,t_b\,|\,x_a,\eta_a,t_a;\tau) \sim \int_{x(t_a)=x_a,\,\eta=\eta_b}^{x(t_b)=x_b,\,\eta=\eta_a} \mathscr{D}[x(t)]\mathscr{D}[\eta(t)] \times\delta\left[\int \mathrm{d}s(\dot{x}(s) - f(x(s)) - g(x(s))\eta(s))\right]\mathrm{e}^{\mathscr{S}_{q,3}} \tag{21}$$

with

$$\mathcal{S}_{q,3} = \int_{t_a}^{t_b} \mathrm{d}s \left(\frac{\tau^2}{4D} \left[\dot{\eta}(s) + \tau^{-1} \left[\frac{\mathrm{d}}{\mathrm{d}\eta} V_q(\eta(s)) \right]^2 \right] \right) \tag{22}$$

and $\delta[\int \mathrm{d}s(\dot{x} - f(x) - g(x)\eta)]$ indicates that, at each instant of time, we have

$$\eta(s) = \frac{(\dot{x}(s) - f(x(s)))}{g(x(s))} . \tag{23}$$

The last condition makes trivial the integration over $\eta(s)$. It corresponds to replacing $\eta(s)$ and $\dot{\eta}(s)$ by Eq. (22) and by

$$\dot{\eta}(s) = -\frac{1}{g(x)^2} \frac{\mathrm{d}g(x)}{\mathrm{d}x} \dot{x}(s)(\dot{x}(s) - f(x(s))) + \frac{1}{g(x)} \left(\ddot{x}(s) - \frac{\mathrm{d}}{\mathrm{d}x} f(x(s)\dot{x}(s)) \right) , \tag{24}$$

respectively. The resulting stochastic action corresponds to a *non-Markovian* description as it involves $\ddot{x}(s)$. In order to obtain an *effective* Markovian approximation we resort to the same kind of approximations and arguments used in relation with colored Gaussian noise [9,10,36,38], that allows us to get a result resembling the UCNA. In simple words, such an approximation corresponds to neglecting all contributions including $\ddot{x}(s)$ and/or $\dot{x}(s)^n$ with $n > 1$. Doing this we get the approximate relation

$$\dot{\eta} + \tau^{-1} \left[\frac{\mathrm{d}}{\mathrm{d}\eta} V_q(\eta) \right] \approx -\frac{1}{g(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\dot{x} - f(x) \frac{\mathrm{d}}{\mathrm{d}x} \ln g(x)\dot{x} \right) + \frac{1}{\tau g(x)} \frac{\dot{x}(s) - f(x(s))}{1 + \frac{\beta(q-1)}{2}(\frac{f(x)}{g(x)})^2} - \frac{1}{\tau g(x)} \frac{\beta(q-1) f(x)^2 \dot{x}(s)}{(1 + \frac{\beta(q-1)}{2}(\frac{f(x)}{g(x)})^2)^2} . \tag{25}$$

As in the case of UCNA, this approximation will give reliable results for small values of τ.

The final result for the transition probability is

$$P_q(x_b, \eta_b, t_b \,|\, x_a, \eta_a, t_a; \tau) = \int_{x(t_a)=x_a,\, \eta=\eta_b}^{x(t_b)=x_b,\, \eta=\eta_a} \mathcal{D}[x(t)] \mathrm{e}^{\mathcal{S}_{q,4}} \tag{26}$$

with (for the simple case $g(x) = 1$)

$$\mathcal{S}_{q,4} = \frac{1}{4D} \int_{t_a}^{t_b} \mathrm{d}s \left(\left[-\tau \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{[1 - \frac{\beta(q-1)}{2} f(x)^2]}{[1 + \frac{\beta(q-1)}{2} f(x)^2]^2} \right] \dot{x} - \frac{f(x)}{[1 + \frac{\beta(q-1)}{2} f(x)^2]} \right)^2 . \tag{27}$$

It is immediate to recover some known limits. For $\tau > 0$ and $q \to 1$ we get the known Gaussian colored noise result (Ornstein–Uhlenbeck process) [9], while for $q \neq 1$ and $\tau \to 0$ we find the case of Gaussian white noise.

The FPE for the evolution of the probability $P(x,t)$ corresponding to the above indicated path-integral representation is

$$\partial_t P(x,t) = -\partial_x[A(x)P(x,t)] + \tfrac{1}{2}\partial_x^2[B(x)P(x,t)]\,, \tag{28}$$

where

$$A(x) = \frac{U'}{\left(\frac{1-(\tau/2D)\,(q-1)U'^2}{1+(\tau/2D)\,(q-1)U'^2}\right) + \tau U''[1+(\tau/2D)\,(q-1)U'^2]} \tag{29}$$

and

$$B(x) = D\left(\frac{[1+(\tau/2D)\,(q-1)U'^2]^2}{\tau U''[1+(\tau/2D)\,(q-1)U'^2]^2 + [1-(\tau/2D)\,(q-1)U'^2]}\right)^2 . \tag{30}$$

The stationary distribution of the FPE in Eq. (28) results in

$$P^{st}(x) = \frac{\aleph}{B}\exp[-\Phi(x)]\,, \tag{31}$$

where $\aleph$ is the normalization factor, and

$$\Phi(x) = 2\int \frac{A}{B}\,\mathrm{d}y\,. \tag{32}$$

The indicated FPE and its associated stationary distribution allow us to obtain the MFPT through a Kramers-like approximation. This quantity is the necessary ingredient to work in a large variety of problems.

4. Mean first passage time

The MFPT can be obtained, in a Kramers-like approximation [6–8,39] from

$$T(x_0) = \int_a^{x_0} \frac{\mathrm{d}y}{\Psi}\int_{-\infty}^{y} \frac{\mathrm{d}z\,\Psi}{B}\,, \tag{33}$$

where

$$\Psi(x) = \exp\left(-2\int \mathrm{d}y\,\frac{A}{B}\right) . \tag{34}$$

We will focus on polynomial-like forms for the potential and adopt

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2}\,. \tag{35}$$

For this kind of potential the normalization constant $\aleph$ diverges for any value of $\tau > 0$. This can be seen from the asymptotic behavior of $\Phi(x)$, for $x \to \infty$: $\Phi(x) \to 0$, while $B^{-1} \to \infty$, resulting in an ill-defined stationary probability density in Eq. (31).

In order to find approximate relations for the MFPT, and other related quantities, we assume that Eq. (28) is valid only for values of x near the wells and when the dispersion of the η process is finite, that is $\langle \eta^2 \rangle < \infty$ (or $q \in (-\infty, \frac{5}{3})$). Such a cutoff is justified a posteriori, analyzing the probability distributions that can be obtained from the simulations.

In order to obtain the MFPT and related quantities, we have integrated Eq. (33), replacing the potential given by Eq. (35) in the expressions for $A(x)$ and $B(x)$.

It is worth remarking here that the relevant quantity is not the white noise intensity D but the non-Gaussian noise intensity D_{nG}. Both quantities are related through (for instance see Eq. (11))

$$D_{nG} = 2D(5 - 3q)^{-1} .$$

We will use D_{nG} in all our results.

5. Theoretical results and simulations

In this section we analyze the different results for the MFPT and PDF obtained in the previous section, as a function of the noise intensity D for different situations: fixed

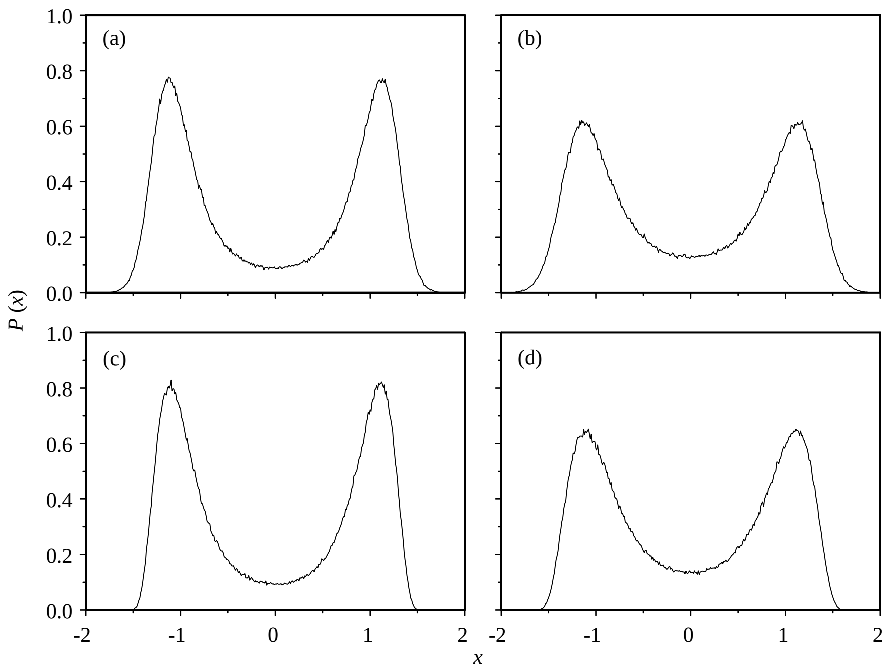
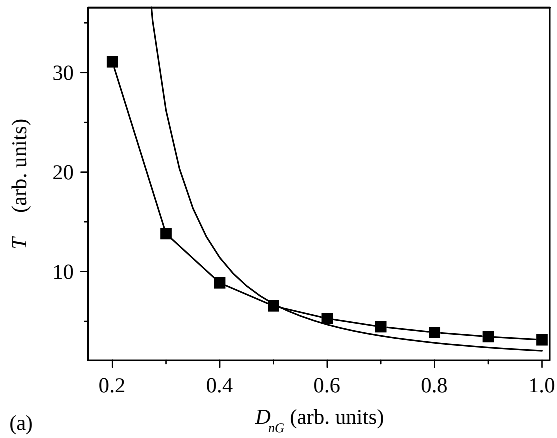


Fig. 2. PDF as a function of the variable x for: (a) $q = 1.25$, $\tau = 1$; (b) $q = 1.25$, $\tau = 0.5$; (c) $q = 0.75$, $\tau = 1$; (d) $q = 0.75$, $\tau = 0.5$.



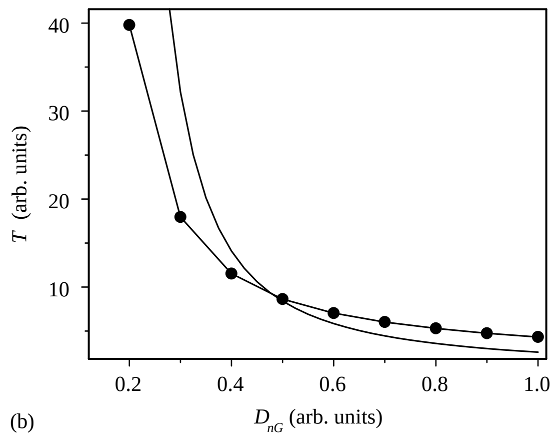


Fig. 3. MFPT as a function of the noise intensity for $q=1$ and (a) $\tau=0.025$ (squares indicate simulations while the full line indicates theory); (b) $\tau=0.1$ (simulations are indicated by dots and theory by a broken line).

q and various τ, fixed τ and various q, etc. In order to test our theoretical predictions we have carried out numerical simulations to calculate both the MFPT and the PDF.

Starting from the initial condition $(x,\eta)=(-1,0)$, we consider MFPT as the time in which the variable x reaches the value 0. The numerical simulations have been carried out considering Eqs. (1) and (2), which in discrete-time representation were characterized by the Heun method [40] using the random variables generated by the Box–Mueller algorithm [41]. All our simulations were performed using a time step $\Delta t=10^{-4}$ and averaging over 5×10^4 realizations.

Fig. 2 shows the results for the PDF obtained from the simulations for the parameters indicated in the caption. We can see that the assumption in Eq. (28) becomes a good approximation only for values of x near the wells. The effect of the non-Gaussian noise $(q\neq1)$ on the PDF is apparent.

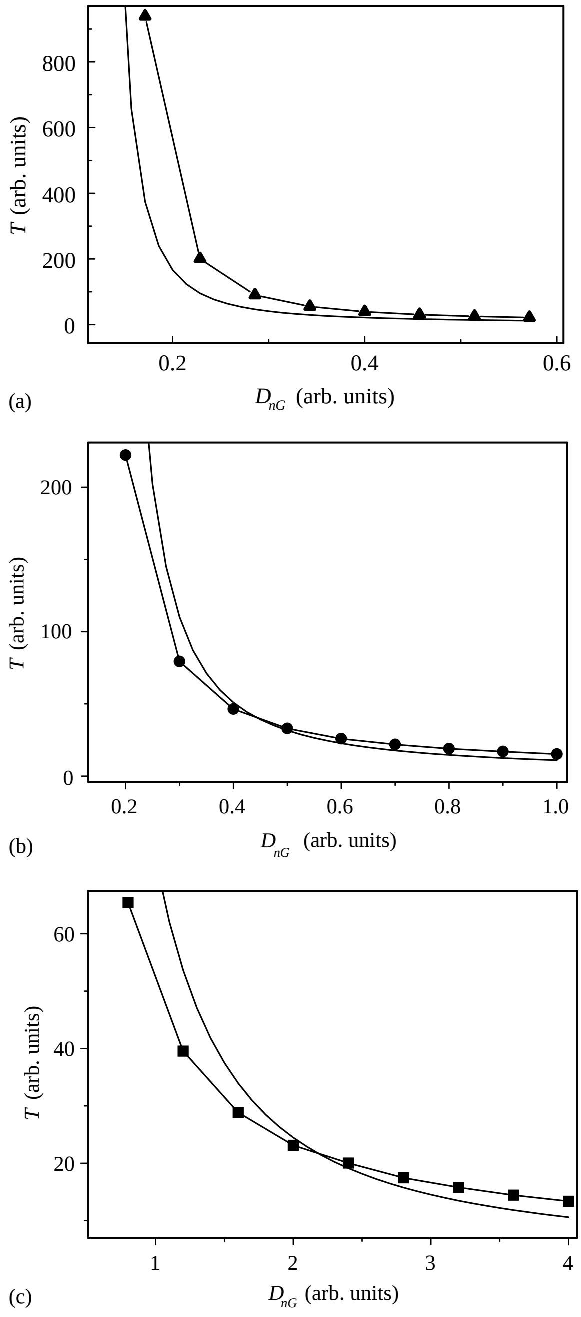


Fig. 4. MFPT as a function of the noise intensity for $\tau = 1$ and (a) $q = 0.5$ (simulations are indicated by triangles and theory by a broken line); (b) $q = 1$, circles (simulations) and full line (theory); (c) $q = 1.5$, squares (simulations) and dotted line (theory).

Figs. 3 and 4 show both simulation and theoretical results for the MFPT. In the case of fixed q, that is in Fig. 3, we see good qualitative agreement between theory and simulations, and a reasonable agreement with the results of previous works for the case of Gaussian noises [42–46]. Fig. 4 depicts the other situation, that is fixed τ and different values of q, where the theory shows a behavior similar to the numerical experiments for $D_{nG} \leqslant 0.5$. For larger values of D_{nG}, the noise intensity becomes the order of the barrier height and the Kramers-like approximation breaks down. For this reason the theoretical results, presenting a crossing between curves with different values of q depart slightly from the behavior indicated by simulations.

6. Conclusions

We have studied the problem of diffusion in a double well potential driven by a non-Gaussian colored noise source. We started analyzing a particular class of Langevin equations (Eq. (2)) having non-Gaussian stationary distribution functions [22,23]. In order to obtain an effective (UCNA-like) Markovian approximation to the original non-Markovian problem, we have approached Eqs. (1) and (2) exploiting a scheme, based on a path-integral description of the problem for Gaussian colored noise, analogous to that used for the obtention of the UCNA and its generalizations [9,10,36,38].

Such an effective Markovian approximation allows us to obtain analytical expressions for some relevant quantities, particularly the MFPT. The comparison of the theoretical results for the MFPT with those obtained from Monte Carlo simulations show a good agreement indicating that the analytical expressions can be used to predict qualitatively the behavior of systems submitted to such kinds of non-Gaussian colored noises.

Regarding the behavior of T, the MFPT, for $D_{nG} \to 0$ (or $D \to 0$), as we use a Kramers-like scheme, the present effective Markovian approximation gives the typical exponential dependence. That is, plotting $\ln T$ vs. $1/D_{nG}$ we find a linear behavior, only the slope depends on q. The analysis of such a dependence through detailed simulations could be extremely time consuming and it is doubtfully can give a sound answer. If we look at the original system Eqs. (1) and (2), even though it is well known that the underlying two-dimensional potential (even for the colored Gaussian case) cannot be determined in general, the form of the force in Eq. (2) (or η's potential, see Eq. (3)), suggests the possibility that for $q \neq 1$ such a dependence could be potential. However, such a study is beyond the scope of the present work.

Preliminary studies on the phenomenon of stochastic resonance [47] have shown the strong effect of such type of noise making possible a significant enhancement of the system response, an aspect of great technological relevance. In addition, this kind of noise source could have biological implications as some experimental results in sensory systems [19,20] offer strong indications that the noise source in these systems could be non-Gaussian. A detailed study of the phenomenon of stochastic resonance in systems driven by such types of noise will be the subject of further work.

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