Fluctuation and Noise Letters
Vol. 5, No. 1 (2005) L63–L71

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TRANSFER OF INFORMATION IN PARRONDO'S GAMES

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Received 2 July 2004
Revised 20 November 2004
Accepted 21 November 2004
Communicated by Juan Parrondo

We study the relation between the discrete-time version of the flashing ratchet known as Parrondo's games and a compression technique used very recently with thermal ratchets for evaluating the transfer of information — negentropy — between the Brownian particle and the source of fluctuations. We present some results concerning different versions of Parrondo's games showing, for each case, a good qualitative agreement between the gain and the variation of the entropy.

Keywords: Ratchets; Parrondo's paradox; information theory.

1. Introduction

The field of microscopic Brownian particles has recently focused its attention on new directed transport phenomena under the conditions of (i) broken spatial inversion symmetry and (ii) the contact with a thermal bath in a nonequilibrium situation [1]. The first condition is accomplished through an asymmetric potential, usually a ratchet-like potential. The second condition can be achieved in different ways, either by periodic or stochastic forcing (*pulsating ratchets*), or by additive driving force unbiased on average (*tilting ratchets*). The *flashing ratchet* corresponds to the class of pulsating ratchets and it consists on switching on and off either periodically or stochastically a ratchet potential. This model has been used recently for DNA transport [2] and separation of biological macromolecules [3].

Recently, Arizmendi *et. al* [4] have quantified the transfer of information — negentropy — between the Brownian particle and the nonequilibrium source of fluctuations acting on it. These authors coded the particle motion of a flashing ratchet into a string of 0's and 1's according to whether the particle had moved to the left or to the right respectively, and then compressed the resulting binary file using the Lempel and Ziv algorithm [5]. They obtained in this way an estimation of the entropy per character h, as the ratio between the lengths of the compressed and the original file, for a sufficiently large file length. They applied this method to estimate the entropy per character of the ergodic source for different values of the flipping rate, with the result that there exists a close relation between the current in the ratchet and the net transfer of information in the system. The aim of this paper is to apply this technique to a discrete-time and space version of the Brownian ratchet, known in the literature as Parrondo's paradox.

Parrondo's paradox [6–12] states that a combination of two negatively biased games — losing games — can give rise to a positively biased game — a winning game. Although this paradox appeared as a translation of the physical model of the flashing ratchet into game-theoretic terms, there has been no quantitative demonstration of their relation until very recently [13–15].

More precisely, the paradox is based on the combination of two games (Parrondo's games). One of them, game A, is a simple coin tossing game where the player has a probability p (respectively, $1-p$) of winning (respectively, losing) one unit of capital. The second game, game B, is a capital dependent game, where the probability of winning depends on the capital of the player modulo a given number M. Usually, M is set to 3 and the winning probability is p_i if the capital is equal to i mod 3. One of the possibilities for the numerical values is: $p = \frac{1}{2} - \epsilon$, $p_0 = \frac{1}{10} - \epsilon$, $p_1 = p_2 = \frac{3}{4} - \epsilon$, where ϵ is a small biasing parameter that converts games A and B into losing games. When $\epsilon = 0$ it can be demonstrated that the fairness condition is fulfilled for both games, that is $\prod_{j=0}^{M-1} p_j = \prod_{j=0}^{M-1}(1-p_j)$. As soon as $\epsilon > 0$ this condition no longer applies and A and B are both losing games.

The combination game, game AB, is obtained alternating between game A and game B with probabilities γ and $1-\gamma$, respectively. The corresponding winning probabilities of game AB are $q_0 = \gamma p + (1-\gamma)p_0$ when the capital is multiple of 3, and $q_1 = q_2 = \gamma p + (1-\gamma)p_1$, otherwise. Since it can be checked that $\prod_{j=0}^{M-1} q_j > \prod_{j=0}^{M-1}(1-q_j)$ for $\gamma \in (0,1)$ and not too large ϵ, it turns out that game AB is a winning game in those cases.

Several other versions of the games have been introduced: in the so-called cooperative games [16, 17], one considers an ensemble of interacting players; in the history dependent games [18,19], the probabilities of winning depend on the history of previous results of wins and loses; finally, in the games with self-transition [20], there is a nonzero probability r_i that the capital remains unchanged (not winning or losing) in a given toss of the coins.

Some previous works in the literature have related Parrondo's games and information theory. Pearce, in Ref. [21], considers the relation between the entropy and the fairness of the games, and the region of the parameter space where the entropy of game A is greater than that of B and AB. Harmer *et. al* [22] study the relation between the fairness of games A and B and the entropy rates considering two approaches. The first one calculates the entropy rates not taking into account the correlations present on game B, finding a good agreement between the region of maximum entropy rates and the region of fairness. The second approach introduces these correlations, obtaining lower entropy rates and no significant relation between fairness and entropy rates for game B.

In this paper we aim to relate the current or gain in Parrondo's games with the variation of information entropy of the binary file generated using techniques similar to those of Ref. [4]. In Sec. 2 we show the numerical results coming from simulations of the different versions of Parrondo's games mentioned above, and we offer, in Sec. 3, a theoretical analysis that helps to understand the behavior observed in the simulations.

2. Simulation Results

We have performed numerical simulations of the different versions of the games. In every case, the evolution of the capital of the player has been converted to a string of bits where bit 0 (respectively, 1) corresponds to a decrease (respectively, increase) of the capital after δ_t plays of the games. It will be shown that the delay time δ_t between capital measurements is a relevant parameter.

An estimation of the entropy per character, h, is obtained as the compression ratio obtained with the `gzip` (v. 1.3) program, that implements the Lempel and Ziv algorithm, although it has been stressed by some authors that this is not the best algorithm one can find in the literature. The simplicity in the use of this algorithm (as it is already implemented "for free" in many operating systems) is an added value, as it will become apparent in the following when we consider strings of symbols generated by more than one ergodic source. As suggested in Ref. [4], we expect that the negentropy, $-h$, which accounts for the known information about the system, is related in some way with the average gain in the games.

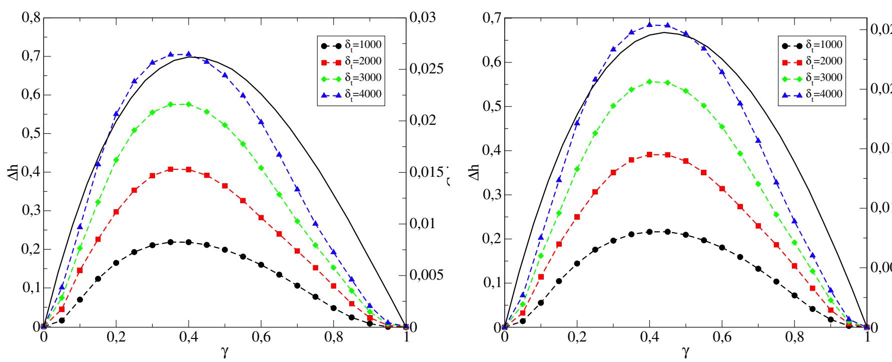

Fig. 1. Comparison of the average gain per game (solid line) with the entropy difference Δh (symbols) as a function of the switching rate γ, for several values of the delay time δ_t, as shown in the legend, and the following versions of the Parrondo's paradox: Left panel: Original Parrondo's combination of games A and B with probabilities: $p = \frac{1}{2}$, $p_0 = \frac{1}{10}$ and $p_1 = \frac{3}{4}$. Right panel: Parrondo's combination of games A and B including self-transitions. The values for the probabilities are: $p = \frac{9}{20}$, $r = \frac{1}{10}$, $p_0 = \frac{3}{25}$, $r_0 = \frac{2}{5}$, $p_1 = \frac{3}{5}$ and $r_1 = \frac{1}{10}$ (see Ref. [20] for the choice of these parameters).

In Fig. 1 we compare the average gain in game AB with the value of the entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ as a function of the probability γ and for different delay times δ_t. We find indeed a qualitative agreement between the increase in the gain and the decrease in entropy as the γ parameter is varied. This decrease in the entropy of the system implies that there exists an increase in the amount of known information about the system. Notice that the compression rate depends on δ_t, and

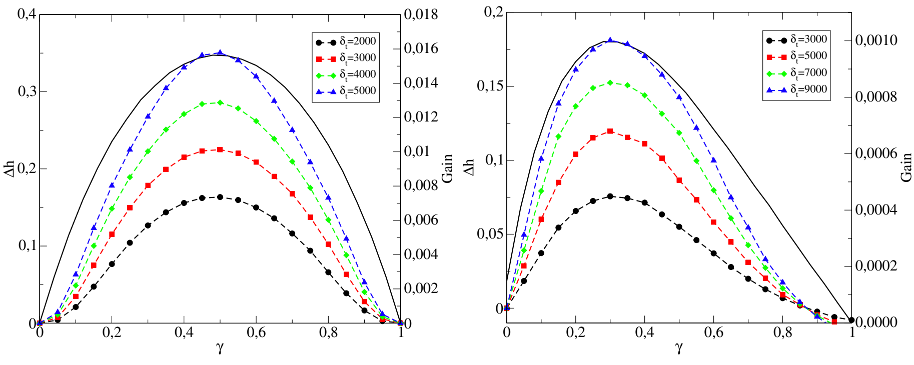

Fig. 2. Same as Fig. 1 in other versions of Parrondo's paradox: Left panel: History dependent games, alternating between two games with probabilities: $p_1 = \frac{9}{10}$, $p_2 = p_3 = \frac{1}{4}$, $p_4 = \frac{7}{10}$; $q_1 = \frac{2}{5}$, $q_2 = q_3 = \frac{3}{5}$ and $q_4 = \frac{2}{5}$ (see Ref. [18] for the choice of these parameters). Right panel: Cooperative Parrondo's games with probabilities: $p = \frac{1}{2}$, $p_1 = 1$, $p_2 = p_3 = \frac{16}{100}$, $p_4 = \frac{7}{10}$ and $N = 150$ players (see Ref. [16] for the choice of these parameters).

that the γ value for which there is the maximum decrease in entropy agrees with the value for the maximum gain in the games. This agreement is similar to the one observed when applying this technique to the Brownian flashing ratchet [4].

Similar results are obtained in other cases of Parrondo's games. For instance, in the right panel of Fig. 1 we compare the average gain and the entropy difference in the games with self-transition [20]. Again in this case the maximum gain coincides with the γ value for the minimum entropy per character for all values of δ_t.

Finally, in Fig. 2 we present the comparison in the case of the history dependent games [18] (left panel), and cooperative games [16] (right panel), showing all of them the same features as in the previous cases. We conclude that there exists a close relation between the entropy and the average gain. In the next section we will develop a simple argument that helps explaining this relation.

3. Theoretical Analysis

Shannon, in his seminal work [23], defines the entropy per character of a text produced by an ergodic source as the following expression[a]:

$$H = -\sum_i p_i \cdot \log(p_i) \tag{1}$$

where p_i denotes the probability that the source will emit a given symbol a_i, and the sum is taken over all possible symbols that the source can emit. For instance, if we consider game A as a source of two symbols, 0 (losing) and 1 (winning) , the Shannon entropy according as a function of the probability p of emitting symbol 1 (i.e. the probability of winning) is $H(p) = -p\log p - (1-p)\log(1-p)$. In Fig. 3 we compare this expression with the compression factor h obtained using the `gzip` algorithm. As shown in this figure, in this case of a single source, the

[a]Units are taken such that all logarithms are base 2.

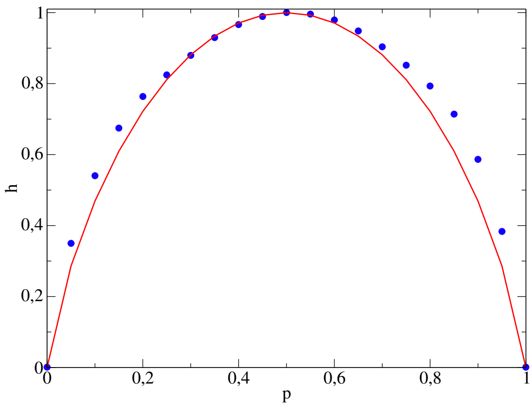

Fig. 3. Comparison between the theoretical value obtained for the Shannon entropy — solid line — with the numerical values — circles — obtained with the `gzip` algorithm for a single source emitting two symbols with probability p.

compression factor of the `gzip` algorithm does give a good approximation to the Shannon entropy.

If the source is a mixed source, that is, composed by M independent sources each one appearing with a probability Π_j and with entropy H_j, the entropy reads [23]

$$H = \sum_{j=1}^{M} \Pi_j \, H_j = -\sum_{j,i} \Pi_j \, p_i^j \, \log(p_i^j). \tag{2}$$

where p_i^j denotes the probability of emitting the symbol a_i by source j.

From now on, we restrict our analysis to the case of the original Parrondo's paradox combining games A and B, as explained in the previous section. The combined games AB can be considered as originated by two sources depending on whether the capital is a multiple of 3 or not. The probability of emitting symbol 1 when using the first source is q_0, whereas the same probability is q_1 when using the second source.

We first consider the case $\delta_t = 1$, i.e. we store the capital after each single play of the games. According to the previous discussion, the Shannon entropy for the combined game AB is:

$$H = -\Pi_0[q_0 \log(q_0) + (1-q_0)\log(1-q_0)] - (1-\Pi_0)[q_1 \log(q_1) + (1-q_1)\log(1-q_1)] \tag{3}$$

being Π_0 the stationary probability than in a given time the capital is a multiple of 3. This can be computed using standard Markov chain theory, with the result [24]:

$$\Pi_0 = \frac{1 - q_1 + q_1^2}{3 - q_0 - 2q_1 + 2q_0 q_1 + q_1^2}\,. \tag{4}$$

In Fig. 4 we compare the Shannon entropy H given by the previous formula with the numerical compression factor h as a function of the probability γ of mixing

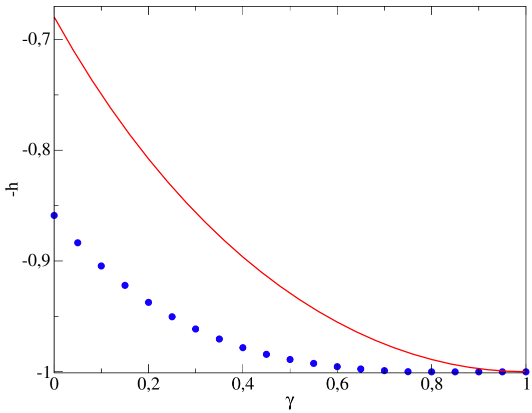

Fig. 4. Plot of Shannon negentropy (solid line) for the combination game AB according to expression 3, together with the numerical values (circles) obtained with the compression factor of the `gzip` algorithm in the case when $\delta_t = 1$ step.

games A and B. Although certainly not as good as in the case of a single game, in this case, the `gzip` compression factor gives a reasonable approximation to the Shannon entropy of the combined game AB. It is worth noting that in this case of $\delta_t = 1$ the entropy increases with γ, corresponding to a decrease of the information known about the system. In order to relate the entropy difference with the current gain, we need to consider larger values for δ_t.

For $\delta_t \gg 1$ the system gradually loses its memory about its previous state. Therefore, the different measures are statistically independent and they can be considered as generated by a single ergodic source. For this single source, the probability of winning after one single play of the games is $p_w = \Pi_0\, q_0 + (1 - \Pi_0)\, q_1$. However, we are interested in calculating the winning probability $p_>$ after δ_t plays. In order to have a larger capital after δ_t plays it is necessary that the number of wins overcomes the number of losses in single game plays. The distribution of the number of wins follows a binomial distribution and the probability $p_>$ is given by:

$$p_> = \sum_{k=0}^{\frac{\delta_t}{2}} \binom{\delta_t}{k} \cdot p_w^{\delta_t - k} \cdot (1 - p_w)^k . \qquad (5)$$

The corresponding Shannon entropy for this single source is:

$$H = -p_> \cdot \log(p_>) - (1 - p_>) \cdot \log(1 - p_>). \qquad (6)$$

We compare in Fig. 5 the Shannon entropy coming from this formula and the one obtained by the compression ratio of the `gzip` program for two different values of $\delta_t = 500, 1000$. In both cases, there is a reasonable agreement between both results. Moreover, as shown in Figs. 1 and 2 the entropy follows closely the average gain of the combined games.

As a conclusion, we have quantified the amount of the transfer of information (negentropy) in the case of Parrondo's games, considered as a discrete-time and

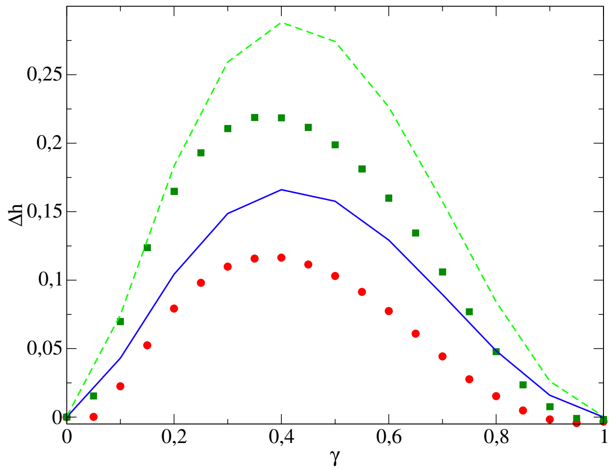

Fig. 5. Plot of Shannon entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ according to formulas 6 and 5 for $\delta_t = 500$ (solid line) and $\delta_t = 1000$ (dashed line) together with the numerical curves obtained with the compression ratio of the `gzip` algorithm for the same values of $\delta_t = 500$ (circles) and $\delta_t = 1000$ (squares).

space version of the flashing ratchet. This effect takes place in every existing version of the games analyzed, showing its robustness, and it is the equivalent of the same result obtained in the case of the Brownian ratchets. In the case of the original Parrondo's paradox mixing two games, A and B, we have computed the entropy by considering that the capital originates from a combination of two ergodic sources, reflecting the different winning probabilities when the capital is a multiple of three or not. We have shown that the entropy behaves very differently for low and high values of the delay parameter δ_t, while for $\delta_t = 1$ there is a monotonic dependence on the switching parameter γ, the relation between the gain and the current is only apparent for large values of δ_t. Our paper offers a new and hopefully enlightening approach to understand Parrondo's paradox. This approach differs (and complements) from previous work [21, 22] in that we consider the capital of the player as the information source.

Acknowledgments

We thank C. M. Arizmendi for several discussions, including the sharing of useful information about the method. We acknowledge financial support from MCYT (Spain) and FEDER through projects BFM2000-1108 and BFM2001-0341-C02-01. P.A. is supported by a grant from the Govern Balear.

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