

Direct transition to high-dimensional chaos through a global bifurcation

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Abstract. – In the present work we report on a genuine route by which a high-dimensional (with $d \gtrsim 4$) chaotic attractor is created directly, *i.e.*, without a low-dimensional chaotic attractor as an intermediate step. The high-dimensional chaotic set is created in a heteroclinic global bifurcation that yields an infinite number of unstable tori. The mechanism is illustrated using a system constructed by coupling three Lorenz oscillators. So, the route presented here can be considered a prototype for high-dimensional chaotic behavior just as the Lorenz model is for low-dimensional chaos.

Introduction. – The field of Nonlinear Dynamics has achieved a state of maturity in the study and characterization of the transitions exhibited by low-dimensional dissipative dynamical systems [1]. In particular, transitions to chaotic behavior in these systems appear to take place through a few well-known routes (or scenarios): the period doubling cascade, the intermittency route, the routes involving the destruction of quasiperiodic tori or the crisis route (see, e.g., [1–3] for a survey). Another possibility is that chaos appears through a global connection to a fixed point [4], *e.g.* Shil'nikov or Lorenz chaos [5]. Characterizing these scenarios is important because they present universal features, independently of the physical system involved.

Contrastingly, not many studies have been published about the transitions to chaotic attractors with dimension $d > 3$, *i.e.*, high-dimensional chaos. Putting aside routes starting from a low-dimensional chaotic attractor (for instance, in the transition to hyperchaos where a second Lyapunov exponent becomes positive [6, 7]), only a few cases have been reported in the literature. These involve generalizations of low-dimensional routes to chaos through an extra oscillation: period doubling cascade of a torus [8] (instead of a limit cycle) or a Lorenz-type homoclinic connection to a limit cycle [9] (instead of a fixed point). In these two

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cases the dimension of the high-dimensional chaotic attractor is (roughly) *one* unit above the chaotic attractor arising from the corresponding low-dimensional route. In this framework, the following relevant question arises: does there exist a *bona fide* direct transition to H-D chaos? *i.e.* a mechanism that cannot be reduced to a low-dimensional route with an extra oscillation (unlike in [8,9])?

High-dimensional ($d > 3$) chaotic attractors *live* in spaces of large dimension $d \geq 4$, and the existence of routes to (H-D) chaos that cannot be envisaged from low-dimensionality is, therefore, plausible. Strictly one should be speaking here of *intermediate-dimensional* rather than high-dimensional attractors, as often in the literature the dimension is by far larger than three. For instance, (very) high-dimensional attractors are usually related to the regime of spatio-temporal chaos in representative nonlinear partial differential equations (like the Kuramoto-Sivashinsky equation [10]). The aspect that marks an important difference here is that these regimes have been characterized mostly using statistical techniques [11], while here we are proposing, instead, a geometric analysis, for which the detailed study of phase spaces of dimension, say, 4 and 5 is already quite a challenge. So, in this sense we present the first example of a genuine route to H-D chaos, allowed by the high enough dimensionality of the embedding space.

System and overall picture. – The model studied here consists of three Lorenz oscillators [5,12] coupled unidirectionally. The evolution equations (an autonomous 9-dimensional dynamical system) read:

$$\left. \begin{aligned} \dot{x}_j &= \sigma(y_j - x_j) \\ \dot{y}_j &= R \underline{x}_j - y_j - x_j z_j \\ \dot{z}_j &= x_j y_j - b z_j \end{aligned} \right\} \quad j = 1, \dots, N = 3, \quad (1)$$

where $\underline{x}_j = x_{j-1}$ for $j \neq 1$, introduces the coupling and periodic boundary conditions are used: $\underline{x}_1 = x_3$.

The study of (1) has been suggested by the results of the experiment with three coupled Lorenz oscillators modeled by electronic circuits [13,14]. By increasing R , synchronized chaos among the three oscillators gives rise to high-dimensional chaos (a Chaotic Rotating Wave, CRW [14,15]) and, finally, quasiperiodic and periodic behaviors. Here we focus on the transition between “order” and H-D chaos, obtained when R is decreased.

A high-frequency periodic component (a rotating wave into the ring) is present along the route. Hence, to reduce the dimensionality of the problem, most of our presentation is based on an appropriate Poincaré section: $Im(X_1) = 0$, $Im(\dot{X}_1) > 0$ (we use the discrete Fourier modes, suggested by the circulant symmetry of the system, defined as $X_k = 1/N \sum x_j \exp[2\pi i(j-1)k/N]$; analogously for y and z coordinates). Figure 1 depicts the coordinates X_0 of the intersections of the trajectories into the different attractors. For large R , the system exhibits a Periodic Rotating Wave (PRW) [13,14,16], which is a periodic motion with $2\pi/3$ phase difference between adjacent oscillators. This state exhibits a pitchfork bifurcation at $R_{pitch} \approx 39.25$, in which the single PRW attractor leads to two twin asymmetric PRW attractors. By further decreasing R , the system exhibits two consecutive Hopf bifurcations at $R_{h1} \approx 35.26$ and $R_{h2} \approx 35.0955$ giving rise to two stable symmetry related 3D tori [17]. The robust existence of \mathbb{T}^3 attractors as well as the absence of appreciable frequency lockings in the \mathbb{T}^2 attractors stems from: 1) The disparity of the frequencies, such that resonances correspond to rationals with large denominators (and therefore narrow Arnol’d tongues [18]); 2) The cyclic symmetry of the system: the absence of lockings is a general feature of modulated rotating waves (like our \mathbb{T}^2) in systems with rotational symmetry [19], so although this result only holds exactly in the continuum limit, one expects some inhibition of the lockings.

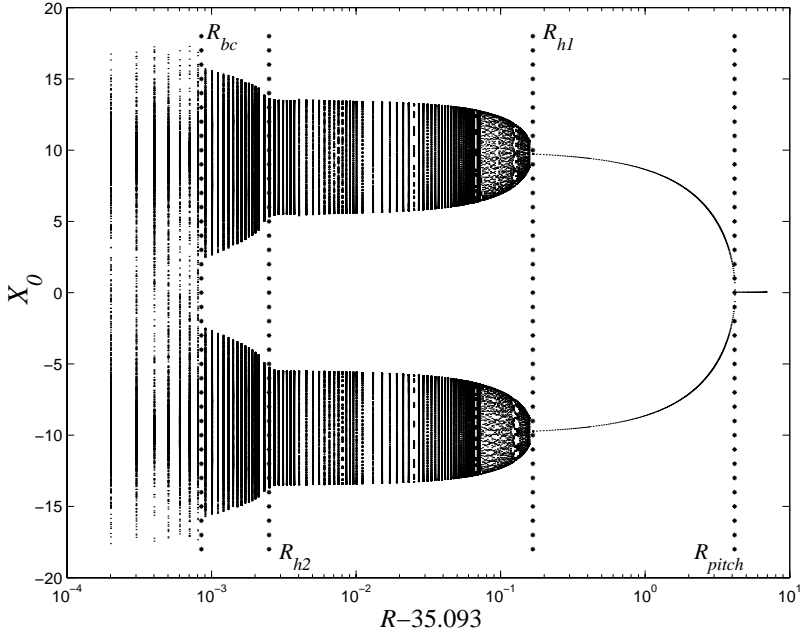


Fig. 1 – Bifurcation diagram representing the X_0 coordinates of the intersection with the Poincaré section (see text) as a function of the parameter R . The logarithmic scale has been adopted in the x -axis to better resolve attractors existing in quite different ranges. Parameters in eq. (1): $\sigma = 20$, $b = 3$.

The H-D chaotic attractor appears at $R_{bc} \approx 35.09384$, and as may be seen in fig. 1 the dynamics recovers the reflection symmetry lost in the pitchfork bifurcation. The dimension of the chaotic attractor can be estimated by means of a direct calculation of the correlation dimension [20] that yields $D_2 = 3.96 \pm 0.05$, for $R = 35.05$. Therefore, we must envisage a route that is able to create a chaotic attractor with dimension around four. Taking this into account, we list some evidences that allowed us to understand the mechanisms involved in the creation of the H-D chaotic attractor:

1) We observe that the average chaotic transient diverges at R_{bc} . The divergence follows a power law, typical for boundary crises [21], that convert chaotic attractors into chaotic transients: $\langle \tau \rangle \sim (R - R_{bc})^\gamma$, $\gamma = -1.53 \pm 0.06$.

2) The \mathbb{T}^3 attractors disappear at twin saddle-node bifurcations [22] at $R_{sn} \approx 35.09367$. Therefore, as $R_{sn} < R_{bc}$, there exists a small range of *coexistence* between the 3D tori and the H-D chaotic attractor. Then, we conclude that the \mathbb{T}^3 attractors are not involved in the birth of the chaotic attractor.

3) A value $R_{expl} \approx 35.11$ is found to define a transition below which there are chaotic transients (in which, for some initial conditions, the trajectory approaches both \mathbb{T}^2 attractors, in a non-periodic manner, before being eventually attracted by one of them). We observed that this point coincides (at least approximately) with a reordering of the unstable manifolds of the PRWs.

Theoretical analysis. – These evidences lead us to conclude that the transition from periodic to chaotic behavior occurs following the set of bifurcations shown in fig. 2. It includes the creation of the H-D chaotic set in a “heteroclinic explosion” followed by its conversion into an attractor through a boundary crisis.

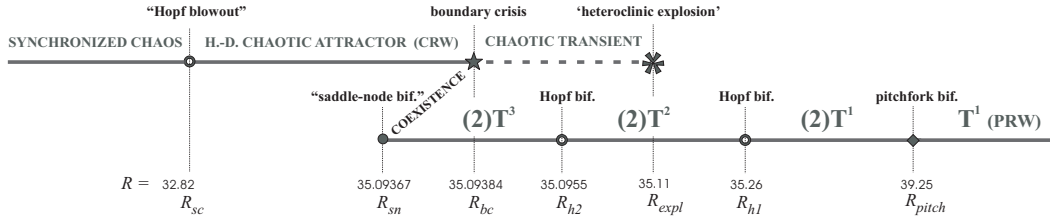


Fig. 2 – Diagram representing schematically the transitions from synchronized chaos (left) to a PRW (right).

Figure 3 illustrates geometrically the route to H-D chaos (note that we consider a Poincaré section that reduces the dimensions of all the attractors by one): the centered PRW (a) becomes unstable through a pitchfork bifurcation (a→b) and two symmetry-related PRWs appear (b). At a supercritical Hopf bifurcation (b→c) the 2D tori appear. When R is further decreased the 2D tori become focus-type, and, as a result, the unstable manifold of the asymmetric PRW forms a “whirlpool” [23] when approaching the T^2 (d). At R_{expl} a double heteroclinic connection between the asymmetric PRWs and the symmetric one occurs (e). At this point the chaotic set, with a dense set of unstable 3D tori, and thus a dimension above 4, is created. In (f) the two simplest unstable 3D tori are represented with dotted lines; because of the heteroclinic birth one of the frequencies of these tori is very small (formally zero at R_{expl}). Notice that the plot shows several forbidden intersections of manifolds that are unavoidable because of the projection onto \mathbb{R}^3 (analogous “visual” effect occurs when the Lorenz system is projected onto \mathbb{R}^2 , x - z plane). Twin secondary Hopf bifurcations (f→g) render unstable

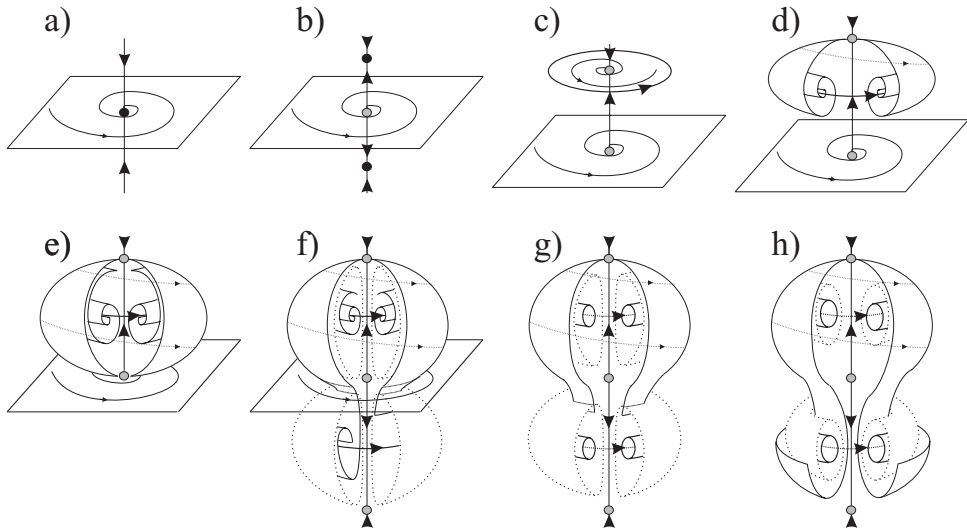


Fig. 3 – Representation (a Poincaré section projected on \mathbb{R}^3) of the proposed heteroclinic route to create the high-dimensional chaotic attractor. Black and gray points correspond to stable and unstable fixed points (cycles in the global phase space), respectively.

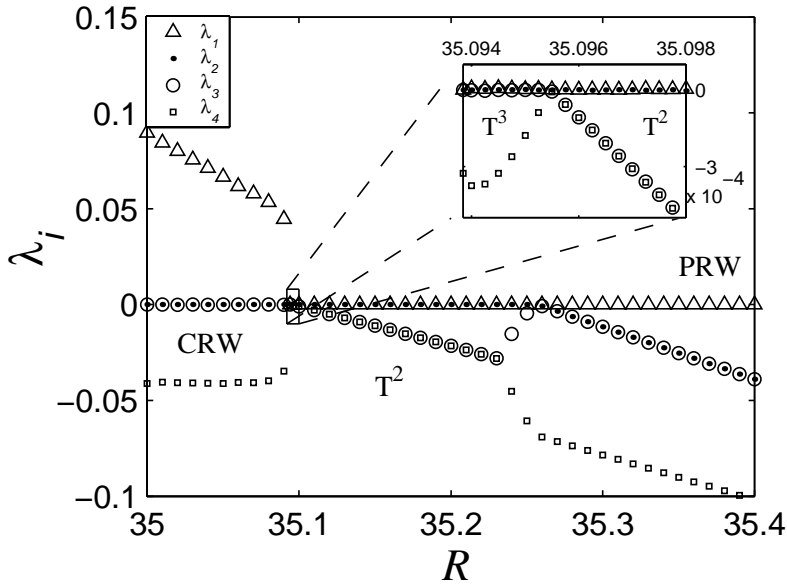


Fig. 4 – The four largest Lyapunov exponents ($\lambda_{i=1-4}$) as a function of R . Four regions can be distinguished according to their spectra. Note that in the chaotic region we obtain $\lambda_2 \simeq \lambda_3 \simeq 0$ and $\lambda_1 \gtrsim |\lambda_4|$ which implies, according to the Kaplan-Yorke conjecture, an information dimension $D_1 \gtrsim 4$.

the 2D tori and give rise to two stable 3D tori (g). For smaller R , the unstable manifolds of the asymmetric PRWs do not connect to the stable 3D tori (h), and the chaotic set becomes attracting. This last step (g→h) is analogous to the boundary crisis occurring in the Lorenz system [21]. Finally, it is to be stressed that the reflection symmetry plays a fundamental role; whereas the existence of stable \mathbb{T}^3 is not needed (focus-type \mathbb{T}^2 suffice to induce whirlpools that finally become 2D heteroclinic connections, Fig. 3(d,e)).

Lyapunov spectrum. – Let us see now how the route shown above relates to the Lyapunov spectrum. Figure 4 shows the leading Lyapunov exponents (LEs) in the transition from periodic behavior to chaos. Please, notice that a similar Lyapunov spectrum has been obtained considering $N = 4$ and a more general coupling (cf. fig. 5(a) in [8]). In the chaotic region only one exponent is positive whereas two vanish and one of the negative LEs is close to zero. It is remarkable that $\lambda_1 \gtrsim |\lambda_4|$, which implies $D_1 \gtrsim 4$, according to the Kaplan-Yorke conjecture [24].

Also, the fifth to ninth (non-leading) LEs for the chaotic attractor and the twin 3D tori are very similar (see table I) which suggests that the 3D tori and the H-D chaotic attractor “live” in the same four-dimensional subspace (as we have implicitly assumed above). And thus, thinking in terms of some kind of generalization of the Birman-Williams theorem [26],

TABLE I – The five smallest Lyapunov exponents for both, chaotic (CRW) and three-frequency quasiperiodic (\mathbb{T}^3), attractors coexisting at $R = 35.0938$.

	$\lambda_i(\text{CRW})$	$\lambda_i(\mathbb{T}^3)$
$\lambda_5 = \lambda_6$	-5.255	-5.203
$\lambda_7 = \lambda_8$	-18.612	-18.652
λ_9	-24.273	-24.290

the template of the chaotic attractor can be visualized as a 4D branched manifold. In the same way, the (butterfly) Lorenz attractor may be understood as a two-dimensional branched manifold, with a “tear point” at the origin [27].

Further remarks and conclusions. – Some further comments are in order with respect to the relation between the LEs and the geometric sketch depicted in Fig. 3. It shows a mechanism, through heteroclinic connections, to create a chaotic attractor containing an infinity of unstable tori (UT) —recall that only the two simplest ones are drawn— instead of unstable periodic orbits (UPOs). If, as it occurs in our system, there is an additional frequency, 3D-UT are created instead. In consequence, there are three neutral directions and the chaotic attractor should have three null LEs. Figure 4 shows instead two vanishing and one slightly negative LEs. This shift is a consequence of the fact that a generic perturbation on the mechanism shown in fig. 3 will destroy its symmetry (and consequently its simplicity). In analogy to previous works [28, 29] dealing with the effect of non-symmetric terms on the normal form of codimension-two points, we expect (generic) homoclinic connections to replace heteroclinic connections. A double (“figure-eight”) homoclinic to the symmetric PRW as well as homoclinic connections to the asymmetric PRWs will occur [30] Consequently, in the perturbed scenario an infinity of UT (instead of 3D-UT) are created. This explains the absence of a third vanishing Lyapunov exponent. But it is important to emphasize that as long as the exact mechanism is closely related to the one shown in fig. 3, the largest negative LE is close enough to zero ($|\lambda_4| < \lambda_1$) to get an information dimension above *four*.

In this letter, we have reported the creation of a high-dimensional chaotic (but not hyper-chaotic) attractor without intermediate low-dimensional chaos. The structure of the global bifurcations, underlying this route, leads to the emergence of a chaotic attractor with dimension $D_1 \gtrsim 4$ (or $D_1 \gtrsim 3$ if the fast rotating wave present all along the route is considered to increase trivially the dimension in one unit). A characteristic of the emerging chaotic attractor is the presence of a very low-frequency component, reminiscent of the heteroclinic birth of the chaotic set (this may have been observed in fluid convection experiments [32]). We believe that the knowledge of the direct routes to high-dimensional chaos may lead to a re-thinking of previous experimental results, as well as to understand and design future experiments.

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