Synchronization in Complex Networks: a Comment on two recent PRL papers

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(Dated: June 20, 2006)

I show that the conclusions of [Hwang, Chavez, Amann, & Boccaletti, PRL **94**, 138701 (2005); Chavez, Hwang, Amann, Hentschel, & Boccaletti, PRL **94**, 218701 (2005)] can be deduced from previous publications.

PACS numbers: 89.75Hc

Recent interest on dynamical networks regards the synchronizability properties of networks with coupled identical oscillators [1]. According to the standard formalism in the study of synchronization in a class of coupled identical oscillators [1], the eigenratio $R = \lambda_N/\lambda_2$, defined from the eigenvalues of the laplacian matrix is a measure of the synchronizability of the network, and has to be smaller than a certain dynamical ratio defined from the system under the study for the network to synchronize. In [2] it was shown that networks that are heterogeneous in the degree distribution are more difficult to synchronize, and as a consequence this applies to scale-free networks (SFNs). A way to overcome this effect is to reduce the weight of the highly connected nodes, i.e., to construct a weighted, and hence asymmetric, network, as shown in [3, 4].

More precisely, the idea is that synchronizability is improved in this type of networks if the weights are such that the total strength of input connections is the same for all the oscillators in the network [3]. Thus, in Ref. [3] the strength of connections is weighted such that $w_{i\to j} = 1/k_j^{\beta}$ (and analogously for $w_{j\to i}$). It was proven [3] that synchronizability is maximum for $\beta = 1$, and for this case it follows immediately that the total strength of input connections $\sum_j w_{j\to i}$ is the same for any node i of the network, while for output connections $\sum_i w_{j\to i}$ the distribution of strengths is identical to the degree distribution (cf. comment reference [30] in [4]). On the other hand, because of the asymmetry $w_{i\to j} = 1/k_j^{\beta} < w_{j\to i} = 1/k_i^{\beta}$ when $k_j > k_i$ and $\beta > 0$, then it is clear that the strength of the output connections is positively correlated with the degree of the node, implying that when synchronizability is improved the dominant coupling direction is from high-degree nodes to low-degree ones.

In particular, in the 1024-node random SFN [5] considered in [3] it is shown, Fig. 2, that a *ten-fold*, i.e. 1000% improvement in synchronizability is attained, compared to the unweighted case considered in Ref. [2]. Moreover, as the number of oscillators N grows, this synchronization enhancement improves even further (check the ratio for weighted networks, Eq. [4] of Ref. [3], with the unweighted case, Eq. 2 in Ref. [2]). This improvement in synchronizability is shown to be significant when the total strength of input connections is equal for all the oscillators in the network.

In more recent work, Ref. [6], (with the *familiar* title *Synchronization is Enhanced in Weighted Complex Networks*) the same idea has been suggested: a weighting scheme that lowers the eigenratio R by lowering the connection strength of the most highly connected nodes, while satisfying also the condition uncovered in Ref. [3] that the total strength of input connections is the same for all the oscillators in the network. Relaxing constraints on the individual connections, they were able to improve the synchronizability uncovered by Motter *et al.* by a factor of about 1.2, i.e., 20 %, or less (cf. Fig.2 in Ref. [6]). In addition, their result is purely numerical, which does not allow a both systematic comparison as a function of N and also elucidating that whether this improvement is generic in parameter space.

In closely connected work, Ref. [7], a more refined version of the weighted and directed coupling of [6] has been introduced. The main conclusion of Ref. [7] is that *propensity for synchronization is enhanced in networks of asymmetrically coupled units* and that in growing SFNs *such enhancement is particularly evident when the dominant coupling direction is from older to younger nodes.* But for the kind of growing SFNs considered in Ref. [7] it is well known that older nodes have a larger degree (more connections)[8]. So, the conclusion is that replacing the words *propensity for synchronization* by *synchronizability* and *age* by *number of connections* (or degree) the conclusions of Hwang *et al.* follow immediately from [3].

In addition, Ref. [3, 4] are considerably more complete than both [7] and [6], as in [3, 4] the authors not only consider the ability to synchronize but also the cost involved in the connections. Moreover, the conclusions in [3, 4] are supported by analytical results (what allows to understand better the effects of N, etc.), and they formulate the problem for general output functions H (e.g., the case of H linear is not useful in the study of coupled maps [4]).

To summarize, I think I have convincingly argued that the main ideas and conclusions of the two recent PRL papers [6, 7] follow immediately from Ref. [3, 4], that moreover are more general and give analytical support of the results.

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[8] This was shown analytically in the very first paper about scale-free networks [5]: $k_i(t) = m(t/t_i)^{1/2}$