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Discrete–time ratchets, the Fokker–Planck equation and Parrondo's games

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Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

We present a detailed analysis of the so called *Parrondo's games*. They were originally devised as a pedagogical example of the flashing ratchet, but the relation was merely qualitative. Basically it is based on the combination of two games, both of them losing games when played alone, whereas if we combine them either periodically or randomly we obtain a winning game.

In this work we establish a quantitative relation between the parameters describing Parrondo's games and the physical model of the Brownian ratchet. In the following sections of the present Chapter we will introduce some basic concepts on stochastic processes –Sec. 1.2–, Markov processes –Sec. 1.3–, and we will also explain some basic concepts about information theory –Sec. 1.5– that will be used in a next Chapter. We will also present briefly the theory of Brownian motion and explain in more detail the *flashing ratchet* as well as the *ratchet effect* in Chapter 2; in this Chapter we also explain in detail the Parrondo's games, as well as other versions that exist in the literature.

In Chapter 3 a new variation of the games is introduced: Parrondo's games with self–transition. Chapter 4 is dedicated to the relation between Parrondo's games and the flashing ratchet model, showing that they can be related in a rigorous way; in Chapter 5 we will explain the relation between Parrondo's games and information theory. Finally in Chapter 6 we will present the main conclusions of the work.

1.2 Stochastic processes

Generally we can define a stochastic process as a system that evolves probabilistically in time, or more explicitly, a system where there exists at least one time–dependent random variable. Denoting this stochastic variable as $\mathbf{X}(t)$, we can measure its actual value $x_1, x_2, x_3, \ldots$ at different times $t_1, t_2, t_3, \ldots$ and so we can obtain the joint probability density of the variable $\mathbf{X}(t)$

$$P(x_1, t_1; x_2, t_2; x_3, t_3; \ldots) \tag{1.1}$$

which denotes the probability that we measured the value x_1 at time t_1, value x_2 at time t_2,..., etc.

Using these probability density functions (PDF) we can also define *conditional probability densities* through

$$P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1; y_2, \tau_2; \ldots) = \frac{P(x_1, t_1; x_2, t_2; \ldots; y_1, \tau_1; y_2, \tau_2; \ldots)}{P(y_1, \tau_1; y_2, \tau_2; \ldots)}, \tag{1.2}$$

where it's been assumed that the times are ordered, i.e., $t_1 \geq t_2 \geq t_3 \geq \ldots \geq \tau_1 \geq \tau_2 \geq \ldots$.

In order to have a stochastic process completely determined, we should know all possible probability density functions of the kind of Eq. (1.1). For the kind of processes that are determined solely by such a knowledge, they will be known as *stochastic separable process*.

The simplest stochastic process is that of complete independence

$$P(x_1, t_1; x_2, t_2; x_3, t_3; \ldots) = \prod_i P(x_i, t_i) \tag{1.3}$$

which means that the value of $\mathbf{X}$ at time t is completely independent of its values at previous – or posterior – times.

The next step is to consider processes where the future state of the system depends on its actual state. This kind of processes are known in the literature as *Markov processes*.

1.3 Markov processes

This class of processes are characterized by the so called *Markov property*. A Markov process can be defined as a stochastic process with the property that for any set of *successive* times, i.e. $t_1 \geq t_2 \geq t_3 \geq \ldots \geq \tau_1 \geq \tau_2 \geq \ldots$, one has

$$P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1; y_2, \tau_2; \ldots) = P(x_1, t_1; x_2, t_2; \ldots \mid y_1, \tau_1). \tag{1.4}$$

This previous statement means that we can define everything in terms of simple conditional probabilities $P(x_1, t_1 \mid y_1, \tau_1)$. For instance, $P(x_1, t_1; x_2, t_2 \mid y_1, \tau_1) = P(x_1, t_1 \mid x_2, t_2; y_1, \tau_1)P(x_2, t_2 \mid y_1, \tau_1)$ and using the Markov property (1.4) we find

$$P(x_1, t_1; x_2, t_2; y_1, \tau_1) = P(x_1, t_1 \mid x_2, t_2)P(x_2, t_2 \mid y_1, \tau_1) \tag{1.5}$$

and for the general case it can be written

$$P(x_1,t_1;x_2,t_2;x_3,t_3;\dots x_n,t_n) = P(x_1,t_1 \mid x_2,t_2) \quad P(x_2,t_2 \mid x_3,t_3)\dots \\ \dots P(x_{n-1},t_{n-1} \mid x_n,t_n) \quad P(x_n,t_n) \tag{1.6}$$

provided that $t_1 \geq t_2 \geq t_3 \geq \dots \geq t_n$.

There are many processes in nature where this property appears. One of the most studied processes that has been described using this Markov property is the Brownian motion, that we present next in more details.

1.3.1 Brownian motion

The botanist Robert Brown discovered in 1827 that small pollen particles suspended in water were found to be in a very animated and irregular motion. Initially it was supposed to represent some manifestation of life, though after some studies this option was rejected, as the same behavior was also observed in other fine particles suspension –minerals, glass .... The solution to this mysterious movement had to await a few decades, until a satisfactory explanation came through the work of Albert Einstein in 1905 [1]. The same explanation was independently developed by Smoluchowski [2], who was responsible for much of the later systematic development and for much of the experimental verification of Brownian motion theory.

Einstein's work had primarily two main premises:

- The motion of the particles is caused by the exceedingly frequent impacts on the pollen grain of the incessantly moving molecules of liquid in which it is suspended.
- The motion of these particles can only be described probabilistically in terms of frequent and statistical independent impacts, due to the erratic and irregular (and so complicated) movement that the particles carry out.

This process is the best known example of Markov process. We have the picture of a particle that makes random jumps back and forth over a given set of coordinates, for instance over the X–axis in one dimension. The jumps may have any length, but the probability for large jumps falls off rapidly. Moreover, the probability is symmetrical in space and independent of the starting point.

Hence, we can explain in a reduced form the basic steps that Einstein did in order to derive his Brownian motion theory.

The first point to consider is that each individual particle executes a motion which is totally uncorrelated from the motion of all other particles; it will also be considered that the displacement of the same particle, but taken at different time intervals, are also independent processes – as long as these time intervals are not taken too small.

Then a characteristic time interval τ can be introduced, which is small compared to the observation time intervals, but large enough so that the approximation of independent successive time intervals τ is correct.

Now we consider n particles suspended in a liquid. In a time interval τ, the x–coordinate of the particles will increase by an amount Δ, where this quantity may have different values – either positive or negative – for different particles in the same time interval. We will also consider that there exists a certain distribution law for Δ, given by the function $\phi(\Delta)$. The number of particles that will shift his position with an interval between Δ and $\Delta + d\Delta$ will be given by the expression

$$dn = n\,\phi(\Delta)\,d\Delta \tag{1.7}$$

where

$$\int_{-\infty}^{\infty} \phi(\Delta)d\Delta = 1 \tag{1.8}$$

The function ϕ is only distinct from zero for small values of Δ, and it also follows the property

$$\phi(\Delta) = \phi(-\Delta) \tag{1.9}$$

which implies that there exists no preferred direction of movement for the particles.

We can now study how the diffusion coefficient depends on ϕ. Let $P(x,t)$ be the number of particles per unit volume at (x,t). We compute the distribution of particles at time $t+\tau$ from the distribution at time t. From the definition of the function $\phi(\Delta)$, we can obtain the number of particles which at time $t+\tau$ are found between the points x and $x+dx$. One obtains

$$P(x,t+\tau) = \int_{-\infty}^{\infty} P(x-\Delta,t)\phi(\Delta)d\Delta. \tag{1.10}$$

But since τ is very small, we can Taylor expand $P(x,t+\tau)$

$$P(x,t+\tau) = P(x,t) + \tau\,\frac{\partial P}{\partial t}. \tag{1.11}$$

Besides, we can also Taylor expand the function $P(x-\Delta,t)$ in powers of Δ

$$P(x-\Delta,t) = P(x,t) - \Delta\frac{\partial P(x,t)}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 P(x,t)}{\partial x^2} + \dots \tag{1.12}$$

Introducing the results from Eq. (1.11,1.12) into the integral Eq. (1.10) we obtain the following expression

$$P + \frac{\partial P}{\partial \tau}\tau = P\int_{-\infty}^{\infty}\phi(\Delta)\,d\Delta - \frac{\partial P}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)\,d\Delta + \frac{\partial^2 P}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)\,d\Delta. \tag{1.13}$$

Due to the symmetry property Eq. (1.9), the odd terms of Eq. (1.13) – second term, fourth term, etc. – vanish, whereas for the remaining terms, that is, first term, third term, etc. each one is very small compared to the previous one. Introducing Eq. (1.8) in the last equation, setting

$$\frac{1}{\tau}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\,\phi(\Delta)\,d\Delta = D, \tag{1.14}$$

and keeping only the first and third terms on the right hand side,

$$\frac{\partial P}{\partial t} = D\,\frac{\partial^2 P}{\partial x^2}\ldots. \tag{1.15}$$

We can clearly identify the latter equation as the diffusion equation, and D as the diffusion coefficient. The solution of the latter equation for an initial condition at $t = 0$ given by $n(x) = n\,\delta(x)$ is

$$P(x,t) = \frac{n}{\sqrt{4\pi D}}\frac{\mathrm{e}^{-\frac{x^2}{4Dt}}}{\sqrt{t}} \tag{1.16}$$

which is a Gaussian function centered at the origin. Using this result we calculate the averages

$$\langle x\rangle = 0 \tag{1.17}$$

$$\langle x^2\rangle = 2Dt. \tag{1.18}$$

This result was derived by Einstein assuming a discrete–time assumption, that is , that the impacts occurred only at times $0, \tau, 2\tau, \ldots,$ and both Eqs. (1.15,1.16) are to be regarded as only approximations, where τ is considered so small that t can be thought as being continuous.

1.3.2 Langevin's equation

After Einstein presented his theory about Brownian motion, Langevin[3] presented another method quite different from Einstein's work. In brief, his theory can be explained as follows.

From statistical mechanics it was already known that the mean kinetic energy of a Brownian particle at equilibrium should reach a value

$$< \frac{1}{2}mv^2 >= \frac{1}{2}kT \tag{1.19}$$

where T denotes the absolute temperature, k is the Boltzmann constant, m the mass and v the velocity of the Brownian particle.

We can distinguish two different forces acting on the particle, namely,

- A viscous drag. Supposing that the expression of the force is analogous to the macroscopic hydrodynamic equation, for a low Reynolds number we can write down the following expression for the drag force $-6\pi\eta a\frac{dx}{dt}$, η being the viscosity and a the diameter of the particle, assuming it to be spherical.

- A fluctuating force ξ coming from the consideration of the impacts of the fluid particles upon the Brownian particle. The unique consideration about this force is that it can be either positive or negative with the same probability. The ensemble may consist on many particles in the same field, far enough from each other so that they cannot influence mutually. Or it may also be considered as a unique particle, where the time intervals between measurements are large enough not to influence each other.

The stochastic properties of ξ are given regardless of the velocity v of the particle. Its average vanishes, $<\xi>=0$, and its autocorrelation function reads

$$<\xi(t)\xi(t')>=\delta(t-t') \tag{1.20}$$

The latter expression comes from the consideration that successive collisions are uncorrelated and practically instantaneous.

Writing down Newton's equation of motion for the particle we get

$$m\frac{d^2x}{dt^2}=-6\pi\eta a\frac{dx}{dt}+\xi \tag{1.21}$$

This equation is usually known as Langevin equation. Multiplying Eq. (1.21) by x, and after a little algebra we obtain

$$\frac{m}{2}\frac{d^2}{dt^2}(x^2)-mv^2=-3\pi\eta a\frac{d(x^2)}{dt}+x\,\xi \tag{1.22}$$

where $v=\frac{dx}{dt}$. Averaging over a large number of particles and making use of Eq. (1.19) we obtain an equation for $<x^2>$

$$\frac{m}{2}\frac{d^2}{dt^2}<x^2>+3\pi\eta a\frac{d}{dt}<x^2>=kT, \tag{1.23}$$

where the term $<x\xi>$ has been set to zero due to the irregularity of the fluctuating force ξ. This assumption implies that the variation suffered by the x variable can be considered as independent from the variation that the fluctuating force ξ experiences[1]

$$<x\xi>=<x><\xi> \tag{1.24}$$

[1]This can be thought as equivalent to the assumption made by Einstein when he considers that for a sufficiently large time interval τ, the displacements Δ suffered by the Brownian particle within two successive time intervals are independent.

The general solution to Eq. (1.23) is

$$\frac{d}{dt} < x^2 >= \frac{kT}{3\pi\eta a} + C\,\mathrm{e}^{\frac{-6\pi\eta a t}{m}} \tag{1.25}$$

where C is an arbitrary constant.

Considering that the exponential in Eq. (1.25) decays very rapidly, we can dismiss this term and so the solution for the average square distance $< x^2 >$ reads

$$< x^2 > - < x_0^2 >= \left(\frac{kT}{3\pi\eta a}\right) t \tag{1.26}$$

Now we can compare Eq. (1.26) with Eq. (1.18) to obtain the following relation

$$D = \frac{kT}{6\pi\eta a} = \mu kT \tag{1.27}$$

where μ is the mobility of the Brownian particle.

This important result, known as the *fluctuation–dissipation* theorem, relates a quantity D pertaining to statistically unpredictable dynamical fluctuations to a quantity which involves deterministic, steady state properties.

1.4 The Fokker–Planck equation

This section aims to be a brief explanation on how to obtain the time evolution of the probability density function for the system under consideration. Its name comes from the work of Fokker [4] and Planck [5]. The former studied Brownian motion in a radiation field and the latter attempted to build a complete theory of fluctuations based on it.

1.4.1 Derivation of the Fokker–Planck equation

If we consider a Markov process, we can write a master equation as

$$\frac{\partial P(x,t)}{\partial t} = \int \left\{W(x \mid x')\,P(x',t) - W(x' \mid x)\,P(x,t)\right\} dx' \tag{1.28}$$

where the term $W(x \mid x')$ denotes the transition probability between states x and x'. $P(x,t)$ denotes the probability of finding the system at position x at time t, and must be normalized, that is

$$\int_{-\infty}^{\infty} dx\,P(x,t) = 1 \tag{1.29}$$

If x corresponds to a discretized variable, the master equation takes the form

$$\frac{dP_n(t)}{dt} = \sum_n \left\{W_{nn'}P_{n'}(t) - W_{n'n}P_n(t)\right\}. \tag{1.30}$$

Written in this form clearly the master equation is a gain–loss equation. The first term on the right hand side of Eq. (1.30) corresponds to the gain of state n due to transitions from different states n' to n, whereas the second term is a loss term due to the transitions from the state n to other states n'.

Planck derived the Fokker–Planck equation as an approximation to the master equation (1.28). He expressed the transition probability $W(x \mid x')$ as a function of the size r of the jump and of the starting point

$$W(x \mid x') = W(x'; r), \quad r = x - x'. \tag{1.31}$$

Then (1.28) can be rewritten in the form

$$\frac{\partial P(x,t)}{\partial t} = \int W(x-r; r) P(x-r,t) dr - P(x,t) \int W(x; -r) dr \tag{1.32}$$

At this stage two assumptions are made,

- Only small jumps occur, i.e., $W(x'; r)$ is a sharply peaked function of r but varies slowly with x'. Then there will exist some $\delta > 0$ such that

$$W(x'; r) \approx 0 \qquad for \mid r \mid > \delta \tag{1.33}$$

$$W(x' + \Delta x; r) \approx W(x'; r) \quad for \mid \Delta x \mid < \delta. \tag{1.34}$$

- The second assumption is that the solution $P(x,t)$ also varies slowly with x, making possible a Taylor expansion of the term $P(x-r,t)$ in terms of $P(x,t)$ obtaining

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= \int W(x;r)P(x,t)\, dx - \int r \frac{\partial}{\partial x} \{W(x;r)P(x,t)\}\, dr \\ &+ \frac{1}{2} \int r^2 \frac{\partial^2}{\partial x^2} \{W(x;r)P(x,t)\}\, dr - P(x,t) \int W(x;-r) dr. \end{aligned} \tag{1.35}$$

The first and fourth terms on the right hand side of Eq. (1.35) vanish, whereas the other two remaining terms are named as

$$F(x) = \int_{-\infty}^{\infty} r\, W(x;r) dr \tag{1.36}$$

$$D(x) = \int_{-\infty}^{\infty} r^2\, W(x;r) dr, \tag{1.37}$$

and they correspond to the first and second jump moments of $W(x;r)$, respectively. The first jump moment corresponds to the so called *drift term* –$F(x)$–, and the second moment to the *diffusion term* –$D(x)$ –. Then the final result is

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x)P(x,t)] \tag{1.38}$$

In conclusion, we have derived the Fokker–Planck equation starting from the master equation governing the transitions between different states from the system.

1.4.2 The Fokker–Planck equation in one dimension

For a one dimension we can write the following Fokker–Planck equation – as derived in the previous section –

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x,t)\,P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[D(x,t)\,P(x,t)\right]. \tag{1.39}$$

Here the term $F(x,t)$ is known as the drift term and it is obtained from a potential as $F(x,t) = -\frac{\partial V(x,t)}{\partial x}$, and $D(x,t)$ is the diffusion term. The stochastic process whose probability density function obeys Eq. (1.39) is equivalent to the stochastic process described by the Ito stochastic differential equation

$$\dot{x} = F(x,t) + \sqrt{D(x,t)}\,\xi(t) \tag{1.40}$$

where $\xi(t)$ is a gaussian white noise of mean zero and correlation given by $<\xi(t)\xi(t')>= \delta(t-t')$.

Defining a *probability current* $J(x,t)$ as

$$J(x,t) = F(x,t)\,P(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[D(x,t)\,P(x,t)] \tag{1.41}$$

Eq. (1.39) can rewritten in the form of a continuity equation

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial}{\partial x}J(x,t) = 0 \tag{1.42}$$

1.4.3 Boundary conditions

The Fokker–Planck equation is a second–order parabolic differential equation, and in order to find a solution we need an initial condition as well as some boundary conditions where the variable x is constrained. For a more general case, in more than one dimension, we can write

$$\partial_t P(\mathbf{x},t) = -\sum_i \frac{\partial}{\partial x_i}F(\mathbf{x},t)\,P(\mathbf{x},t) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}D(\mathbf{x},t) \tag{1.43}$$

which can also be written as a continuity equation

$$\frac{\partial P(\mathbf{x},t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i}J_i(\mathbf{x},t) = 0 \tag{1.44}$$

Eq. (1.44) has the form of a local conservation law, and so it can be rewritten in an integral form. Considering a region R with boundary $\mathbf{S}$ we have

$$\frac{\partial P(R,t)}{\partial t} = -\int_{\mathbf{S}} dS\,\hat{\mathbf{n}}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x},t) \tag{1.45}$$

where we have defined the total probability in the region R as $P(R,t) = \int_R d\mathbf{x}\, P(\mathbf{x},t)$, and $\hat{\mathbf{n}}(\mathbf{x})$ is an outward vector pointing normal to $\mathbf{S}$. Eq. (1.45) indicates that the total loss of probability in the region R is given by the surface integral of $\mathbf{J}(\mathbf{x},t)$ over the region R. The current $\mathbf{J}(\mathbf{x},t)$ also has the property that a surface integral over any surface $\mathbf{S}$ gives us the net flow of probability across that surface. Depending on the existing boundary conditions, we will impose different conditions, such as

Reflecting barrier In this case there is no flow of probability through the surface $\mathbf{S}$, which can be thought of as the particle not leaving region R. In this case it is required

$$\hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x},t) = 0,\ \forall\, \mathbf{x} \in \mathbf{S} \tag{1.46}$$

Absorbing barrier For this case when the particle reaches one of either boundaries, it is removed from the system. As a consequence, the probability of finding the particle in the boundary is strictly zero,

$$P(\mathbf{x},t) = 0, \forall \mathbf{x} \in \mathbf{S} \tag{1.47}$$

Periodic boundary conditions The process takes place in a closed interval $[\mathbf{a},\mathbf{b}]$, where the two end points are identified with each other. This implies the following set of conditions to be fulfilled

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{b}^-} P(\mathbf{x}+\mathbf{m}L,t) &= \lim_{\mathbf{x}\to\mathbf{a}^+} P(\mathbf{x}+\mathbf{m}L,t) \\ \lim_{\mathbf{x}\to\mathbf{b}^-} J(\mathbf{x}+\mathbf{m}L,t) &= \lim_{\mathbf{x}\to\mathbf{a}^+} J(\mathbf{x}+\mathbf{m}L,t). \end{aligned} \tag{1.48}$$

where the quantity $\mathbf{m}L$ accounts for a displacement in any direction equal to the periodicity of the system.

1.4.4 Stationary properties

For a homogeneous process, the drift and diffusion terms are time independent. Then, returning to the $1D$ case, in the stationary state $\frac{\partial P(x,t)}{\partial t} = 0$ and so $P(x,t) = P^s(x)$ becomes independent of time. From Eq. (1.39) we have

$$\frac{d}{dx}[F(x)P(x)] - \frac{1}{2}\frac{d^2}{dx^2}[D(x)P(x)] = 0. \tag{1.49}$$

And using Eq. (1.42) we have $\frac{dJ(x)}{dx} = 0$, whose trivial solution is given by $J(x) = J = Constant$.

If the process takes place in the interval (a,b), it must be accomplished that $J(a) = J(x) = J(b) = J$; so if one of the boundary conditions is reflecting, it means that both of them must be reflecting, and then $J = 0$.

If the boundaries are not reflecting, the condition of constant current requires them to be periodic. In that case we may use the boundary conditions given by (1.48).

1.4.4.a Zero–current case

If $J = 0$, Eq. (1.49) can be rewritten as

$$F(x)\,P^s(x) = \frac{1}{2}\frac{d}{dx}[D(x)P^s(x)] \tag{1.50}$$

with solution

$$P^s(x) = \frac{\mathcal{N}}{D(x)}\,\mathrm{e}^{2\int_a^x dx'\,\frac{F(x)}{D(x)}} \tag{1.51}$$

$\mathcal{N}$ being a normalization constant ensuring that $\int_a^b dx\,P^s(x) = 1$.

1.4.4.b Periodic boundary conditions

For the case where we have a non–zero current Eq. (1.42) can be written as

$$F(x)\,P^s(x) - \frac{1}{2}\frac{d}{dx}[D(x)\,P^s(x)] = 0 \tag{1.52}$$

In this case the current J is completely determined by the boundary conditions

$$\begin{aligned} P^s(a) &= P^s(b) && (1.53)\\ J(a) &= J(b). && (1.54) \end{aligned}$$

For calculating the stationary probability density function $P^s(x)$ we can integrate Eq. (1.52) to obtain

$$P^s(x) = P^s(a)\left[\frac{\int_a^x \frac{dx'}{\psi(x')}\frac{D(b)}{\psi(b)} + \int_x^b \frac{dx'}{\psi(x')}\frac{D(a)}{\psi(a)}}{\frac{D(x)}{\psi(x)}\int_a^b \frac{dx'}{\psi(x')}}\right] \tag{1.55}$$

and the current is determined through

$$J = \left[\frac{D(b)}{\psi(b)} - \frac{D(a)}{\psi(a)}\right]\frac{P^s(a)}{\int_a^b \frac{dx'}{\psi(x')}} \tag{1.56}$$

1.4.5 Particle current

Once the stationary probability density function (1.55) and the probability current (1.56) are obtained, the next quantity of interest is the *particle current* $<\dot{x}>$, defined as the ensemble average over the velocities. Its relation with the probability current $J(x,t)$ is

$$J(x,t) := <\dot{x}(t)\delta(x - x(t))> \tag{1.57}$$

from where we derive

$$<\dot{x}> = \int_{-\infty}^{\infty} dx J(x,t) \tag{1.58}$$

and using Eq. (1.42) can be written as

$$<\dot{x}> = \frac{d}{dt}\int_{-\infty}^{\infty} dx\,xP(x,t). \tag{1.59}$$

1.5 Information Theory

The information theory was introduced in the seminal paper by Shannon [6] in 1948. Basically this work studies certain problems of the transmission of messages through channels involving communication systems. Mainly these communication systems can be divided in three categories: discrete, continuous and mixed. By a discrete system it is meant one where the signal and the message are a sequence of discrete symbols – for example, the telegraphy. A continuous system is one where the message and the signal are both continuous, e.g., the television. The last one is the mixed system, where both discrete and continuous variables appear, for instance the pulse code-modulation (PCM) for the transmission of speech.

The case of our interest here deals with discrete systems. Basically we can distinguish three main parts: the information source, the communication channel (through where the signal is transmitted) and the receiver. Generally, a discrete channel will mean a system where a sequence of choices from a finite set of elementary symbols $\alpha_1, \ldots, \alpha_n$ can be transmitted from one point to another.

1.5.1 Discrete and ergodic sources

We can think of the information source as generating the message, symbol by symbol. It will choose successively symbols according to certain probabilities depending, in general, on preceding choices as well as the particular symbols in question.

We may define an *ergodic source* as a source that generates strings of symbols $\alpha_1, \alpha_2, \ldots$ with the same statistical properties. Thus the symbols frequencies obtained from particular sequences will, as the length of the message increase, approach definite limits independent of the particular sequence.

In some cases a message L that is not homogeneous statistically speaking, can be considered as composed of pieces of messages coming from various pure ergodic sources $L_1, L_2, L_3, \ldots$ that is

$$L = \Pi_1\, L_1 + \Pi_2\, L_2 + \Pi_3\, L_3 + \ldots \tag{1.60}$$

where Π_i corresponds to the probability of the component source L_i.

1.5.1.a Shannon Entropy

For a single source we may define the entropy as

$$H = -\sum_i p^j \log(p^j) \tag{1.61}$$

where p^j denotes the probability of emitting a given symbol α_j. This quantity was introduced by Shannon for measuring, in some sense, how much lack of information is produced by such a source. It can also be regarded as a measure of how much “choice” is involved in the selection of the symbol emitted by the source or of the uncertainty of the outcome.

The information entropy represents the average information content of a message. Some of its most interesting properties are

1. $H = 0$ if and only if all the p^i but one are zero, this one having the value unity. Thus only when we are certain of the outcome does H vanish. Otherwise H is positive.

2. For a given n, H is a maximum and equal to $\log n$ when all the p^i are equal, i.e.: $\frac{1}{n}$.

3. Any change toward equalization of the probabilities $p^1, p^2, \ldots, p^n$ increases H.

The Shannon entropy gives the minimum transfer rate – bit rate – at which a message can be transmitted without losing any information content. For instance, we can consider an information source that emits only two symbols, either 1 or 0 with probability p and $q = 1 - p$ respectively. The corresponding expression for the entropy of the source reads

$$H = -p \log p - q \log q = -p \log p - (1 - p) \log(1 - p) \tag{1.62}$$

In Fig. 1.1 we plot the entropy as a function of the probability p of emitting the symbol 1. It can be appreciated how the entropy of the message generated by the source acquires its maximum when $p = \frac{1}{2}$, corresponding to the value where both symbols have the same probability of being emitted, and therefore the uncertainty of the resulting message is maximum.

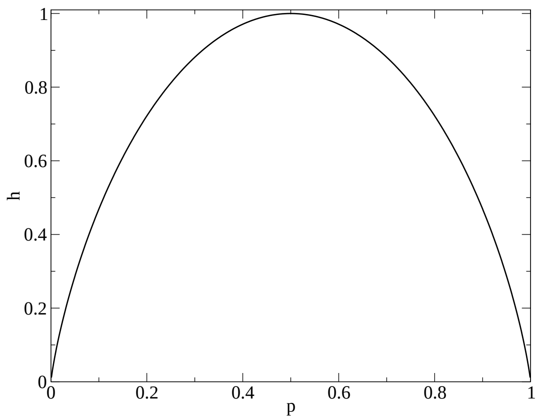

Figure 1.1: Plot of the variation of the entropy of the source when varying the probability p of emitting symbol 1.

If we now consider a source L, composed itself of a mixture of different sources L_i with probability Π_i, the resulting entropy of the system will depend on the entropy of each individual source in the following way

$$H = \sum_i \Pi_i H_i = -\sum_{i,j} \Pi_i p_i^j \log p_i^j \tag{1.63}$$

where p_i^j denotes the probability of emitting a symbol α_j by the source L_i.

1.5.1.b Entropy of a message

Given a message composed of a set of symbols $\alpha_1, \alpha_2, \ldots$, successive approximations of the actual entropy of the message can be obtained. As a first step, it can be considered that all the symbols have been emitted by the source with a fixed and independent probability. Therefore, we can measure the frequencies of all the symbols of the alphabet present in the message, estimating from them their probabilities using Eq. (1.61).

Next thing to consider are the so–called *block entropies*. We must calculate the probabilities of words constructed with symbols from the alphabet $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, and thereafter obtain their corresponding block entropies

$$H_n = -\sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1, \ldots, \alpha_n) \log[p(\alpha_1, \ldots, \alpha_n)]. \tag{1.64}$$

This quantity measures the average amount of information contained in a word of length n. From Eq. (1.64) we can then evaluate the differential entropy

$$\begin{aligned} h_n &= H_n - H_{n-1} \\ &= -\sum_{\alpha_1,\ldots,\alpha_n} p(\alpha_1, \ldots, \alpha_n) \log[p(\alpha_n|\alpha_1, \ldots, \alpha_{n-1})], \end{aligned} \tag{1.65}$$

that gives the new information of the n-th symbol if the preceding $(n-1)$ symbols are known; $p(\alpha_n|\alpha_1, \ldots, \alpha_{n-1})$ is the conditional probability for α_n being conditioned on the previous symbols $\alpha_1, \ldots, \alpha_{n-1}$. The Shannon entropy is then

$$h = \lim_{n\to\infty} h_n \tag{1.66}$$

The latter expression gives the average amount of information per symbol if all correlations are taken into account, and the limit approaches monotonically the actual value of h from above, i.e., all the h_n are upper bounds on h.

For a numerical estimation of Eq. (1.64) we must count the number of times that the word $\alpha_1, \ldots, \alpha_n$ is contained in the message, and then obtain its probability with $\frac{n_{\alpha_1,\ldots,\alpha_n}}{N}$, where N is the total length of the message.

The actual problem of evaluating the Shannon entropy in this way is that the number of possible words increases exponentially as the length of the word n increases. In order to obtain good statistical results when calculating the word probabilities we must have a sufficiently long message when evaluating the probabilities of large words[2], which in fact is a considerable inconvenient.

There exist other ways of evaluating the entropy of a message. An interesting algorithm developed by A. Lempel and J. Ziv [7] permits the calculation of the entropy of a message, and it will be explained in the next section.

1.6 Lempel and Ziv algorithm

In 1977, Abraham Lempel and Jakob Ziv created the lossless[3] compressor algorithm LZ77. This algorithm is present in programs such as `gzip`,`arj`, etc. It was later modified by Terry Welch in 1978 becoming the LZW algorithm, and this is the algorithm commonly found today.

It was originally designed to obtain the algorithmic complexity of a binary string[4] [8]. Basically it is a dictionary based or substitutional encoding/decoding algorithm, creating a dictionary during the process of encoding and decoding of a certain message.

For a useful example of how the algorithm[5] works, we will encode/decode the following binary string 10010110100111011100101, of length $n = 23$.

1.6.0.c Encoding process

First, we will partition the chain into words $B_1, B_2, ..$ of variable block length – Lempel & Ziv parsing –

$$10010110100111011100101 \tag{1.67}$$

So we obtain the following words: $B_1 = 1$, $B_2 = 0$, $B_3 = 01$, $B_4 = 011$, $B_5 = 010$, $B_6 = 0111$, $B_7 = 01110$, $B_8 = 0101$.

This words are then coded as *(prefix+newbit) =(pointer to the last occurrence, newbit)*: $(01) = (0 + 1) = (2, 1)$, $(011) = (01 + 1) = (3, 1)$, $(010) = (01 + 0) = (3, 0)$, $(0111) = (011 + 1) = (4, 1)$, $(01110) = (0111 + 0) = (6, 0)$, $(0101) = (010 + 1) = (5, 1)$. We have then the following pairs

$$(0,1) \quad (0,0) \quad (2,1) \quad (3,1) \quad (3,0) \quad (4,1) \quad (6,0) \quad (5,1) \tag{1.68}$$

---

[2] the necessary length of the message also increments exponentially with n

[3] it assures that the original information can be exactly reproduced from the compressed data

[4] *Algorithmic complexity* of a binary string is the length in bits of the shortest computer program able to reproduce the string and to stop afterward

[5] we will make use of the LZ78 algorithm, which is simpler than its original LZ77

Once the pairs for each B_j are obtained, we replace each pair *(i,s)* by the integer $I_j = 2i + s$.

$$\begin{array}{ll} (0,1) \rightarrow I_1 = 20+1 = 1 & (3,0) \rightarrow I_5 = 23+0 = 6 \\ (0,0) \rightarrow I_2 = 20+0 = 0 & (4,1) \rightarrow I_6 = 24+1 = 9 \\ (2,1) \rightarrow I_3 = 22+1 = 5 & (6,0) \rightarrow I_7 = 26+0 = 12 \\ (3,1) \rightarrow I_4 = 23+1 = 7 & (5,1) \rightarrow I_8 = 25+1 = 11 \end{array} \tag{1.69}$$

Each integer I_j is then expanded to base two, and the binary expansions are padded with zeroes on the left so that the total length of bits is $\lceil \log_2(2j) \rceil$, where the brackets $\lceil\,\rceil$ denote the upper integer value of $log_2(2j)$. We obtain in this way the strings W_j.

j	I_j	**Binary string**	$\lceil \log_2(2j) \rceil$	W_j	**Binary string**
1	1	1	$\lceil \log_2(2) \rceil = 1$	W_1	1
2	0	0	$\lceil \log_2(4) \rceil = 2$	W_2	00
3	5	101	$\lceil \log_2(6) \rceil = 3$	W_3	101
4	7	111	$\lceil \log_2(8) \rceil = 3$	W_4	111
5	6	110	$\lceil \log_2(10) \rceil = 4$	W_5	0110
6	9	1001	$\lceil \log_2(12) \rceil = 4$	W_6	1001
7	12	1100	$\lceil \log_2(14) \rceil = 4$	W_7	1100
8	11	1011	$\lceil \log_2(16) \rceil = 4$	W_8	1011

Finally we just need to concatenate the binary words W_j to obtain the encoded string: 1001011110110100111001011. Clearly, the length of the encoded string is not much shorter than the original in this case, but it must be kept in mind that the algorithm becomes *optimal* as the length of the string increases[6]

1.6.0.d Decoding process

The decoding process is much simpler than the encoding. We just need to know the size alphabet of the source that created the string. From the previous section we obtained the encoded string 1001011110110100111001011 with an alphabet equal to 2.

The first thing to do is to divide the string in blocks of size $\lceil \log_2(2j) \rceil$: $1 \cdot 00 \cdot 101 \cdot 111 \cdot 0110 \cdot 1001 \cdot 1100 \cdot 1011$; then convert these blocks into integer form :$1, 0, 5, 7, 6, 9, 12, 11$; we divide by the size alphabet, 2 in this case, and we keep the quotient q and remainder r, (q, r) : $(0, 1)$, $(0, 0)$, $(2, 1)$, $(3, 1)$, $(3, 0)$, $(4, 1)$, $(6, 0)$, $(5, 1)$.

Finally we convert these pairs into words using the same formalism than in the encoding process, and we join them to obtain the original binary string 10010110100111011100101.

[6]Because the length of the words B_j that will be substituted increases linearly with the binary string, whereas the length of the words W_j increases logarithmically.

1.6.0.e Properties of the LZ algorithm

An important property of the LZ algorithm is that it relates the compression factor with the entropy of the compressed string.

Defining the *compression factor* (CF) as the ratio between the compression length and the original length $l(S) = n$ of a string S we have

$$CF = \frac{c(S)}{l(S)} = \frac{c(S)}{n} \tag{1.70}$$

and defining the *optimality ratio* $\gamma(n)$ as the ratio between the compression factor and the entropy per character h of the source

$$\gamma(n) = \frac{CF}{h} \tag{1.71}$$

it is said that the compression is *asymptotically optimal* if $\gamma(n) \to 1$ as $n \to \infty$.

Lempel and Ziv showed that their dictionary–based algorithms $LZ77$, $LZ78$ give asymptotically optimal compression for strings generated by an ergodic stationary process, that is, as the length of the file to compress $n \to \infty$ the ratio of the length of the compressed file with n tends to the entropy per character h.

This algorithm together with the previous definitions explained above will be used in Sec. 5 for establishing a relation between Parrondo's games and information theory.

Chapter 2

The Brownian ratchet

2.1 Brownian ratchet

2.2 Smoluchowski–Feynman ratchet

Is it possible to obtain useful work out of unbiased random fluctuations? In the case of macroscopic devices we can find many ways of accomplishing this task, for example a wind–mill, the self–winding wristwratch, etc. But when dealing with the microscopic world, this case becomes more subtle. A clear example of this problem was illustrated in the conference talk by Smoluchowsky in Münster 1912 (and published as a proceedings–article in [9]) and later popularized and extended in Feynman's Lectures on Physics [10].

2.2.1 Ratchet and pawl

The ratchet and pawl consists on an axis with a paddle located at one end, and a circular saw with a ratchet–like shape on the other end, see Fig. 2.1 for details. This device is surrounded by a thermal bath at equilibrium at temperature T. If left alone, the system would perform a rotatory Brownian motion due to the collisions of the gas molecules with the paddles.

We can modify this picture by introducing a pawl in order to rectify this random fluctuations. Hence in this way rotations would be favored in one precise direction, allowing the saw–teeth to rotate clockwise – as depicted in Fig. 2.1 –, whereas it would block the saw–teeth to rotate in the counter–clock direction. So it seems that this gadget would perform a net rotation clockwise, and if a weight is added to the axis it could even perform some work lifting the weight.

Based on the previous reasoning we could conclude that the device constructed this way would constitute a *perpetuum mobile* of the second kind, therefore violating the second law of thermodynamics. However, this naive expectation is wrong. In spite of the asymmetry of the device, no preferred motion is possible. The reason is the following: due to the microscopic size of

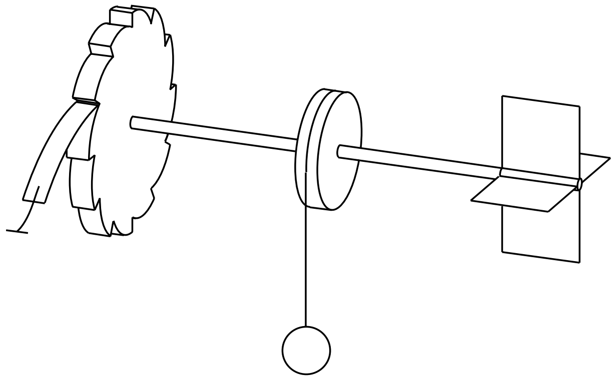

Figure 2.1: Plot of the ratchet and pawl device.

the machine, not only the paddles are subjected to the fluctuations due to the collisions with the gas particles, but also the pawl is exposed to them. These collisions of the particles with the pawl would, occasionally, lift the pawl. Then the ratchet could rotate counter–clockwise as it would not have any opposing force. As a result the ratchet and pawl device would have no preferred direction of rotation. This Smoluchowski–Feynman's ratchet and pawl device was introduced as a pedagogic example of the second law of thermodynamics.

We can modify the previous picture by considering that the gas surrounding the paddles and the gas surrounding the ratchet are not at the same temperature. In this case an equilibrium situation no longer exists. This second model was introduced by Feynman [10], and later revised by Parrondo [11].

A simplified stochastic model known as Brownian ratchet will be presented in the next section, capturing the essential features of the Smoluchowski–Feynman's ratchet and pawl device.

2.2.2 Brownian ratchet

We will consider the motion of a Brownian particle of mass m under the effect of a potential $V(x,t)$ that can be time–dependent, a friction force $-\eta\dot{x}(t)$, a force $F(t)$ exerted by an external agent and a stochastic force $\sqrt{D(x,t)}\xi(t)$, where $D(x,t) = 2\eta k_B T(x,t)$ is the noise strength or noise intensity, proportional to the temperature. Newton's equation of motion for this system can be expressed as

$$m\ddot{x}(t) + V'(x,t) = -\eta\dot{x}(t) + F(t) + \sqrt{D(x,t)}\xi(t). \tag{2.1}$$

The terms on the left hand side account for the deterministic, conservative part, whereas the terms on the right hand side account for the dissipative terms due to the interaction of the Brownian particle with its environment and the external agent. Usually the time–dependent external force $F(t)$ is split in two terms, a constant term F and a time–dependent term $y(t)$, and so it can be written as $F(t) = F + y(t)$.

The potential $V(x,t)$ used in Eq. (2.1) must fulfill the following conditions

- *Periodicity*. It must be periodic with period L, that is, $V(x,t) = V(x+L,t)$ for all x and t.
- *Asymmetry*. This asymmetry can be established in many ways, the simplest consisting on spatial asymmetry, that occurs when for any value of x there exists no Δx such that $V(-x,t) = V(x+\Delta x,t)$, in some sense this condition accounts for some kind of spatial anisotropy. A typical example of an asymmetric potential is

$$V(x,t) = V_0 \left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{4\pi x}{L}\right)\right] \cdot W(t). \tag{2.2}$$

where the function $W(t)$ represents the time dependence of the potential, if there is any.

The stochastic force or thermal noise $\xi(t)$ generally is considered to be *Gaussian white noise* of zero mean $<\xi(t)>= 0$ and correlations

$$<\xi(t)\xi(s)>= \delta(t-s) \tag{2.3}$$

For the systems we will study, the inertia term $m\ddot{x}(t)$ is negligible, and so Eq. (2.1) can be written as

$$\eta\dot{x}(t) = -V'(x(t),t) + F + y(t) + \sqrt{D(x,t)}\xi(t). \tag{2.4}$$

The latter equation can be considered as a generalized equation describing the dynamics of an overdamped Brownian particle.

2.2.2.a *Reduced probability variables*

As our interest is focused mainly on transport in periodic systems, we can introduce the *reduced probability density* and *reduced probability current* as

$$\hat{P}(x,t) := \sum_{n=-\infty}^{\infty} P(x+nL,t), \tag{2.5}$$

$$\hat{J}(x,t) := \sum_{n=-\infty}^{\infty} J(x+nL,t). \tag{2.6}$$

And from Eqs. (1.29,1.58) we get

$$\hat{P}(x+L,t) = \hat{P}(x,t), \tag{2.7}$$

$$\int_0^L dx\hat{P}(x,t) = 1, \tag{2.8}$$

$$<\dot{x}> = \int_0^L dx\hat{J}(x,t) \tag{2.9}$$

As $P(x,t)$ is solution of the Fokker–Planck equation (1.39), it follows from the periodic condition introduced above, $V(x,t) = V(x+L,t)$, that $P(x+nL,t)$ is also solution for any integer value n. Introducing expressions (2.5) and (2.6) into the Fokker–Planck equation (1.39), it can be rewritten as a continuity equation

$$\frac{\partial \hat{P}(x,t)}{\partial t} + \frac{\partial \hat{J}(x,t)}{\partial x} = 0 \tag{2.10}$$

$$\hat{J}(x,t) = F(x,t)\,\hat{P}(x,t) - \frac{1}{2}\frac{\partial}{\partial x}[D(x,t)\,\hat{P}(x,t)] \tag{2.11}$$

Therefore, in order to obtain the particle current is sufficient to solve the Fokker–Planck equation (1.39) with periodic boundary conditions, together with the initial conditions. Besides, operating with $\int_{x_0}^{x_0+L} dx\, x \ldots$ on both sides of Eq. (2.10) we obtain

$$< \dot{x} >= \frac{d}{dt}\left[\int_{x_0}^{x_0+L} dx\, x\hat{P}(x,t)\right] + L\hat{J}(x_0,t), \tag{2.12}$$

where x_0 denotes the initial position of the particle. Essentially, we distinguish two contributions to the particle current: the first term on the right hand side of Eq. (2.12) accounts for the motion of the center of mass, and the second term is L times the reduced probability current $\hat{J}(x_0,t)$ measured at the reference point x_0. If the reduced dynamics reach a steady state, characterized by $\frac{d\hat{P}(x,t)}{dt} = 0$, then the reduced probability current $\hat{J}(x_0,t) = \hat{J}^{st}$ becomes independent of x_0 and t, and the particle current becomes

$$< \dot{x} >= L\hat{J}^{st} \tag{2.13}$$

The particle current can also be calculated through the time-averaged velocity of a single realization $x(t)$ of the stochastic process described by Eq. (2.1), i.e.

$$< \dot{x} >= \lim_{t\to\infty} \frac{x(t)}{t} \tag{2.14}$$

independent of the initial condition $x(0)$.

2.2.2.b Ratchet effect

The so–called ratchet effect takes place when a given set of conditions are accomplished.

- We must have a spatially periodic system.

- Second, there must be some asymmetry in the system, for example spatial asymmetry.

- Last but not least, our system must be out of equilibrium.

Depending on the way these conditions are accomplished, we will distinguish different types of ratchets.

2.2.3 Classes of ratchets

There are two main groups of ratchets that can be derived from Eq. (2.4). The first group considers those systems where the term $y(t) = 0$, these are the *pulsating ratchets*; the second group considers those where $W(t) = 0$, and they are known as *tilting ratchets*.

2.2.3.a Pulsating ratchets

Within this group, we can also distinguish the following types of ratchets

Fluctuating potential ratchets They are obtained when the time dependence of the potential $W(t)$ is additive, that is $V(x,t) = V(x)[1 + W(t)]$. This group contains as a special case the *on–off* ratchet, also known as *flashing ratchet*, consisting on $W(t)$ having only two possible values: $+1$ (ON state) and -1 (OFF state).

Travelling potential ratchets They have potentials of the form $V(x,t) = V(x - W(t))$.

2.2.3.b Tilting ratchets

This group is characterized by $W(t) = 0$, and so the potential is time–independent $V(x,t) = V(x)$. Within this group we will distinguish three types of ratchets depending on the time dependence of $y(t)$ in Eq. (2.4)

Fluctuating force ratchets They are obtained when $y(t)$ is a stochastic process.

Rocking ratchet It is obtained when $y(t)$ is periodic.

Asymmetrically tilting ratchet We explained before that one essential ingredient for the ratchet effect was the existence of an asymmetry in the system. If our potential $V(x)$ is symmetric the source of asymmetry can be introduced through the term $y(t)$, imposing it to be non–symmetric.

From all these different kinds of ratchets, we will now focus on the *flashing ratchet* model and analyze it a little closer.

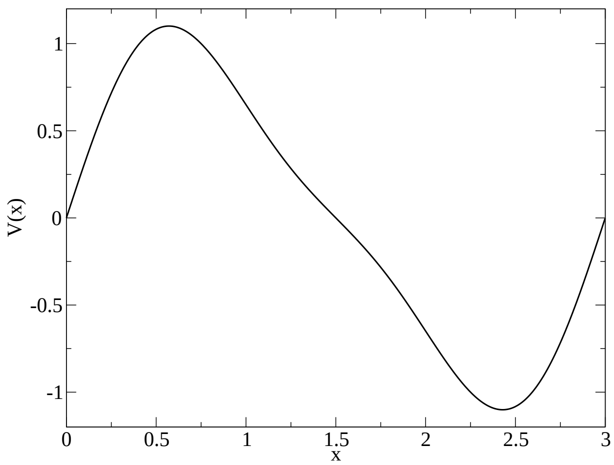

Figure 2.2: Plot of the asymmetric potential (2.2) with the parameters $L = 3$ and $V_0 = 1$.

2.2.4 The flashing ratchet

This system is characterized by a Brownian particle subjected to a potential that is switched on and off either periodically or stochastically – depending on the time dependence of the function $W(t)$. This scheme was introduced by Ajdari and Prost [12]. The model can be described through the equation

$$\eta \dot{x}(t) = -V'(x(t))\,[1 + W(t)] + \sqrt{D(x,t)}\xi(t) \tag{2.15}$$

where $V(x)$ is a spatially periodic and asymmetric potential, and usually a potential such as the one in Eq. (2.2) is used – in Fig. 2.2 we can see a plot of the potential for the parameters $L = 3$ and $V_0 = 1$. The function $W(t)$ is restricted to two values ± 1, switching on and off the potential, and $D(x,t) = 2\eta k_B T(x,t)$ is the noise strength.

The *ratchet mechanism* can be explained as follows. Imagine a landscape with a couple of Brownian particles moving freely. At a certain instant, a ratchet–like potential is switched on: $W(t) = 1$, and the particles (assuming the thermal energy $k_B T$ to be much smaller than the potential amplitude) are eventually confined to one of the potential wells located at x_0, see Fig. 2.3. When the potential is switched off: $W(t) = -1$, the particles are subjected only to the thermal noise $\xi(t)$ and start to diffuse.

If we let the particles diffuse for a large enough time interval, a small fraction of them will reach the vicinity of the next potential well [1] at $x_0 + L$. Repeating this cycle many times, a net current of particles is obtained $\langle \dot{x} \rangle > 0$. In Fig. 2.4 – left panel – we see the plot of the net current vs the natural logarithm of the flip rate[2] γ for a single Brownian particle. It can be

[1] due to the asymmetry in the potential of Fig. 2.3, is more likely that the particles will reach the potential well located on the right than the one on the left, as the distance is shorter in the former case.

[2] The flip rate γ accounts for the probability of switching the potential on or off per time unit.

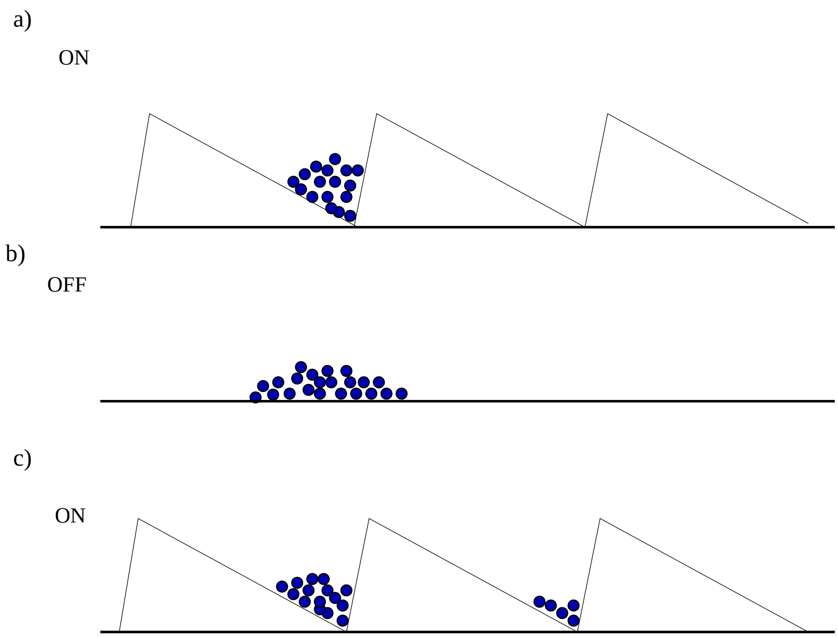

Figure 2.3: Three different stages of the *on-off* cycle for the case of the flashing ratchet. In case a) the potential is **on** and the particles get trapped in a potential well; in stage b) the potential is **off** and the particles spread due to diffusion; finally, in stage c) some particles have diffused up to the vicinity of the next potential well, and so when the potential is **on** again, there are a certain number of particles located in the next potential well. The flux of particles due to the asymmetry in the potential in this case is to the right.

clearly identified the existence of an optimal switching rate that produces the maximum current.

We can modify this picture introducing an external force F acting against the particle. Even with this opposing force applied on the particle, the *ratchet effect* is still present for sufficiently small values of F. We see in Fig. 2.4 – right panel – how the current is positive and different from zero up to a value of the applied force $F = F_0$, being F_0 the so–called *stopping force*. It is worth noting that for this case, the particle is doing work against the external force applied.

2.3 A discrete–time flashing ratchet : Parrondo's games

In many physical and biological systems, combining processes may lead to counter-intuitive dynamics. For example, in control theory, the combination of two unstable systems can cause them to become stable [13]. In the theory of granular flow, drift can occur in a counter-intuitive direction [14, 15]. Also the switching between two transient diffusion processes in random media can form a positive recurrent process [16]. Other interesting phenomena where physical processes drift in a counter-intuitive direction can be found (see for

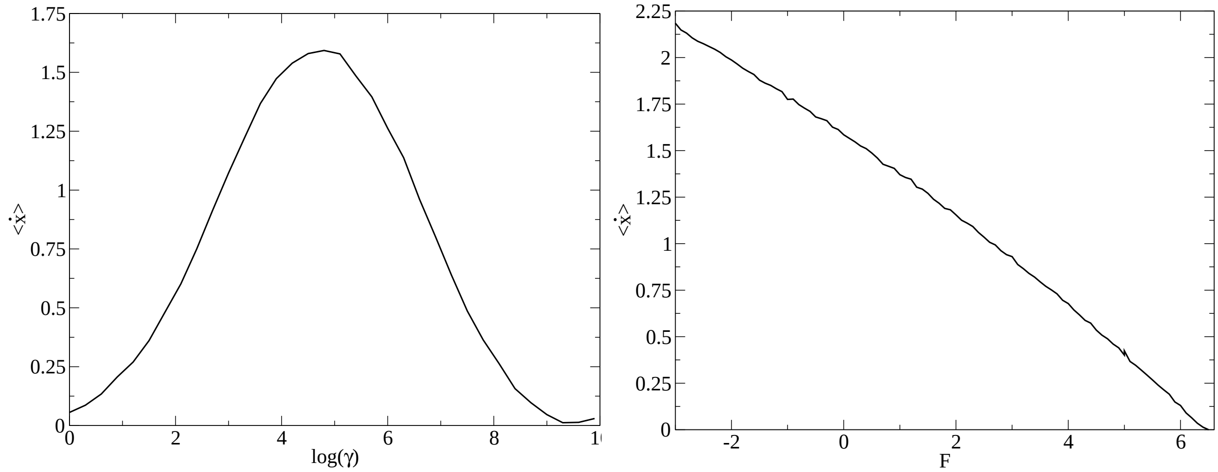

Figure 2.4: Left panel: Plot of the average particle current versus the logarithm of the flip rate. Right panel: Plot of the average particle current versus the applied external force F.

example [12, 17, 18, 19, 20].

The Parrondo's paradox [21, 22, 23] is based on the combination of two negatively biased games – losing games – which when combined give rise to a positively biased game, that is, we obtain a winning game. This paradox is a translation of the physical model of the Brownian ratchet into game-theoretic terms. These games were first devised in 1996 by the Spanish physicist Juan M. R. Parrondo, who presented them in unpublished form in Torino, Italy [24]. They served as a pedagogical illustration of the flashing ratchet, where directed motion is obtained from the random or periodic alternation of two relaxation potentials acting on a Brownian particle, none of which individually produce any net flux.

These games have attracted much interest in other fields, for example quantum information theory [25, 26, 27, 28], control theory [29, 30], Ising systems [31], pattern formation [32, 33, 34], stochastic resonance [35], random walks and diffusions [36, 37, 38, 39, 40], economics [41], molecular motors in biology [42, 43] and biogenesis [44]. They have also been considered as quasi-birth-death processes [45] and lattice gas automata [46].

Parrondo's two original games are as follows. Game A is a simple tossing coin game, where a player increases (decreases) his capital in one unit if heads (tails) show up. The probability of winning is denoted by p and the probability of losing is $1 - p$.

Game B is a capital dependent game, where the probability of winning depends upon the actual capital of the player, modulo a given integer M. Therefore if the capital is i the probability of winning p_i is taken from the set $\{p_0, p_1, \ldots, p_{M-1}\}$ as $p_i = p_{i \bmod M}$. In the original version of game B, the number M is set equal to three and the probability of winning can take only two values, p_1, p_2, according to whether the capital of the player is a multiple of three or not, respectively. The numerical values corresponding to the original Parrondo's games [21] are:

$$\begin{cases} p = & \frac{1}{2} - \epsilon, \\ p_1 = & \frac{1}{10} - \epsilon, \\ p_2 = & \frac{3}{4} - \epsilon, \end{cases} \tag{2.16}$$

where ϵ is a small biasing parameter introduced to control the three probabilities. For a value of ϵ equal to zero, both games are fair games, whereas if ϵ is small and positive both games are losing. In both cases, the combined game results in a winning game.

Intuitively, we could think of a potential representing games A and B – for the simplest case of $\epsilon = 0$ – through the following reasoning: the winning and losing probabilities for game A are independent of the site and equal to $\frac{1}{2}$. Therefore it would be equally likely a forward or a backward transition. Then the barriers of the potential that one would find would be of equal height, as depicted in Fig. 2.5a.

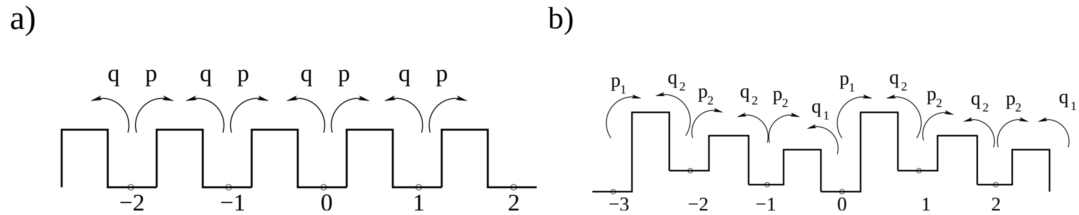

Figure 2.5: Left panel: schematic potential related to game A. Right panel: Schematic potential related to game B.

For the case of game B, we must take into account the dependence of the winning probabilities with the current capital of the player. When the capital is multiple of three the winning probability is very small, i.e. $p_1 = \frac{1}{10}$, this translates into a high potential barrier between this site and the one located on the right. However, for the sites that correspond to the capital of the player not being multiple of three the winning probability is rather high, $p_2 = \frac{3}{4}$, and so the potential barriers must be placed in a way that it is favored a forward transition than a backward transition. One possible way of depicting the potential is found in Fig. 2.5b.

In Fig. 2.6 we can see a plot of the average gain for a player that alternates between games A and B, either periodically or stochastically. For both kind of alternations, it can be seen that the resulting game is a winning game. When the player alternates periodically between games A and B, it follows a fixed sequence of plays for game A and B. For example, the sequence $[3, 2]$ implies that the player will play game A three times in a row, followed by game B two times. The case of random mixing between games is obtained as follows: the player will decide on each time step if he plays game A or B with probability γ and $1 - \gamma$ respectively[3]. In Fig. 2.6 we have plotted the random case for a

[3] From now on the randomized game will be referred to as game AB.

value of $\gamma = \frac{1}{2}$.

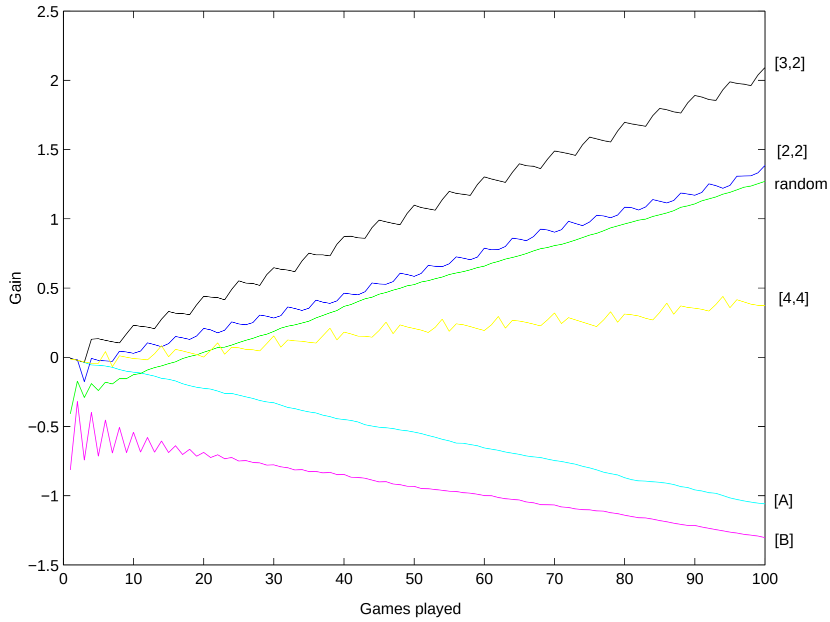

Figure 2.6: Plot of the average gain over 100 plays of either game A or B alone – both of them losing games –, although any combination of them, either periodic or stochastic results in a winning game. The notation $[a, b]$ indicates, for the periodic case, that we play a times game A, followed by b times game B. For the random case games A and B are alternated with a probability $\gamma = \frac{1}{2}$.

2.3.1 Theoretical analysis of the games

One way of analyzing these games is through discrete–time Markov chains [54]. Each value of capital is represented by a state, and the transition probabilities between these states are determined by the rules of the games. In this section we will analyze the games A, B and the randomized game AB with this technique in order to obtain the stationary probability distributions.

2.3.1.a Analysis of game B

The discrete–time Markov chain that represents game B is shown in Fig. 2.7, where the states 0, 1 and 2 represent the value of capital modulo three. The transition probabilities between these three states will be given by the winning and losing probabilities for each state.

The set of equations that describe the evolution with the number of games played –denoted by n– of the probabilities Π_0^B, Π_1^B and Π_2^B of finding the capital of the player in the state 0, 1 and 2 respectively are

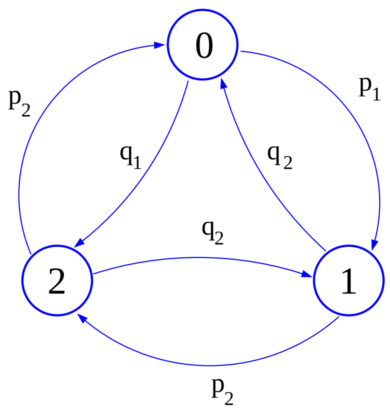

Figure 2.7: Diagram representing the different states of game B, as well as the allowed transitions between these states.

$$
\begin{aligned}
\Pi_0^B(n+1) &= p_2\,\Pi_2^B(n) + (1-p_2)\,\Pi_1^B(n) & (2.17)\\
\Pi_1^B(n+1) &= p_1\,\Pi_0^B(n) + (1-p_2)\,\Pi_2^B(n) & (2.18)\\
\Pi_2^B(n+1) &= p_2\,\Pi_1^B(n) + (1-p_1)\,\Pi_0^B(n). & (2.19)
\end{aligned}
$$

Defining the vector $\Pi^B(n) = \left[\Pi_0^B(n), \Pi_1^B(n), \Pi_2^B(n)\right]^T$ we can rewrite the previous set of equations in a matrix form as $\Pi^B(n+1) = \mathbb{P}_B \Pi^B(n)$, where we have also defined a transition matrix for game B as

$$
\mathbb{P}_B = \begin{bmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{bmatrix} \tag{2.20}
$$

Our objective is to obtain the stationary probabilities, that occurs when the distribution of capital in the states 0, 1 and 2 does not change from one game to the next. This implies that the distribution of probabilities is independent of the number of games played n and invariant under the action of the matrix $\mathbb{P}_B$, i.e., $\Pi^B = \mathbb{P}_B\Pi^B$. So we have to solve the equation $(\mathbb{I} - \mathbb{P}_B)\Pi^B = 0$. The solution is

$$
\Pi^B = \frac{1}{D}\begin{bmatrix} 1-p_2+p_2^2 \\ 1-p_2+p_1p_2 \\ 1-p_1+p_1p_2 \end{bmatrix}, \tag{2.21}
$$

where $D = 3 - p_1 - 2p_2 + 2p_1p_2 + p_2^2$ is a normalization constant. Introducing the probabilities for game B described in (2.16) for $\epsilon = 0$ we obtain

$$
\Pi^B = \frac{1}{13}\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}, \tag{2.22}
$$

2.3.1.b Analysis for game A

For the simplest case of game A we can make use of the previous result obtained for game B, as we need only to substitute the winning probabilities p_1 and p_2 by p. The result for the stationary probabilities Π^A obtained for $\epsilon = 0$ is

$$\Pi^A = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \qquad (2.23)$$

A result that is in some sense logical, because due to the fact that all the transition probabilities between states are the same, all the states will have the same stationary probability.

2.3.1.c Analysis for the randomized game

Recalling that the randomized game is based on the combination of games A and B with probability γ and $1-\gamma$ respectively, we can define an equivalent set of probabilities p'_1, p'_2 that define this combined game denoted by AB. This probabilities are

$$p'_1 = \gamma\, p + (1-\gamma)\, p_1 \qquad (2.24)$$
$$p'_2 = \gamma\, p + (1-\gamma)\, p_2. \qquad (2.25)$$

And so we can also make use of the results obtained in Sec. 2.3.1.a with the previous set of probabilities for obtaining the stationary probabilities of the randomized game AB. For the case of $\epsilon = 0$ and a mixing probability $\gamma = \frac{1}{2}$ we obtain

$$\Pi^{AB} = \frac{1}{709} \begin{bmatrix} 245 \\ 180 \\ 284 \end{bmatrix}. \qquad (2.26)$$

2.3.1.d Average winning probabilities

There are different ways of obtaining the average winning probabilities for these games, or equivalently, the conditions under which the games are losing, fair or winning. One of them uses the stationary probability distribution obtained in the previous sections for games A, B and the randomized game AB. The average winning probability p_{win} over all the states is then defined as

$$p_{win} = \sum_{i}^{M-1} p_i\, \Pi_i. \qquad (2.27)$$

Then, a game will be fair on average if $p_{win} = \frac{1}{2}$, losing if $p_{win} < \frac{1}{2}$ and winning if $p_{win} > \frac{1}{2}$. Substituting the set of winning probabilities (2.16) for $\epsilon = 0$ and the stationary probabilities for games A, B and AB given by Eqs. (2.22),(2.23) and 2.26 respectively, we obtain

$$p^A_{win} = \frac{1}{2} \quad (2.28)$$

$$p^B_{win} = \frac{1}{2} \quad (2.29)$$

$$p^{AB}_{win} = 0.5144. \quad (2.30)$$

This reflects what we has been previously presented, where games A and B are fair and the combined game AB is winning. For a more general case the following set of conditions must be fulfilled in order to reproduce the same effect

$$\frac{1-p}{p} > 1 \quad (2.31)$$

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1 \quad (2.32)$$

$$\frac{(1-p'_1)(1-p'_2)^2}{p'_1 p'^2_2} < 1. \quad (2.33)$$

2.3.1.e Rates of winning

With the stationary probabilities obtained for the games it is possible to find the rate of winning as a function of the number of games played, $r(n)$. The rate of winning can be obtained by subtracting the probability of losing from the probability of winning. Thus, we have

$$\frac{d\langle X_n\rangle}{dn} \equiv r = \sum_{i=0}^{M-1} 2\,\Pi_i\,p_i - 1 \quad (2.34)$$

For the simplest case of game A, the rate of winning is $r_A = 2p - 1$. For game B the corresponding rate of winning is $r_B = 2\,p_2 - 1 + 2\,\Pi_0\,(p_1 - p_2)$. Substituting the set of probabilities (2.16) with $\epsilon \neq 0$ we obtain

$$r_A = -2\,\epsilon \quad (2.35)$$

$$r_B = -1.74\,\epsilon + 0.119\epsilon^2 + \mathcal{O}(\epsilon^3) \quad (2.36)$$

$$r_{AB} = -1.74\epsilon + 0.119\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (2.37)$$

It can be checked that for values of small and positive values of ϵ both rates of winning of games A and B are negative, whereas the rate of winning for the randomized game AB is positive.

2.4 Other classes of Parrondo's games

Although the original game B was based on a modulo rule, there are other versions of Parrondo's games where this rule has been replaced by a different rule. For example, in [47] a rule based on the previous history of the player is used; also combinations of two history dependent games can be found in [48].

Effects of cooperation between players in Parrondo's games have been considered by Toral [50], where the probabilities of game B depend on the actual state of the neighbors of the player. A redistribution of capital between the players has been considered [51]. Other variations of collective games have recently appeared [52, 53].

Regarding the way the games are alternated, we find chaotic alternation instead of random alternation in [49]. In the following sections we will briefly present some of these games.

2.4.1 History dependent games

These games combine two games: game A is identical to the capital dependent games, for every state there will be a winning probability of p, and a losing probability of $1-p$. However, instead of game B a new game B' is defined, where the probabilities depend on the two previous results of winnings and losings. If the subscript $t-1$ refers to the previous game and $t-2$ to the game prior to that, depending on the previous results we will make use of different coins, leading to different winning probabilities in each case,

time $t-2$	time $t-1$	Winning probability
Loss	Loss	p_1
Loss	Win	p_2
Win	Loss	p_3
Win	Win	p_4

(2.38)

These probabilities can be parameterized using the following set of probabilities,

$$p = \frac{1}{2} - \epsilon \tag{2.39}$$

$$p_1 = \frac{9}{10} - \epsilon \tag{2.40}$$

$$p_2 = p_3 = \frac{1}{4} - \epsilon \tag{2.41}$$

$$p_4 = \frac{7}{10} - \epsilon. \tag{2.42}$$

These values for the probabilities reproduce the same effect than in the capital dependent games, i.e. for $\epsilon = 0$ both games are fair and for $\epsilon > 0$ they are losing; however, any sort of combination between both gives rise to a winning game, see Fig. 2.8.

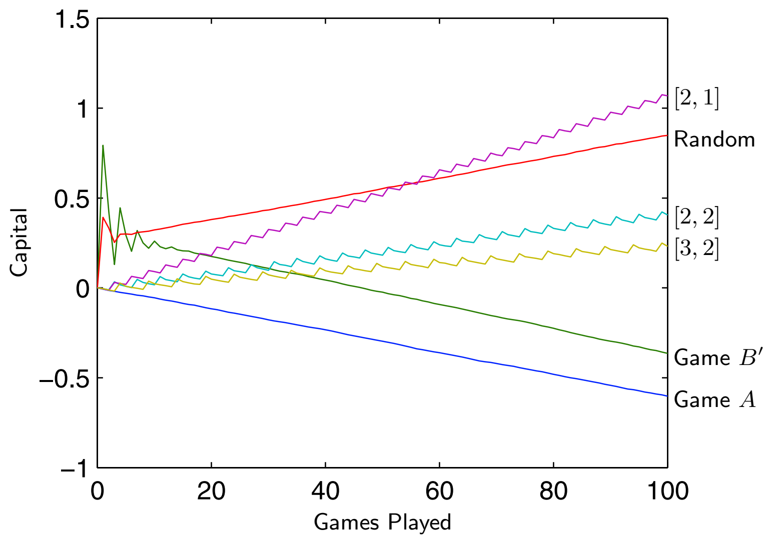

Figure 2.8: Plot of the average gain of a single player versus the number of plays for Parrondo's history dependent games A and B, as well as a periodic and a random combination of them. Simulations were performed using the probabilities defined in (2.39) with $\epsilon = 0.003$.

2.4.1.a Analysis of the games

We can also analyze these games in terms of discrete–time Markov chains. Game A is exactly the same as in the capital dependent games, so we need only to analyze the new game B'. For this game we will distinguish four different states, each one corresponding to a different history of wins and losses of the player, see Fig. 2.9, and the transition probabilities will be given by the winning probabilities $\{p_1, p_2, p_3, p_4\}$ for every state.

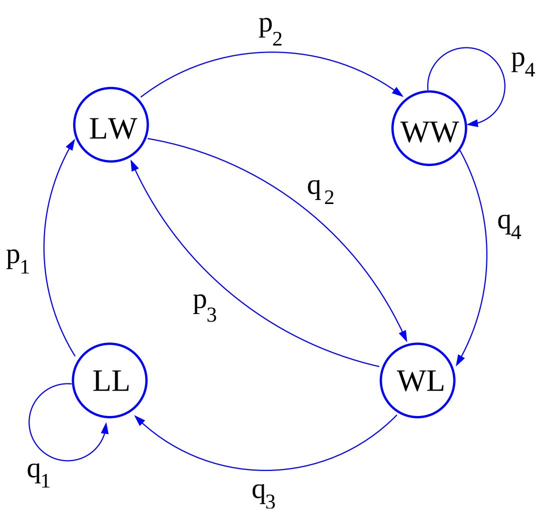

Figure 2.9: Diagram representing the different states of the history dependent game B, as well as the allowed transitions between these states.

The corresponding transition matrix for game B' is

$$\mathbb{P}_{B'} = \begin{bmatrix} 1-p_1 & 0 & 1-p_3 & 0 \\ p_1 & 0 & p_3 & 0 \\ 0 & 1-p_2 & 0 & 1-p_4 \\ 0 & p_2 & 0 & p_4 \end{bmatrix} \tag{2.43}$$

with the rows and columns representing the four states LL, LW, WL and WW, labelling from the top left corner.

When randomly mixing the games, the probabilities are given by $p_i' = \gamma\, p + (1-\gamma)\, p_i$ for $i = 1, \ldots, 4$ and γ is the mixing probability.

The corresponding stationary probabilities Π_0, Π_1, Π_2, Π_3, and Π_4 of finding the player in any of the four states are

$$\Pi^{B'} = \frac{1}{D'} \begin{bmatrix} (1-p_3)(1-p_4) \\ p_1(1-p_4) \\ p_1(1-p_4) \\ p_1 p_2 \end{bmatrix}. \tag{2.44}$$

where $D' = p_1p_2 + (1+2p_1-p_3)(1-p_4)$. Using the probabilities for game B' defined in (2.39) with $\epsilon = 0$ we obtain

$$\Pi^{B'} = \frac{1}{22} \begin{bmatrix} 5 \\ 6 \\ 6 \\ 5 \end{bmatrix}. \tag{2.45}$$

Then the average winning probability (2.27) will be $p_{win}^{B'} = \frac{5}{22} \cdot \frac{9}{10} + 2\left(\frac{3}{22} \cdot \frac{1}{4}\right) + \frac{5}{22} \cdot \frac{7}{10} = \frac{1}{2}$

Finally, for a more general case the following set of conditions needs to be fulfilled in order to reproduce the same effect

$$\begin{aligned} \frac{1-p}{p} &> 1, \\ \frac{(1-p_3)(1-p_4)}{p_1p_2} &> 1, \\ \frac{(1-p_3')(1-p_4')}{p_1'p_2'} &< 1. \end{aligned} \tag{2.46}$$

2.4.2 Collective games

As already mentioned, the collective games were introduced by Toral [50],[51]. In one version of these collective games [50] – *cooperative games* – one of the games depends on the actual state of the neighbors of a given player i; the other version of the collective games [51] – *capital redistribution*–, consist in a redistribution of capital between a given set of players. We will now briefly explain both games.

2.4.2.a Cooperative games

It can be described as follows: a group of N players are arranged in a circle so each that player has two neighbors. Game A remains unchanged, thus does not have any dependencies; game B depends on the state of the neighbors to the left and right of a player. This gives the possible states as $\{LL, LW, WL, WW\}$, where each pair is the previous state of the left and right neighbor respectively. For each of these states, a different coin will be used by a randomly selected player i according to the following table

player $i-1$	player $i+1$	Winning probability
Loser	Loser	p_1
Loser	Winner	p_2
Winner	Loser	p_3
Winner	Winner	p_4

(2.47)

The games are classified according to the behavior of the total capital $C(t) = \sum_i C_i(t)$. Therefore, a winning game is one for which the average value of the total capital $C(t)$ increases with time, and similarly for losing and fair games. In Fig. 2.10 we can see the average gain per player $\frac{\langle C(t)\rangle}{N}$ for a set of probabilities given by $p = 0.5$, $p_1 = 1$, $p_2 = p_3 = 0.16$, and $p_4 = 0.7$. We can see how the same effect as in the previous games is again reproduced: playing game A or B reports no winnings on average, whereas when alternating between both the average capital per player increases with time.

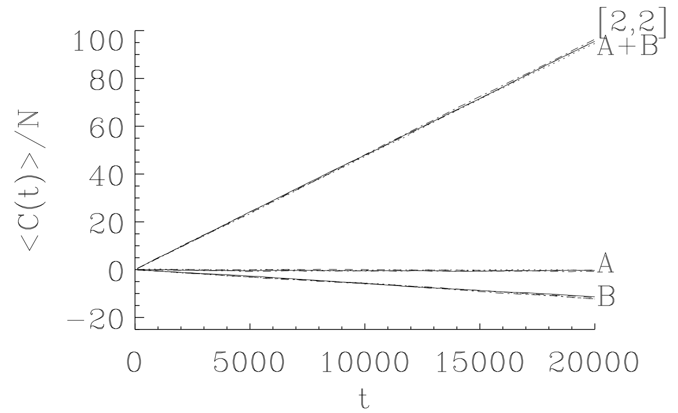

Figure 2.10: Average capital per player, $\frac{\langle C(t)\rangle}{N}$, versus time, t, measured in units of games per player. The probabilities defining the games are: $p = 0.5$, $p_1 = 1$, $p_2 = p_3 = 0.16$, $p_4 = 0.7$. These results show that game A is fair, game B is a losing game, but that when games A are B are played in random succession (game AB) or in the [2, 2] alternation $AABBAABB\ldots$, the result is a winning game. We show in this graph results for $N = 50, 100, 200$ players.

2.4.2.b Capital redistribution between players

The novelty of these games is that game A has been substituted by different versions A' and A''. Game A' consists in player i giving away one unit of capital to a randomly selected player; for game A'', instead of giving it to a random player, player i gives one unit of capital to one of its neighbors with a probability that is proportional to the capital difference, i.e. the probability of player i of giving one unit of capital to player $i-1$ is $p(i \to i-1) \propto max[C_i, C_{i\pm 1}, 0]$ with $p(i \to i-1) + p(i \to i+1) = 1$ and where C_i denotes the current capital of player i.

The mechanism of plays can be described as follows: we have a set of N players, and each time step a random player i is chosen for playing. In one version of these collective games, he will choose to player either game A' or the capital dependent game B; another version involves choosing between game A' and the history dependent game B', already explained in a previous section. A final version includes game A'' and the capital dependent game B.

In Fig. 2.11 the evolution of the average capital per player is plotted versus time for the three versions explained previously. In all the cases, the same effect is reproduced, i.e., the combination of any version of game A with any other of game B turns to be a winning game. This result emphasizes the fact that it is better for an individual player to redistribute part of its capital between other players.

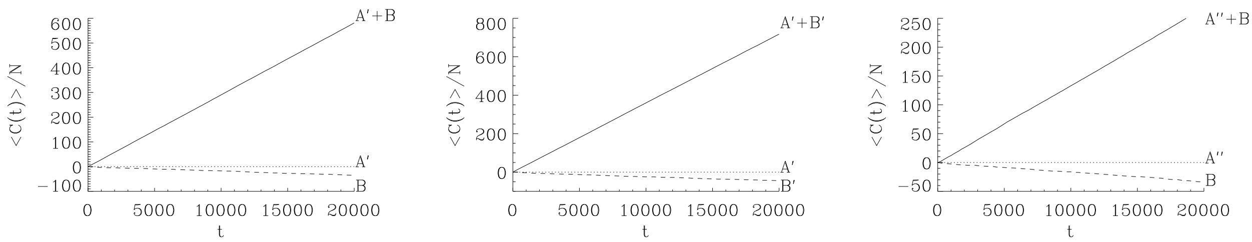

Figure 2.11: Average gain per player for the collecitve Parrondo's games based on the redistribution of capital between the players. Left panel: combination of new game A' with the original game B. Central panel: combination of new game A' with the history dependent game B'. Right panel: combination of new game A'' with the original game B.

Chapter 3

Parrondo's games with self–transition

3.1 Parrondo's games with self–transition

Games A and B appearing in the Parrondo's paradox can be thought of as diffusion processes under the action of a external potential. However, they do not have the more general form of a natural diffusion process, because the capital will always change with every game played, whereas in a general diffusion process the particle can either move up or down or remain in the same position at a given time. In this section we present a new version of Parrondo's games, where a new transition probability is taken into account. We introduce a *self-transition* probability, that is, the capital of the player now can remain the same after a game played with a probability r_i, taken from the set $\{r_0, r_1, \ldots, r_{M-1}\}$ as $r_i = r_{i \bmod M}$. Again, for simplicity, we will only consider the case of $M = 3$ with just two possible self-transition probabilities, r_1, r_2, depending only on the capital being a multiple of three or not.

As we will show, the significance of this new version is a natural evolution of Parrondo's games, which can now be rigorously derived from the Fokker-Planck equation, based on a physical flashing ratchet model.

3.2 Analysis of the new Parrondo games with self-transitions

3.2.1 Game A

We start with the new game A, where the probability of winning is p, the probability of remaining with the same capital will be denoted as r, and we lose with probability $q = 1 - r - p$.

Following the same reasoning as [23] we will calculate the probability f_j that our capital reaches zero in a finite number of plays, supposing that initially we have a given capital of j units. From Markov chain analysis [54] we have that

- $f_j = 1$ for all $j \geq 0$, and so the game is either fair or losing; or
- $f_j < 1$ for all $j > 0$, in which case the game can be winning because there is a certain probability that our capital can grow indefinitely.

We are looking for the set of numbers $\{f_j\}$ that correspond to the minimal non-negative solution of the equation

$$f_j = p \cdot f_{j+1} + r \cdot f_j + q \cdot f_{j-1}\,, \quad j \geq 1 \tag{3.1}$$

with the boundary condition

$$f_0 = 1\,. \tag{3.2}$$

With a subtle rearrangement, (3.1) can be put in the following form

$$f_j = \frac{p}{1-r} \cdot f_{j+1} + \frac{q}{1-r} \cdot f_{j-1}\,. \tag{3.3}$$

Whose solution, for the initial condition (3.2), is $f_j = A \cdot [(\frac{1-p-r}{p})^j - 1] + 1$, where A is a constant. For the minimal non-negative solution we obtain

$$f_j = \min\left[1, \left(\frac{1-p-r}{p}\right)^j\right]. \tag{3.4}$$

So we can see that the new game A is a winning game for

$$\frac{1-p-r}{p} < 1, \tag{3.5}$$

is a losing game for

$$\frac{1-p-r}{p} > 1, \tag{3.6}$$

and is a fair game for

$$\frac{1-p-r}{p} = 1. \tag{3.7}$$

3.2.2 Game B

We now analyze the new game B. Like game A, we have introduced the probabilities of a self-transition in each state, that is, if the capital is a multiple of three we have a probability r_1 of remaining in the same state, whereas if the capital is not a multiple of three then the probability is r_2. The rest of the probabilities will follow the same notation as in the original game B, so we have the following scheme

$$\begin{cases} \text{mod}(\text{capital}, 3) = 0 \rightarrow p_1, r_1, q_1 \\ \\ \text{mod}(\text{capital}, 3) \neq 0 \rightarrow p_2, r_2, q_2. \end{cases} \tag{3.8}$$

As in the case of game A, we will follow similar reasoning as [23] but for game B. Let g_j be the probability that the capital will reach the zeroth state in a finite number of plays, supposing an initial capital of j units. Again, from Markov chain theory we have

- $g_j = 1$ for all $j \geq 0$, so game B is either fair or losing; or
- $g_j < 1$ for all $j > 0$, in which case game B can be winning because there is a certain probability for the capital to grow indefinitely.

For $j \geq 1$, the following set of recurrence equations must be solved:

$$\begin{array}{rcll} g_{3j} &=& p_1 \cdot g_{3j+1} + r_1 \cdot g_{3j} + (1 - p_1 - r_1) \cdot g_{3j-1}, & j \geq 1 \\ g_{3j+1} &=& p_2 \cdot g_{3j+2} + r_2 \cdot g_{3j+1} + (1 - p_2 - r_2) \cdot g_{3j}, & j \geq 0 \\ g_{3j+2} &=& p_2 \cdot g_{3j+3} + r_2 \cdot g_{3j+2} + (1 - p_2 - r_2) \cdot g_{3j+1}, & j \geq 0\,. \end{array} \tag{3.9}$$

As in game A, we are looking for the set of numbers $\{g_j\}$ that correspond to the minimal non-negative solution. Eliminating terms g_{3j-1}, g_{3j+1} and g_{3j+2} from (3.9) we get

$$[p_1 p_2^2 + (1-p_1-r_1)(1-p_2-r_2)^2] \cdot g_{3j} = p_1 p_2^2 \cdot g_{3j+3} + (1-p_1-r_1)(1-p_2-r_2)^2 \cdot g_{3j-3}\,. \tag{3.10}$$

Considering the same boundary condition as in game A, $g_0 = 1$, the last equation has a general solution of the form $g_{3j} = B \cdot \left[\left(\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1 p_2^2} \right)^j - 1 \right] + 1$, where B is a constant. For the minimal non-negative solution we obtain

$$g_{3j} = \min \left[1, \left(\frac{(1 - p_1 - r_1)(1 - p_2 - r_2)^2}{p_1 p_2^2} \right)^j \right]. \tag{3.11}$$

It can be verified that the same solution (3.11) will be obtained solving (3.9) for g_{3j+1} and g_{3j+2}, leading all them to the same condition for the probabilities of the games.

As with game A, game B will be winning if

$$\frac{(1 - p_1 - r_1)(1 - p_2 - r_2)^2}{p_1 p_2^2} < 1, \tag{3.12}$$

losing if
$$\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1 p_2^2} > 1, \tag{3.13}$$

and fair if
$$\frac{(1-p_1-r_1)(1-p_2-r_2)^2}{p_1 p_2^2} = 1. \tag{3.14}$$

3.2.3 Game AB

Now we will turn to the random alternation of games A and B with probability γ. This will be named as game AB. For this game AB we have the following (primed) probabilities

- if the capital is a multiple of three

$$\begin{cases} p_1' = \gamma \cdot p + (1-\gamma) \cdot p_1, \\ r_1' = \gamma \cdot r + (1-\gamma) \cdot r_1, \end{cases} \tag{3.15}$$

- if the capital is not multiple of three

$$\begin{cases} p_2' = \gamma \cdot p + (1-\gamma) \cdot p_2, \\ r_2' = \gamma \cdot r + (1-\gamma) \cdot r_2. \end{cases} \tag{3.16}$$

The same reasoning as with game B can be made but with the new probabilities p_1', r_1', p_2', r_2' instead of p_1, r_1, p_2, r_2. Eventually we obtain that game AB will be winning if

$$\frac{(1-p_1'-r_1')(1-p_2'-r_2')^2}{p_1' p_2'^2} < 1, \tag{3.17}$$

losing if
$$\frac{(1-p_1'-r_1')(1-p_2'-r_2')^2}{p_1' p_2'^2} > 1, \tag{3.18}$$

and fair if
$$\frac{(1-p_1'-r_1')(1-p_2'-r_2')^2}{p_1' p_2'^2} = 1. \tag{3.19}$$

The paradox will be present if games A and B are losing, while game AB is winning. In this framework this means that the conditions (3.6), (3.13) and (3.17) must be satisfied simultaneously. In order to obtain sets of probabilities fulfilling theses conditions we have first obtained sets of probabilities yielding *fair* A and B games but such that AB is a winning game, and then introducing a small biasing parameter ϵ making game A and game B losing games, but still keeping a winning AB game. As an example, we give some sets of probabilities that fulfill these conditions:

$$
\begin{array}{llllllll}
(a) & p & = \frac{9}{20} - \epsilon, & r = \frac{1}{10}, & p_1 = \frac{9}{100} - \epsilon, & r_1 = \frac{1}{10}, & p_2 = \frac{3}{5} - \epsilon, & r_2 = \frac{1}{5}, \\
(b) & p & = \frac{9}{20} - \epsilon, & r = \frac{1}{10}, & p_1 = \frac{509}{5000} - \epsilon, & r_1 = \frac{1}{10}, & p_2 = \frac{7}{10} - \epsilon, & r_2 = \frac{1}{20}, \\
(c) & p & = \frac{9}{20} - \epsilon, & r = \frac{1}{10}, & p_1 = \frac{3}{25} - \epsilon, & r_1 = \frac{2}{5}, & p_2 = \frac{3}{5} - \epsilon, & r_2 = \frac{1}{10}, \\
(d) & p & = \frac{1}{4} - \epsilon, & r = \frac{1}{2}, & p_1 = \frac{3}{25} - \epsilon, & r_1 = \frac{2}{5}, & p_2 = \frac{3}{5} - \epsilon, & r_2 = \frac{1}{10}.
\end{array}
\tag{3.20}
$$

3.3 Properties of the Games

3.3.1 Rate of winning

If we consider the capital of a player at play number n, X_n modulo M, we can perform a Discrete Time Markov Chain (DTMC) analysis of the games with a state-space $\{0, 1, \ldots, M-1\}$ (*c.f.* [55]). For the case of Parrondo's games we have $M = 3$, so the following set of difference equations for the probability distribution can be obtained [56]:

$$
\begin{aligned}
P_{0,n+1} &= p_2 \cdot P_{2,n} + r_1 \cdot P_{0,n} + q_2 \cdot P_{1,n} \\
P_{1,n+1} &= p_1 \cdot P_{0,n} + r_2 \cdot P_{1,n} + q_2 \cdot P_{2,n} \\
P_{2,n+1} &= p_2 \cdot P_{1,n} + r_2 \cdot P_{2,n} + q_1 \cdot P_{0,n}
\end{aligned}
\tag{3.21}
$$

which can be put in a matrix form as $\mathbf{P}_{n+1} = \mathbb{T} \cdot \mathbf{P}_n$, where

$$
\mathbb{T} = \begin{pmatrix} r_1 & q_2 & p_2 \\ p_1 & r_2 & q_2 \\ q_1 & p_2 & r_2 \end{pmatrix}
\tag{3.22}
$$

and

$$
\mathbf{P}_n = \begin{pmatrix} P_{0,n} \\ P_{1,n} \\ P_{2,n} \end{pmatrix}.
\tag{3.23}
$$

In the limiting case where $n \to \infty$ the system will tend to a stationary state characterized by

$$
\mathbf{\Pi} = \mathbb{T} \cdot \mathbf{\Pi}
\tag{3.24}
$$

where $\lim_{n\to\infty} \mathbf{P}_n = \mathbf{\Pi}$.

Solving (3.24) is equivalent to solving an eigenvalue problem. As we are dealing with Markov chains, we know that there will be an eigenvalue $\lambda = 1$ and the rest will be under 1 [54]. For the $\lambda = 1$ value we obtain the following eigenvector giving the stationary probability distribution in terms of the games' probabilities.

$$
\mathbf{\Pi} \equiv \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} (1-r_2)^2 - p_2 \cdot (1 - p_2 - r_2) \\ (1-r_1)(1-r_2) - p_2 \cdot (1 - p_1 - r_1) \\ (1-r_1)(1-r_2) - p_1 \cdot (1 - p_2 - r_2) \end{pmatrix}
\tag{3.25}
$$

where D is a normalization constant given by

$$D = (1-r_2)^2 + 2(1-r_1)(1-r_2) - p_2(2-p_2-r_2-r_1-p_1) - p_1(1-p_2-r_2). \quad (3.26)$$

The rate of winning at the n–th step, has the general expression

$$r(n) \equiv E[X_{n+1}] - E[X_n] = \sum_{i=-\infty}^{\infty} i \cdot [P_{i,n+1} - P_{i,n}] \,. \quad (3.27)$$

Using these expressions and by similar techniques to those employed in [57] it is possible to obtain the stationary rate of winning for the new games introduced in the previous section. The results are, for game A:

$$r_A^{st} = 2p + r - 1 \quad (3.28)$$

and for game B

$$\begin{aligned} r_B^{st} &= 2p_2 + r_2 - 1 + [q_2 - p_2 + p_1 - q_1] \cdot \Pi_0 \\ &= \frac{3}{D}\left(p_1 p_2^2 - (1-p_1-r_1)(1-p_2-r_2)^2\right) \end{aligned} \quad (3.29)$$

where D is given by (3.26).

It is an easy task to check that when $r_1 = r_2 = 0$ we recover the known expressions for the original games obtained by [55]. To obtain the stationary rate for the randomized game AB we just need to replace in the above expression the probabilities from (3.15) and (3.16).

Within this context the paradox appears when $r_A^{st} \leq 0$, $r_B^{st} \leq 0$ and $r_{AB}^{st} > 0$. If, for example, we use the values from (3.20d) and a switching probability $\gamma = 1/2$, we obtain the following stationary rates for game A, game B and the random combination AB:

$$\begin{aligned} r_A^{st} &= -2\epsilon, \\ r_B^{st} &= \frac{-\epsilon\,(441 - 120\epsilon + 1000\epsilon^2)}{231 - 40\epsilon + 500\epsilon^2}, \\ r_{AB}^{st} &= \frac{93 - 9828\epsilon + 1920\epsilon^2 - 32000\epsilon^3}{2\,(2499 - 320\epsilon + 8000\epsilon^2)}\,. \end{aligned} \quad (3.30)$$

which yield the desired paradoxical result for small $\epsilon > 0$.

We can also evaluate the stationary rate of winning when both the probability of winning and the self-transition probability for the games vary with a parameter ϵ as $p = p - \frac{\epsilon}{2}$ and $r = r + \epsilon$, so that normalization is preserved. Using the set of probabilities derived from (3.20d), namely $p = \frac{1}{4} - \frac{\epsilon}{2}\,, r = \frac{1}{2} + \epsilon\,, p_1 = \frac{3}{25} - \frac{\epsilon}{2}\,, r_1 = \frac{2}{5} + \epsilon\,, p_2 = \frac{3}{5} - \frac{\epsilon}{2}\,, r_2 = \frac{1}{10} + \epsilon$, the result is:

$$\begin{aligned} r_A^{st} &= 0, \\ r_B^{st} &= \frac{-\epsilon\,(21 - 20\epsilon)}{2\,(77 - 200\epsilon + 125\epsilon^2)}, \\ r_{AB}^{st} &= \frac{31 - 164\epsilon + 160\epsilon^2}{2\,(833 - 2600\epsilon + 2000\epsilon^2)}\,, \end{aligned} \quad (3.31)$$

again a paradoxical result.

A comparison between the expressions for the rates of winning of the original Parrondo games [55] and the new games can be done in two ways. The first one consists in comparing two games with the same probabilities of winning, say original game A with probabilities $p = \frac{1}{2}$ and $q = \frac{1}{2}$ and the new game A with probabilities $p_{\text{new}} = \frac{1}{2}$, $r_{\text{new}} = \frac{1}{4}$ and $q_{\text{new}} = \frac{1}{4}$. In this case we can think of the 'old' probability of losing q as taking the place of the *self-transition* probability r_{new} and the new probability of losing q_{new}. In this way we obtain a higher rate of winning in the new game A than in the original game – remember that the new game A has an extra term r in the rate of winning compared to the original rate, and this extra term is what gives rise to the higher value. The same reasoning applies for game B, leading to the same conclusion.

The other possibility could be to compare the two games with the same probability of losing. In this case, we follow the same reasoning as before, but now we can imagine the 'old' probability of winning as replacing the winning and self-transition probabilities of the new game. What we now obtain is a lower rate of winning for the new game compared to the original one. An easy way of checking this is by rewriting (3.28) and (3.29) as

$$r_A^{st} = p - q \tag{3.32}$$

$$r_B^{st} = \frac{3}{D}\,(p_1 p_2^2 - q_1 q_2^2).$$

So for the same value of q but a lower value of p we obtain a lower value for the rates of game A and B.

We now explore the range of probabilities in which the Parrondo effect takes place. We restrict ourselves to the case $M = 3$ and $\gamma = 1/2$ used in the previous formulae.

The fact that we have introduced three new probabilities complicates the representation of the parameter space as we have six variables altogether, two variables $\{p, r\}$ from game A and four variables $\{p_1, r_1, p_2, r_2\}$ coming from game B. In order to simplify this high number of variables, some probabilities must be set so that a representation in three dimensions will be possible. In our case we will fix the variables $\{r, r_1, r_2\}$ so that the surfaces can be represented in the parameter space $\{p, p_1, p_2\}$.

In Fig. 3.1 we can see the resulting region where the paradox exists for the variables $r = \frac{1}{4}$, $r_1 = \frac{1}{8}$ and $r_2 = \frac{1}{10}$. Some animations have shown that the volume where the paradox takes place, gradually shrinks to zero as the variables r, r_1 and r_2 increase from zero to their maximum value of one.

Another interesting fact that we have encountered, which remains an open question, is the impossibility of obtaining the equivalent parameter space to Fig. 3.1 with the fixed variables p, p_1, p_2 and with the parameter space variables r, r_1, r_2 instead – it is possible to obtain the planes for games A and B, but not for the randomized game AB.

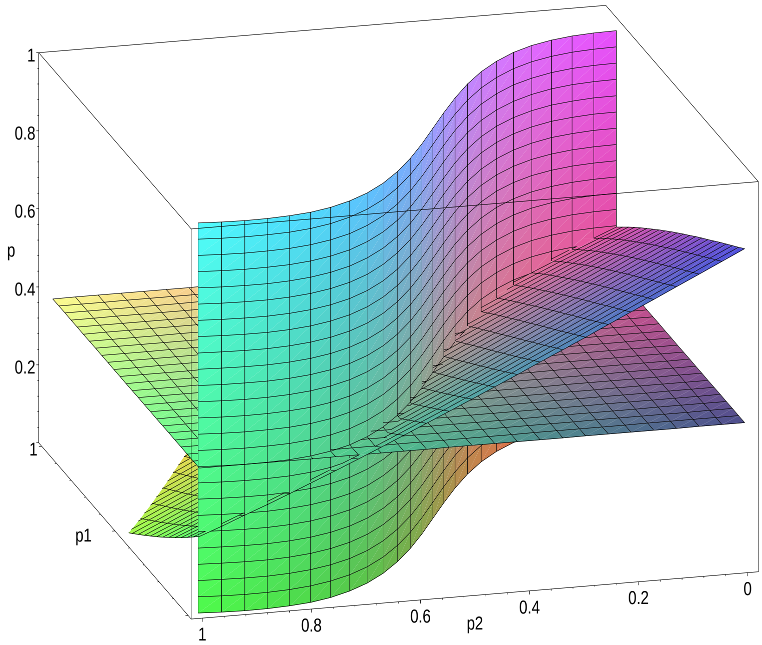

Figure 3.1: Parameter space corresponding to the values $r = \frac{1}{4}$, $r_1 = \frac{1}{8}$ and $r_2 = \frac{1}{10}$. The actual region where the paradox exists is delimited by the plane $p_1 = 0$ and the triangular region situated at the frontal face, where all the planes intersect.

3.3.2 Simulations and discussion

We have analyzed the new games A and B, and obtained the conditions in order to reproduce the Parrondo effect. We now present some simulations to verify that the paradox is present for a different range of probabilities – see Fig. 3.2. Some interesting features can be observed from these graphs. First it can be noticed that the performance of random or deterministic alternation of the games drastically changes with the parameters.

We use the notation $[a, b]$ to indicate that game A was played a times and game B b times. The performance of the deterministic alternations $[3, 2]$ and $[2, 2]$ remain close to one another, as can be seen in Fig. 3.2. However the alternation $[4, 4]$ has a low rate of winning because as we play each game four times, that causes the dynamics of games A and B to dominate over the dynamic of alternation, thereby considerably reducing the gain.

The performance of the random alternation is more variable, obtaining in some cases a greater gain than in the deterministic cases – see Fig. 3.2c.

In figures (3.3a) and (3.3b) a comparison between the theoretical rates of winning for games A, B and AB given by (3.30) and (3.31) and the rates obtained through simulations is presented. It is worth noting the good agreement between both results.

It is also interesting to see how evolves the average gain obtained from the random alternation of game A and game B when varying the mixing param-

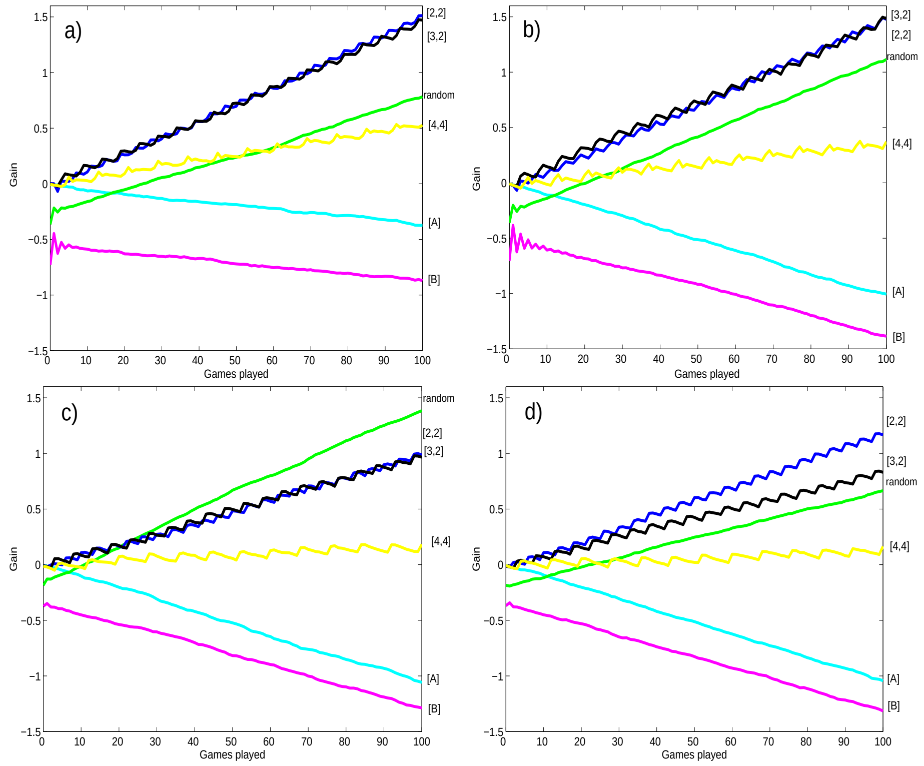

Figure 3.2: Average gain as a function of the number of games played coming from numerical simulation of Parrondo's games with different sets of probabilities. The notation $[a, b]$ indicates that game A was played a times and game B b times. The gains were averaged over 50 000 realizations of the games. a) Simulation corresponding to the probabilities (3.20a) and $\epsilon = \frac{1}{500}$; b) probabilities (3.20b) and $\epsilon = \frac{1}{200}$; c) probabilities (3.20c) and $\epsilon = \frac{1}{200}$; d) probabilities (3.20d) and $\epsilon = \frac{1}{200}$.

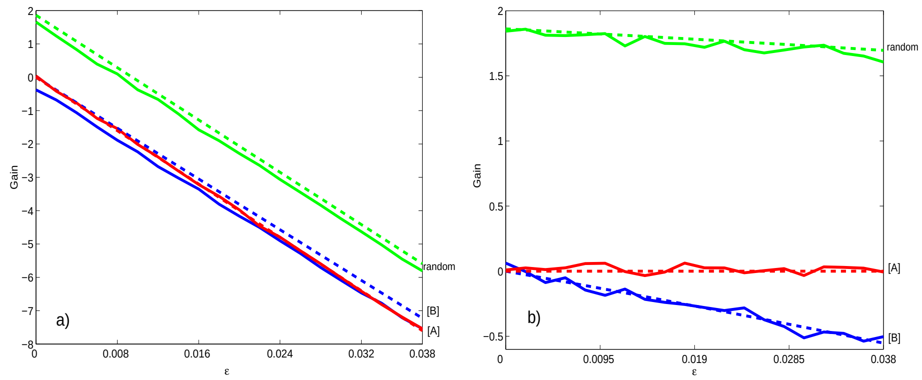

Figure 3.3: Comparison of the theoretical rates of winning – dashed lines – together with the rates obtained through simulations – solid lines. All the simulations were obtained by averaging over $50\,000$ trials and over all possible initial conditions. a) The parameters correspond to the ones used in equations (3.30). b) The parameters correspond to the ones used in equations (3.31).

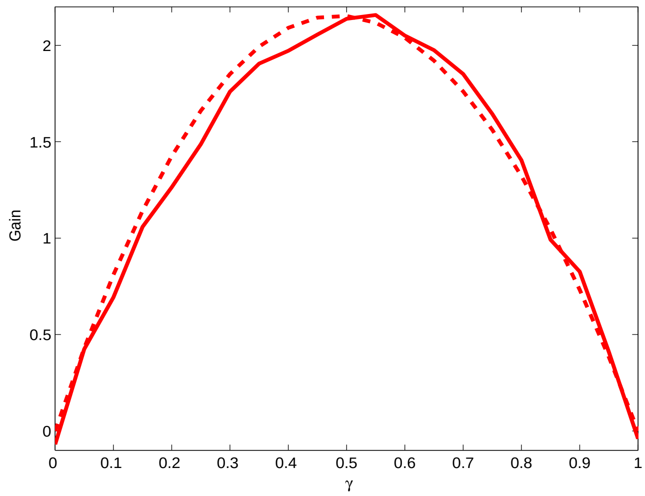

Figure 3.4: Comparison between the theoretical and the simulation for the gain vs gamma, for the following set of probabilities : $p = \frac{1}{3}$, $r = \frac{1}{3}$; $p_1 = \frac{3}{25}$, $r_1 = \frac{2}{5}$ and $p_2 = \frac{3}{5}$, $r_2 = \frac{1}{10}$. The simulations were carried out by averaging over $50\,000$ trials and all possible initial conditions.

eter γ. In Fig. 3.4 we compare both the experimental and theoretical curves. As in the original games, the maximum gain obtained for this set of parameters is obtained for a value around $\gamma \sim \frac{1}{2}$ [56]. For other sets of the game probabilities, though, the optimal γ differs from $\gamma = \frac{1}{2}$.

Chapter 4

Relation between Parrondo's games and the Brownian ratchet

In this chapter we will obtain a quantitative relation between the variables defining a game and the physical variables that determine the Brownian ratchet. Depending on the game considered a different formulation will be obtained : it will be shown that the original Parrondo's games can be derived from a Langevin equation with additive noise, whereas the Parrondo's games with self–transition come from a description using multiplicative noise.

4.1 Additive noise

A discrete time τ can be introduced by considering that every coin toss increases τ by one. If we denote by $P_i(\tau)$ the probability that at time τ the capital is equal to i, we can write the general master equation

$$P_i(\tau+1) = a^i_{-1}P_{i-1}(\tau) + a^i_0 P_i(\tau) + a^i_1 P_{i+1}(\tau) \tag{4.1}$$

where a^i_{-1} is the probability of winning when the capital is $i-1$, a^i_1 is the probability of losing when the capital is $i+1$, and, for completeness, we have introduced a^i_0 as the probability that the capital i remains unchanged (a possibility not considered in the original Parrondo games). In accordance with the rules described before, the probabilities $\{a^i_{-1}, a^i_0, a^i_1\}$ do not depend on time and they satisfy $a^{i+1}_{-1} + a^i_0 + a^{i-1}_1 = 1$ which ensures the conservation of probability: $\sum_i P_i(\tau+1) = \sum_i P_i(\tau)$.

It is a matter of straightforward algebra to write the master equation in the form of a continuity equation:

$$P_i(\tau+1) - P_i(\tau) = -\left[J_{i+1}(t) - J_i(t)\right] \tag{4.2}$$

where the current $J_i(\tau)$ is given by:

$$J_i(\tau) = \frac{1}{2}\left[F_i P_i(\tau) + F_{i-1}P_{i-1}(\tau)\right] - \left[D_i P_i(\tau) - D_{i-1}P_{i-1}(\tau)\right] \tag{4.3}$$

and $F_i = a_{-1}^{i+1} - a_1^{i-1}$, $D_i = \frac{1}{2}(a_{-1}^{i+1} + a_1^{i-1})$. This form is a consistent discretization of the Fokker–Plank equation[58] for a probability $P(x,t)$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{4.4}$$

with a current

$$J(x,t) = F(x)P(x,t) - \frac{\partial[D(x)P(x,t)]}{\partial x} \tag{4.5}$$

with general drift, $F(x)$, and diffusion, $D(x)$. If Δt and Δx are, respectively, the time and space discretization steps, such that $x = i\Delta x$ and $t = \tau\Delta t$, it is clear the identification

$$F_i \longleftrightarrow \frac{\Delta t}{\Delta x}F(i\Delta x), \qquad D_i \longleftrightarrow \frac{\Delta t}{(\Delta x)^2}D(i\Delta x) \tag{4.6}$$

The discrete and continuum probabilities are related by $P_i(\tau) \leftrightarrow P(i\Delta x, \tau\Delta t)\Delta x$ and the continuum limit can be taken by considering that $M = \lim\limits_{\Delta t\to 0, \Delta x\to 0} \frac{(\Delta x)^2}{\Delta t}$ is a finite number. In this case $F_i \leftrightarrow M^{-1}\Delta x F(i\Delta x)$ and $D_i \leftrightarrow M^{-1}D(i\Delta x)$.

From now on, we consider the case $a_0^i = 0$ (this corresponds to the original Parrondo's games). Since $p_i = a_{-1}^{i+1}$ we have $D_i \equiv D = 1/2$, $F_i = -1+2p_i$ and the current $J_i(\tau) = -(1-p_i)P_i(\tau) + p_{i-1}P_{i-1}(\tau)$ is nothing but the probability flux from $i-1$ to i. The stationary solutions P_i^{st} are found solving the recurrence relation (4.3) for a constant current $J_i = J$ with the boundary condition $P_i^{st} = P_{i+L}^{st}$:

$$P_i^{st} = N\mathrm{e}^{-V_i/D}\left[1 - \frac{2J}{N}\sum_{j=1}^{i}\frac{\mathrm{e}^{V_j/D}}{1-F_j}\right], \qquad J = N\frac{\mathrm{e}^{-V_L/D} - 1}{2\sum_{j=1}^{L}\frac{\mathrm{e}^{V_j/D}}{1-F_j}}. \tag{4.7}$$

N is the normalization constant obtained from $\sum_{i=0}^{L-1} P_i^{st} = 1$. In these expressions we have introduced the potential V_i in terms of the probabilities of the games[1]

$$V_i = -D\sum_{j=1}^{i}\ln\left[\frac{1+F_{j-1}}{1-F_j}\right] = -D\sum_{j=1}^{i}\ln\left[\frac{p_{j-1}}{1-p_j}\right] \tag{4.8}$$

The case of zero current $J = 0$, implies a periodic potential $V_L = V_0 = 0$. This reproduces again the condition $\prod_{i=0}^{L-1} p_i = \prod_{i=0}^{L-1}(1-p_i)$ for a fair game. In this case, the stationary solution can be written as the exponential of the potential $P_i^{st} = N\mathrm{e}^{-V_i/D}$. Note that Eq. (4.8) reduces in the limit $\Delta x \to 0$ to $V(x) = -M^{-1}\int F(x)dx$ or $F(x) = -M\frac{\partial V(x)}{\partial x}$, which is the usual relation between the drift $F(x)$ and the potential $V(x)$ with a mobility coefficient M.

[1] In this, as well as in other similar expressions, the notation is such that $\sum_{j=1}^{0} = 0$. Therefore the potential is arbitrarily rescaled such that $V_0 = 0$.

The inverse problem of obtaining the game probabilities in terms of the potential requires solving Eq. (4.8) with the boundary condition $F_0 = F_L$ [2]:

$$F_i = (-1)^i e^{V_i/D} \left[\frac{\sum_{j=1}^{L}(-1)^j [e^{-V_j/D} - e^{-V_{j-1}/D}]}{(-1)^L e^{(V_0 - V_L)/D} - 1} + \sum_{j=1}^{i}(-1)^j [e^{-V_j/D} - e^{-V_{j-1}/D}] \right] \tag{4.9}$$

These results allow us to obtain the stochastic potential V_i (and hence the current J) for a given set of probabilities $\{p_0, \ldots, p_{L-1}\}$, using (4.8); as well as the inverse: obtain the probabilities of the games given a stochastic potential, using (4.9). Note that the game resulting from the alternation, with probability γ, of a *game A* with $p_i = 1/2$, $\forall i$ and a game B defined by the set $\{p_0, \ldots, p_{L-1}\}$ has a set of probabilities $\{p'_0, \ldots, p'_{L-1}\}$ with $p'_i = (1-\gamma)\frac{1}{2} + \gamma p_i$. For the F_i's variables, this relation yields $F'_i = \gamma F_i$, and the related potential V' follows from (4.8).

We give now two examples of the application of the above formalism. In the first one we compute the stochastic potentials of the fair game B and the winning game B', the random combination with probability $\gamma = 1/2$ of game B and a game A with constant probabilities, in the original version of the paradox[21]. The resulting potentials are shown in Fig. 4.1. Note that the potential for game B takes different values at each point $i \mod 3$ even though the probabilities were equal for $i = 1, 2 \mod 3$. The resulting asymmetry in the potential is the required one for the existence of the ratchet effect. On the other hand, the potential of the combined game B' has a non-zero mean slope as it corresponds to a winning game.

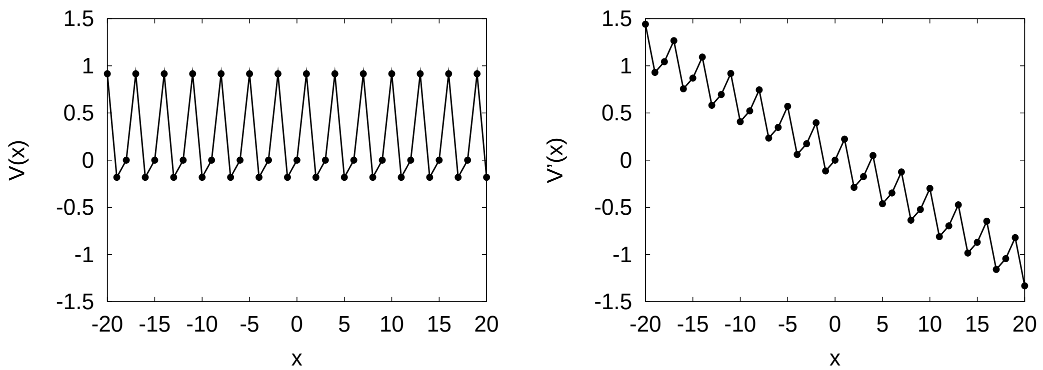

Figure 4.1: Left panel: potential V_i obtained from (4.8) for the fair game B defined by $p_0 = 1/10$, $p_1 = p_2 = 3/4$. Right panel: potential for game B', with $p'_0 = 3/10$, $p'_1 = p'_2 = 5/8$ resulting from the random alternation of game B with a game A with constant probabilities $p_i = p = 1/2$, $\forall i$.

[2] The singularity appearing for a fair game $V_L = V_0$ in the case of an even number L might be related to the lack of ergodicity explicitely shown in [43] for $L = 4$. In this case additional conditions on the potential are required for the existence of a fair game[59]

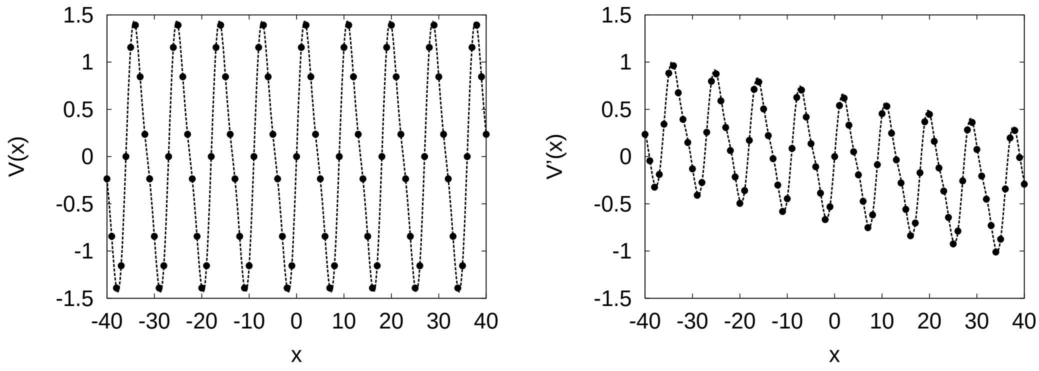

Figure 4.2: Left panel: Ratchet potential (2.2) in the case $L = 9$, $A = 1.3$. The dots are the discrete values $V_i = V(i)$ used in the definition of game B. Right panel: discrete values for the potential V_i' for the combined game B' obtained by alternating with probability $\gamma = 1/2$ games A and B. The line is a fit to the empirical form $V'(x) = -\Gamma x + \alpha V(x)$ with $\Gamma = 0.009525$, $\alpha = 0.4718$.

The second application considers as input the potential (2.2), which has been widely used as a prototype for ratchets[60, 61]. Using (4.9) we obtain a set of probabilities $\{p_0, \ldots, p_{L-1}\}$ by discretizing this potential with $\Delta x = 1$, i.e. setting $V_i = V(i)$. Since the potential $V(x)$ is periodic, the resulting game B defined by these probabilities is a fair one and the current J is zero. Game A, as always is defined by $p_i = p = 1/2$, $\forall i$. We plot in Fig. 4.2 the potentials for game B and for the game B', the random combination with probability $\gamma = 1/2$ of games A and B. Note again that the potential V_i' is tilted as corresponding to a winning game B'. As shown in Fig. 4.3, the current J depends on the probability γ for the alternation of games A and B.

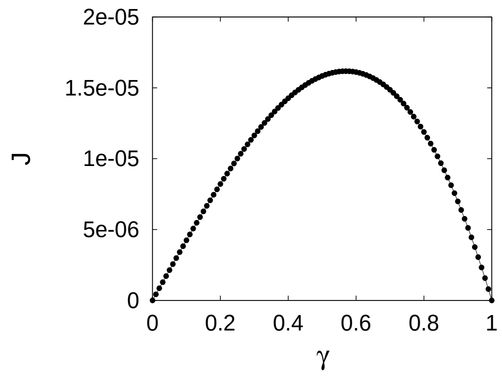

Figure 4.3: Current J resulting from equation (4.7) for the game B' as a function of the probability γ of alternation of games A and B. Game B is defined as the discretization of the ratchet potential (2.2) in the case $A = 0.4$, $L = 9$. The maximum gain corresponds to $\gamma = 0.57$.

4.2 The case of L even

A problem arises when finding the probabilities p_i using (4.9) for a periodic potential (corresponding to a fair game) when the number of points L is even. This is obvious since the periodicity condition $V_L = V_0$ gives a zero value for the denominator $(-1)^L e^{2(V_0-V_L)} - 1$ in (4.9). In order to be able to find solutions for the probabilities, the numerator has to vanish as well. This is equivalent to the condition:

$$\sum_k e^{-2V_{2k}} = \sum_k e^{-2V_{2k+1}} \tag{4.10}$$

which, in terms of the stationary probabilities, becomes:

$$\sum_k P^{st}_{2k} = \sum_k P^{st}_{2k+1}. \tag{4.11}$$

This condition implies that one can have a fair game in the case of an even number L only if the probability of finding an even value for the capital equals that of finding an odd value. To our knowledge, this curious property, which emerges naturally from the relation between the potential and the probabilities, has not been reported previously.

It turns out that one has to be careful when discretizing a periodic potential $V(x)$ in order to preserve this property. Otherwise, there will be no equivalent Parrondo game with zero current. The simple identification $V_i = V(i\lambda)$ might not satisfy this requirement, but we have found that a possible solution is to shift the origin of the x-axis, i.e. setting $V_i = V((i+\delta)\lambda)$ for a suitable value of δ. For example, in Fig. 4.4 we plot the difference

$$d(\delta) = \sum_i e^{-2V((2i+\delta)\lambda)} - \sum_i e^{-2V((2i+1+\delta)\lambda)} \tag{4.12}$$

as a function of δ in the case of the potential (2.2) and $\lambda = 1/4$ (which corresponds to $L = 4$ points per period). We see that there is only one value that accomplishes $d(\delta) = 0$, namely $\delta = -0.068616$.

Once the proper value of δ is found, it follows from Eq. (4.9) that there are infinitely many solutions for the probabilities. They can be found by varying, say, p_0, such that for each value of p_0 we will get a set of probabilities p_i. Solutions satisfying the additional requirement that $p_i \in [0,1]$, $\forall i$, will exist only for a certain range of values of $p_0 \in [0.0025, 0.68]$. Some of the different solutions are plotted in Fig. 4.5. Some numerical values are :

- $p_0 = 0.125$, $p_1 = 0.8167766$, $p_2 = 0.3927740$, $p_3 = 0.7082539$
- $p_0 = 0.25$, $p_1 = 0.6335531$, $p_2 = 0.5289900$, $p_3 = 0.6070749$
- $p_0 = 0.3525$, $p_1 = 0.4833099$, $p_2 = 0.6406871$, $p_3 = 0.5241081$
- $p_0 = 0.50$, $p_1 = 0.2671062$, $p_2 = 0.8014221$, $p_3 = 0.4047168$

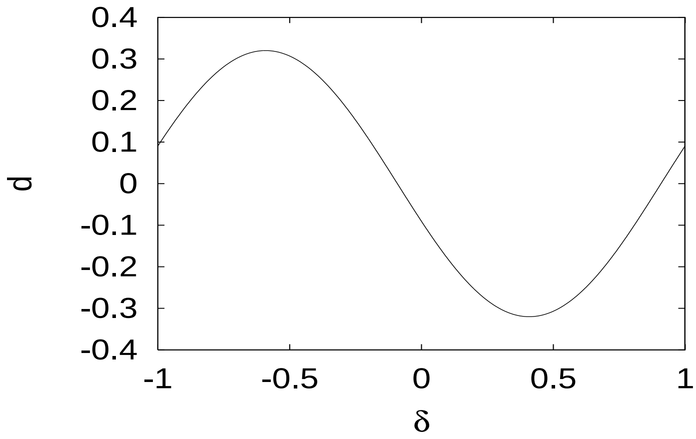

Figure 4.4: Plot of $d(\delta)$ as given by Eq. (4.12) versus displacement δ. The unique zero crossing is at $\delta = -0.068616$.

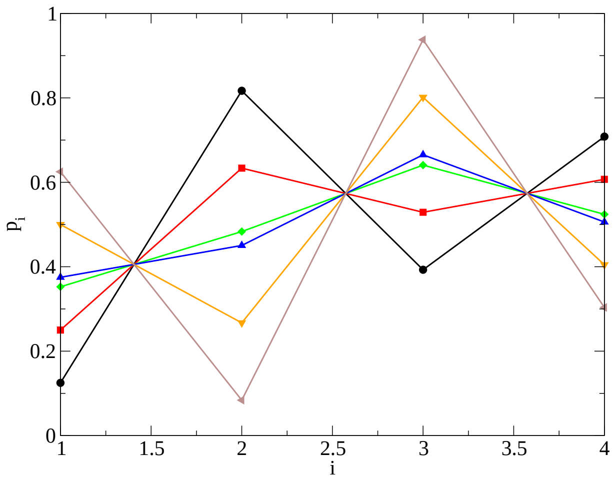

Figure 4.5: Multiple solutions for the probabilities p_i obtained with equation (4.8) for a potential like (2.2) with $A = 0.3$, $\lambda = \frac{1}{4}$, $\delta = -0.068616$ varying the value of p_0. The continuous line corresponds to the "optimal" solution, $p_0 = 0.3525$ (see the text).

An additional criterion to chose between the different sets of probabilities is to impose the maximum "smoothness" in the distribution of the p_i's. For instance, one could minimized the sum $\sum_{i=0}^{L-1}(p_{i+1} - p_i)^2$. In our example this criterion yields $p_0 = 0.3525$ and the other values follow from the previous table.

4.3 Multiplicative Noise

We go now a step forward, and calculate how these previous expressions obtained for the stationary probability, current and the defined potential vary when we consider the case $a_0^i \neq 0$ (which is equivalent to $r_i \neq 0$). As we stated before, considering this term implies that the player has now a certain probability of remaining with the same capital after a round played.

The drift and diffusion terms now read

$$F_i = a_{-1}^{i+1} - a_1^{i-1} = 2p_i + r_i - 1 \tag{4.13}$$

$$D_i = \frac{1}{2}(1 - a_0^i) = \frac{1}{2}(1 - r_i) \tag{4.14}$$

It can be appreciated that now both terms, the diffusion D_i as well as the drift F_i, may vary on every site. Using Eq. (4.3) and considering the stationary case $P_i(\tau) = P_i$ together with a constant current $J_i = J$, we get

$$P_i^{st} = \frac{J}{\frac{1}{2}F_i - D_i} - \left(\frac{\frac{1}{2}F_{i-1} + D_{i-1}}{\frac{1}{2}F_i - D_i}\right) \cdot P_{i-1}^{st}. \tag{4.15}$$

The previous equation has a general form $x_i = a_i + b_i x_{i-1}$, from which a solution can be derived as $x_n = \left[\prod_{k=1}^n b_k\right] \cdot x_0 + \sum_{j=1}^n a_j \cdot \left[\prod_{k=j+1}^n b_k\right]$. Applying the latter result to the stationary probability we have

$$P_n^{st} = \left[\prod_{k=1}^{n} \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k}\right] \cdot P_0^{st} - J \sum_{j=1}^{n} \frac{1}{D_j - \frac{1}{2}F_j} \left[\prod_{k=j+1}^{n} \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k}\right] \tag{4.16}$$

We can solve for the current J using Eq. (4.15) together with the periodic boundary condition $P_L^{st} = P_0^{st}$

$$J = \frac{P_0^{st} \cdot \left(\prod_{k=1}^{L} \left[\frac{\frac{1}{2}F_{k-1} + D_{k-1}}{D_k - \frac{1}{2}F_k}\right] - 1\right)}{\sum_{j=1}^{L} \frac{1}{D_j - \frac{1}{2}F_j} \prod_{k=j+1}^{L} \left[\frac{\frac{1}{2}F_{k-1} + D_{k-1}}{D_k - \frac{1}{2}F_k}\right]} \tag{4.17}$$

An *effective potential* can be defined in a similar way to its continuous analog as

$$V_i = -\sum_{j=1}^{i} \ln\left(\frac{1 + \frac{1}{2}\frac{F_{j-1}}{D_{j-1}}}{1 - \frac{1}{2}\frac{F_j}{D_j}}\right) = -\sum_{j=1}^{i} \ln\left(\frac{\frac{p_{j-1}}{1 - r_{j-1}}}{\frac{1 - p_j - r_j}{1 - r_j}}\right). \tag{4.18}$$

It is important to note that, as in the previous case $a_0^i = 0$, the potential must verify periodic conditions $V_0 = V_L$ when the set of probabilities define a fair game. It is an easy task to check that using Eq. (4.18) together with the periodic boundary condition, what we obtain is the fairness condition for a given set of probabilities defining a game with *self–transition*[62] , that is

$$\prod_{k=1}^{L-1} p_i = \prod_{k=1}^{L-1} q_i = \prod_{k=1}^{L-1} (1 - p_i - r_i) \tag{4.19}$$

By means of Eq. (4.18) we can obtain the stationary probability (4.16) and current (4.17) in terms of the defined potential as

$$P_n^{st} = \mathrm{e}^{-V_n}\left(\frac{D_0 \cdot P_0^{st}}{D_n} - J\sum_{j=1}^{n}\frac{\mathrm{e}^{V_j}}{D_n\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}\right) \tag{4.20}$$

$$J = \frac{P_0^{st}\left[D_0 - D_L \cdot \mathrm{e}^{V_L}\right]}{\sum_{j=1}^{L}\frac{\mathrm{e}^{V_j}}{\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}} \tag{4.21}$$

These are the new expressions which, together with Eqs. (4.13) and (4.14) allow us to obtain the potential, current and stationary probability for a given set of probabilities $\{p_i, r_i, q_i\}$ defining a Parrondo game with *self–transition*. We will now show that the set of Eqs. (4.18),(4.20),(4.21) can be related in a consistent form with the continuous solutions corresponding to the Fokker–Planck equation of a process with multiplicative noise [63].

Given a Langevin equation with multiplicative noise

$$\dot{x} = F[x(t), t] + \sqrt{B[x(t), t]} \cdot \xi(t) \tag{4.22}$$

interpreted in the sense of Ito, we can obtain its associated Fokker–Planck equation given by Eq. (4.4) recalling that $D(x,t) = \frac{1}{2}B(x,t)$. The general solution for the stationary probability density function $P(x,t)$ is given by

$$P^{st}(x) = \frac{\mathrm{e}^{\int^x \Psi(x)dx}}{D(x)} \cdot \left[\mathcal{N} - J\int^x \mathrm{e}^{-\int^{x'}\Psi(x'')dx''}dx'\right] \tag{4.23}$$

where $\mathcal{N}$ is a normalization constant and $\Psi(x) = \frac{F(x)}{D(x)}$. Making use of the periodicity and the normalization condition$P(0) = P(L)$ and $\int_0^x P(x)dx = 1$ we obtain the following expressions for $\mathcal{N}$ and J

$$\mathcal{N} = P(0) \cdot D(0) \qquad J = \frac{P(0)\cdot\left(D(0) - D(L)\mathrm{e}^{\int_0^L \Psi(x)dx}\right)}{\int_0^L \mathrm{e}^{-\int_0^{x'}\Psi(x'')dx''}dx'} \tag{4.24}$$

Comparing the discrete equations for the current and stationary probability (4.20-4.21) with the continuous solutions (4.23-4.24) we have the following equivalences

$$P_0^{st} \cdot D_0 \equiv P(0) \cdot D(0) \tag{4.25}$$

$$D_j \equiv D(x) \tag{4.26}$$

$$\mathrm{e}^{V_n} \equiv \mathrm{e}^{\int^x \Psi(x)dx} \tag{4.27}$$

$$\sum_{j=1}^{n}\frac{\mathrm{e}^{V_j}}{\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)} \equiv \int^x \mathrm{e}^{-\int^{x'}\Psi(x'')dx''}dx' \tag{4.28}$$

It is clear the identification of the terms in Eqs. (4.25) and (4.26). Now we need to demonstrate the equivalence given by Eqs. (4.27) and (4.28). If we define a *discretised function* as $\psi_j = \frac{F_{j-1}}{D_{j-1}}$ and we use the Taylor expansion up to first order of the logarithm $\ln(1+x) \approx x$ already used in the previous section we get

$$\begin{aligned}
V_n &= -\sum_{j=1}^{n} \ln\left(\frac{1+\frac{1}{2}\psi_{j-1}}{1-\frac{1}{2}\psi_j}\right) \approx -\frac{1}{2}\sum_{j=1}^{n}(\psi_{j-1}+\psi_j) = \\
&= -\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1}\psi_k + \frac{1}{2}\psi_n\right) \qquad (4.29)\\
\sum_{j=1}^{n}\frac{e^{V_j}}{1-\frac{1}{2}\psi_j} &= \sum_{j=1}^{n} e^{V_j - \ln\left(1-\frac{1}{2}\psi_j\right)} \approx \sum_{j=1}^{n} e^{-\frac{1}{2}\left(\sum_{k=1}^{j}[\psi_{k-1}+\psi_k]-\psi_j\right)} = \\
&= \sum_{j=1}^{n} e^{-\left(\frac{1}{2}\psi_0+\sum_{k=1}^{j}\psi_k+\frac{1}{2}\psi_j\right)+\frac{1}{2}\psi_j} \qquad (4.30)
\end{aligned}$$

It can be clearly seen that Eq. (4.29) corresponds to the numerical integration of the function $\Psi(x)$ defined previously, but with a $\Delta = 1$ (the difference in the sign is due to the way we have defined our potential). It can be demonstrated that when $\Delta \neq 1$ both expressions agree up to first order in Δ,

$$V_{n\Delta} = -\Delta\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1}\psi_k + \frac{1}{2}\psi_n\right) \qquad (4.31)$$

In the case of Eq. (4.30) what we obtain is nearly the Simpson's numerical integration method but for an extra term. As in the previous case, when $\Delta \neq 1$ then we have up to a first order an extra Δ term,

$$\sum_{j=1}^{n}\frac{e^{V_{j\Delta}}}{1-\frac{1}{2}\psi_{j\Delta}} \approx \Delta\cdot\sum_{j=1}^{n} e^{-\Delta\left(\frac{1}{2}\psi_0+\sum_{k=1}^{j}\psi_{k\Delta}+\frac{1}{2}\psi_{j\Delta}\right)+\frac{1}{2}\Delta\psi_{j\Delta}} \qquad (4.32)$$

So when $\Delta \to 0$ the contribution of the *extra* term can be neglected as compared to that of the sum.

We can also perform the inverse process, that is, to obtain the set of probabilities $\{p_i, r_i, q_i\}$ for a given potential V_i. If we call $A_n = \frac{p_n - q_n}{p_n + q_n}$, we need only to solve Eq. (4.18) for A_n obtaining

$$A_n = (-1)^n \cdot e^{V_n}\left[\frac{\sum_{j=1}^{L}(-1)^j(e^{-V_j}-e^{-V_{j-1}})}{(-1)^L\cdot e^{V_0-V_L}-1} + \sum_{j=1}^{n}(-1)^j\cdot\left(e^{-V_j}-e^{-V_{j-1}}\right)\right] \qquad (4.33)$$

Once these values are obtained, we must solve for the probabilities together with the normalization condition $p_i + r_i + q_i = 1$. As we have a free parameter in the set of solutions, we can fix the r_i values on every site and the rest of parameters can be obtained through

$$p_i = \frac{1}{2}(1 + A_i)(1 - r_i) \tag{4.34}$$

$$q_i = \frac{1}{2}(1 - A_i)(1 - r_i). \tag{4.35}$$

In this way what we have is a method for inverting an *effective* potential, fixing a parameter that in our case is the diffusion in every site (remember that the parameter r_i is related to the diffusion coefficient by Eq. (4.14) or equivalently the temperature.

The fact that we can obtain different sets of probabilities, both describing different dynamics but coming from the same potential $V(x)$, it is not surprising if we take into account that a system with multiplicative noise is equivalent, in the sense that both have the same stationary probability distribution, to another system with additive noise

$$\dot{x} = F(x) + D(x) \cdot \xi(t) \longrightarrow \dot{x} = \bar{F}(x) + \xi(t) \tag{4.36}$$

but with a renormalized drift term $\bar{F}(x)$ given by $\bar{F}(x) = -\frac{\partial \bar{V}}{\partial x}$ where $F(x) = -\frac{\partial V}{\partial x}$ and $\bar{V} = \int \frac{F(x)}{D(x)} dx + \ln D(x)$.

Chapter 5

Parrondo's games and Information theory

Recently, Arizmendi *et. al* [64] have quantified the transfer of information – negentropy – between the Brownian particle and the nonequilibrium source of fluctuations acting on it. These authors coded the particle motion of a flashing ratchet into a string of 0's and 1's according to whether the particle had moved to the left or to the right respectively, and then compressed the resulting binary file using the Lempel and Ziv algorithm [7]. They obtained in this way an estimation of the entropy per character h, as the ratio between the lengths of the compressed and the original file, for a sufficiently large file length. They applied this method to estimate the entropy per character of the ergodic source for different values of the flipping rate, with the result that there exists a close relation between the current in the ratchet and the net transfer of information in the system. The aim of this Chapter is to apply this technique to the discrete–time and space version of the Brownian ratchet, i.e., Parrondo's games.

5.1 Parrondo's games and Information Theory

Some previous works in the literature have related Parrondo's games and information theory. Pearce, in Ref. [65], considers the relation between the entropy and the fairness of the games, and the region of the parameter space where the entropy of game A is greater than that of B and AB. Harmer *et. al* [57] study the relation between the fairness of games A and B and the entropy rates considering two approaches. The first one calculates the entropy rates not taking into account the correlations present on game B, finding a good agreement between the region of maximum entropy rates and the region of fairness. The second approach introduces these correlations, obtaining lower entropy rates and no significant relation between fairness and entropy rates for game B.

In this section we aim to relate the current or gain in Parrondo's games with the variation of information entropy of the binary file generated using techniques similar to those of Ref. [64]. We will present the numerical results coming from simulations of the different versions of Parrondo's games: in the cooperative games[50, 51], one considers an ensemble of interacting players; in the history dependent games[47, 48], the probabilities of winning depend on the history of previous results of wins and loses; finally, in the games with self–transition[62], there is a non–zero probability r_i that the capital remains unchanged (not winning or losing) in a given toss of the coins. Finally, we offer, in Sec. 5.3, a theoretical analysis that helps to understand the behaviour observed in the simulations.

5.2 Simulation results

We have performed numerical simulations of the different versions of the games. In every case, the evolution of the capital of the player has been converted to a string of bits where bit 0 (resp., 1) corresponds to a decrease (resp., increase) of the capital after δ_t plays of the games. It will be shown that the delay time δ_t between capital measurements is a relevant parameter.

An estimation of the entropy per character, h, is obtained as the compression ratio obtained with the `gzip` (v. 1.3) program, that implements the Lempel and Ziv algorithm, although it has been stressed by some authors that this is not the best algorithm one can find in the literature. The simplicity in the use of this algorithm (as it is already implemented "for free" in many operating systems) is an added value, as it will become apparent in the following when we consider strings of symbols generated by more than one ergodic source. As suggested in Ref.[64], we expect that the negentropy, $-h$, which accounts for the known information about the system, is related in some way with the average gain in the games.

In Fig. 5.1 we compare the average gain in game AB with the value of the entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ as a function of the probability γ and for different delay times δ_t. We find indeed a qualitative agreement between the increase in the gain and the decrease in entropy as the γ parameter is varied. This decrease in the entropy of the system implies that there exists an increase in the amount of known information about the system. Notice that the compression rate depends on δ_t, and that the γ value for which there is the maximum decrease in entropy agrees with the value for the maximum gain in the games. This agreement is similar to the one observed when applying this technique to the Brownian flashing ratchet[64].

Similar results are obtained in other cases of Parrondo's games. For instance, in the right panel of Fig. 5.1 we compare the average gain and the entropy difference in the games with self–transition[62]. Again in this case the maximum gain coincides with the γ value for the minimum entropy per character for all values of δ_t.

Finally, in Fig. 5.2 we present the comparison in the case of the history

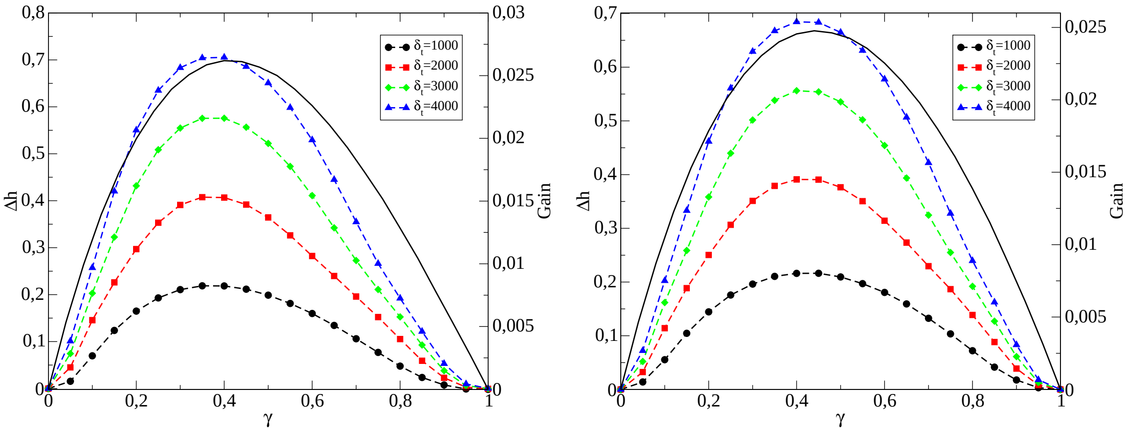

Figure 5.1: Comparison of the average gain per game (solid line) with the entropy difference Δh (symbols) as a function of the switching rate γ, for several values of the delay time δ_t, as shown in the legend, and the following versions of the Parrondo's paradox:
Left panel: Original Parrondo's combination of games A and B with probabilities: $p = \frac{1}{2}$, $p_0 = \frac{1}{10}$ and $p_1 = \frac{3}{4}$.
Right panel: Parrondo's combination of games A and B including self–transitions. The values for the probabilities are: $p = \frac{9}{20}$, $r = \frac{1}{10}$, $p_0 = \frac{3}{25}$, $r_0 = \frac{2}{5}$, $p_1 = \frac{3}{5}$ and $r_1 = \frac{1}{10}$ (see Ref.[62] for the choice of these parameters).

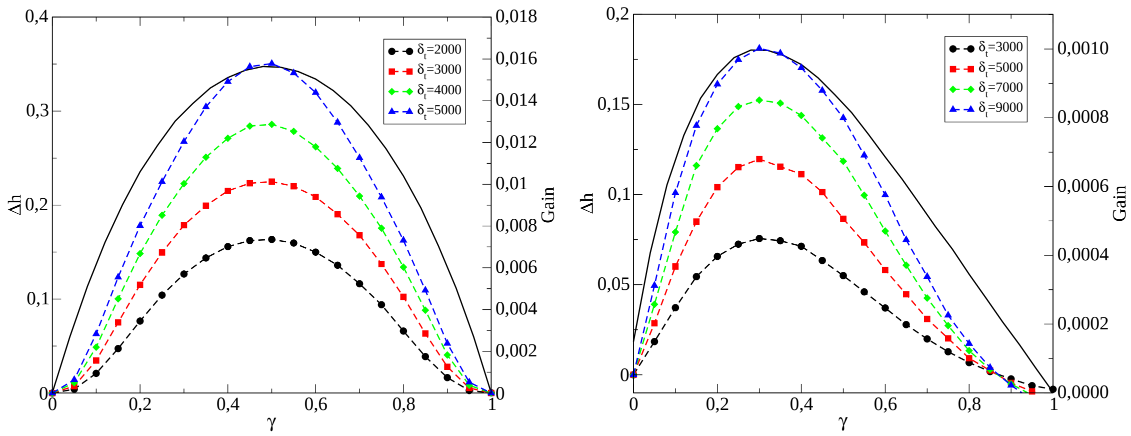

Figure 5.2: Same as Fig. 5.1 in other versions of Parrondo's paradox:
Left panel: History dependent games, alternating between two games with probabilities: $p_1 = \frac{9}{10}$, $p_2 = p_3 = \frac{1}{4}$, $p_4 = \frac{7}{10}$; $q_1 = \frac{2}{5}$, $q_2 = q_3 = \frac{3}{5}$ and $q_4 = \frac{2}{5}$ (see Ref.[47] for the choice of these parameters).
Right panel: Cooperative Parrondo's games with probabilities: $p = \frac{1}{2}$, $p_1 = 1$, $p_2 = p_3 = \frac{16}{100}$, $p_4 = \frac{7}{10}$ and $N = 150$ players (see Ref.[50] for the choice of these parameters).

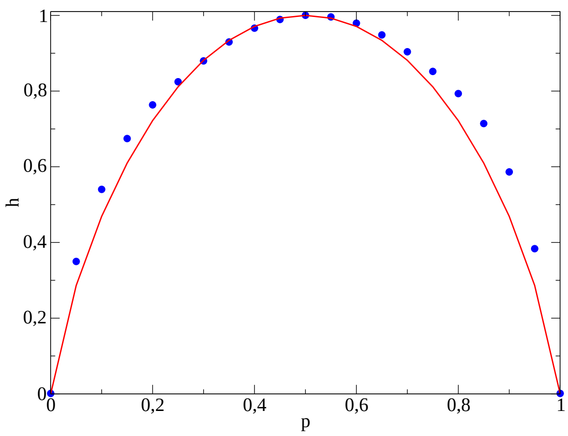

Figure 5.3: Comparison between the theoretical value obtained for the Shannon entropy – solid line – with the numerical values – circles – obtained with the `gzip` algorithm for a single source emitting two symbols with probability p.

dependent games [47] (left panel), and cooperative games [50] (right panel), showing all of them the same features as in the previous cases. We conclude that there exists a close relation between the entropy and the average gain. In the next section we will develop a simple argument that helps explaining this relation.

5.3 Theoretical analysis

As stressed in Sec. 5.1, the entropy per character of a text produced by an ergodic source is[1] $H = -\sum_i p_i \cdot \log(p_i)$, where p_i denotes the probability that the source will emit a given symbol α_i, and the sum is taken over all possible symbols that the source can emit. For instance, if we consider game A as a source of two symbols, 0 (losing) and 1 (winning) , the Shannon entropy according as a function of the probability p of emitting symbol 1 (i.e. the probability of winning) is given by Eq. (1.62). In Fig. 5.3 we compare this expression with the compression factor h obtained using the `gzip` algorithm. As shown in this figure, in this case of a single source, the compression factor of the `gzip` algorithm does give a good approximation to the Shannon entropy.

From now on, we restrict our analysis to the case of the original Parrondo's paradox combining games A and B, as explained in the previous section. The combined games AB can be considered as originated by two sources depending on whether the capital is a multiple of 3 or not. The probability of emitting symbol 1 when using the first source is q_0, whereas the same probability is q_1 when using the second source.

[1]Units are taken such that all logarithms are base 2.

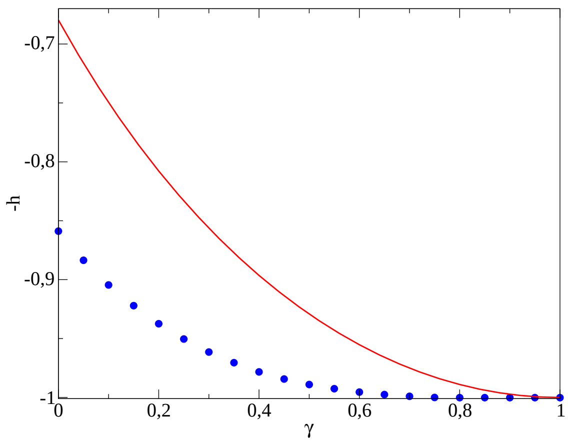

Figure 5.4: Plot of Shannon negentropy (solid line) for the combination game AB according to expression (5.1), together with the numerical values (circles) obtained with the compression factor of the gzip algorithm in the case when $\delta_t = 1$ step.

We first consider the case $\delta_t = 1$, i.e. we store the capital after each single play of the games. According to the expression (1.63) for the entropy of a mixed source, the Shannon entropy for the combined game AB is:

$$H = -\Pi_0[q_0 \log(q_0) + (1 - q_0) \log(1 - q_0)] - (1 - \Pi_0)[q_1 \log(q_1) + (1 - q_1) \log(1 - q_1)] \tag{5.1}$$

being Π_0 the stationary probability than in a given time the capital is a multiple of 3. This can be computed using standard Markov chain theory, with the result[56]:

$$\Pi_0 = \frac{1 - q_1 + q_1^2}{3 - q_0 - 2q_1 + 2q_0q_1 + q_1^2} \tag{5.2}$$

In Fig. 5.4 we compare the Shannon entropy H given by the previous formula with the numerical compression factor h as a function of the probability γ of mixing games A and B. Although certainly not as good as in the case of a single game, in this case, the gzip compression factor gives a reasonable approximation to the Shannon entropy of the combined game AB. It is worth noting that in this case of $\delta_t = 1$ the entropy increases with γ, corresponding to a decrease of the information known about the system. In order to relate the entropy difference with the current gain, we need to consider larger values for δ_t.

For $\delta_t \gg 1$ the system gradually loses its memory about its previous state. Therefore, the different measures are statistically independent and they can be considered as generated by a single ergodic source. For this single source, the probability of winning after one single play of the games is $p_w = \Pi_0 \, q_0 + (1 - \Pi_0) \, q_1$. However, we are interested in calculating the winning probability $p_>$ after δ_t plays. In order to have a larger capital after δ_t plays it is necessary that the number of wins overcomes the number of losses in single game plays. The distribution of the number of wins follows a binomial distribution and the

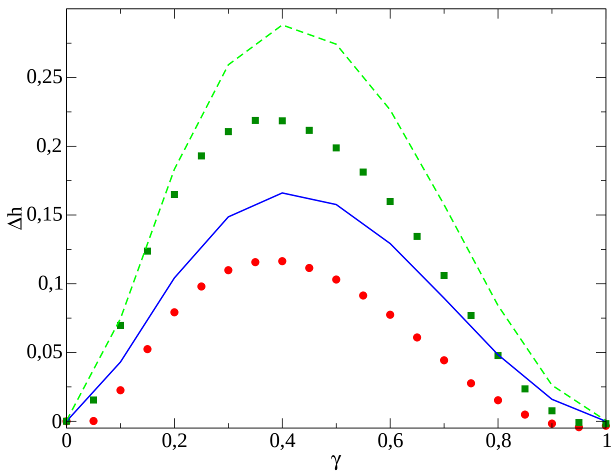

Figure 5.5: Plot of Shannon entropy difference $\Delta h = h(\gamma = 0) - h(\gamma)$ according to formulas (5.4) and (5.3) for $\delta_t = 500$ (solid line) and $\delta_t = 1000$ (dashed line) together with the numerical curves obtained with the compression ratio of the `gzip` algorithm for the same values of $\delta_t = 500$ (circles) and $\delta_t = 1000$ (squares).

probability $p_>$ is given by:

$$p_> = \sum_{k=0}^{\frac{\delta_t}{2}} \binom{\delta_t}{k} \cdot p_w^{\delta_t - k} \cdot (1 - p_w)^k. \tag{5.3}$$

The corresponding Shannon entropy for this single source is:

$$H = -p_> \cdot \log(p_>) - (1 - p_>) \cdot \log(1 - p_>). \tag{5.4}$$

We compare in Fig. 5.5 the Shannon entropy coming from this formula and the one obtained by the compression ratio of the `gzip` program for two different values of $\delta_t = 500, 1000$. In both cases, there is a reasonable agreement between both results. Moreover, as shown in Figs. 5.1 and 5.2 the entropy follows closely the average gain of the combined games.

Chapter 6

Conclusions

We have introduced in Chapter 3 a new family of Parrondo's games with a new probability, i.e., the *self–transition* probability, of which the original Parrondo games are a special case with self-transitions set to zero. New discrete–time Markov chain analysis for this new games were presented in Sec. 3.2, showing that Parrondo's paradox still occurs if the appropriate conditions are fulfilled. New expressions for the rates of winning have been obtained in Sec. 3.3, with the result that within certain conditions a higher rate of winning than in the original games can be obtained. We have also studied how the parameter space where the paradox exists changes with the self-transition variables, and conclude that the parameter space corresponding to the original Parrondo's games is a limiting case of the maximum volume – as the self-transition probabilities increase in value the volume shrinks to zero. However, it is worth noting that despite the volume decreases with increasing the self-transition probabilities, the rates of winning that can be obtained are higher than in the original Parrondo's games.

In Chapter 4 we have written the master equation describing the Parrondo's games as a consistent discretization of the formalism of the Fokker–Planck equation for an overdamped Brownian particle. In this way we can relate the probabilities of the games $\{p_0, \ldots, p_{L-1}\}$ to the dynamical potential $V(x)$. Our approach yields a periodic potential for a fair game and a tilted potential for a winning game. The resulting expressions, in the limit $\Delta x \to 0$ could be used to obtain the effective potential for a flashing ratchet as well as its current. This relation also works in two ways: we can obtain the physical potential corresponding to a set of probabilities defining a Parrondo game, as well as the current and its stationary probability distribution. Inversely, the probabilities corresponding to a given physical potential can also be obtained. Our relations work both in the cases of additive noise or multiplicative noise, showing that the former case is equivalent to the original Parrondo's games, whereas the latter corresponds to the Parrondo's games with self–transition probability.

With the relations introduced for the cases of additive and multiplicative noise, we have now a precise and of general validity connection between individual Brownian ratchets and single Parrondo's games. This work confirms Parrondo's original intuition based on a flashing ratchet is correct with rigour.

Finally, in Chapter 5, we have quantified the amount of the transfer of information (negentropy) in the case of Parrondo's games, considered as a discrete–time and space version of the flashing ratchet. This effect takes place in every existing version of the games analyzed, showing its robustness, and it is the equivalent of the same result obtained in the case of the Brownian ratchets. In the case of the original Parrondo's paradox mixing two games, A and B, we have computed the entropy by considering that the capital originates from a combination of two ergodic sources, reflecting the different winning probabilities when the capital is a multiple of three or not. We have shown that the entropy behaves very differently for low and high values of the delay parameter δ_t, while for $\delta_t = 1$ there is a monotonic dependence on the switching parameter γ, the relation between the gain and the current is only apparent for large values of δ_t. Our paper offers a new and hopefully enlightening approach to understand Parrondo's paradox. This approach differs (and complements) from previous works[65, 57] in that we consider the capital of the player as the information source.

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CURRICULUM VITAE

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Academic Degrees

Degree: M.Sc. in Physics
Center: University of the Balearic Islands (U.I.B.)
Years: 1996/97-2000/01

Current Affiliation

1. Organism: Universitat de les Illes Balears.

2. Center: Institut Mediterrani de Estudis Avançats (*IMEDEA*) *CSIC-UIB*.

3. Department: Física Interdisciplinar.

4. Professional category and starting date: PhD student at the UIB and scholarship holder of a Fellowship from the Government of the Balearic Islands since 01/07/2002. PhD supervisor: Prof. Raúl Toral .

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Awarded Fellowships

1. Collaboration fellowship of the Ministerio de Educación y Cultura with the Physics Department of the UIB during the academic year 2000/01 with the project “About Potential Vorticity” in collaboration with Prof. Sergio Alonso Oroza.
2. Awarded a Leonardo fellowship for working practice at ’Plasma Ireland Ltd.’ (Cork, Ireland) from 01/09/01 to 28/02/02.

Other Merits

1. Collaborator student with Prof. Sergio Alonso Oroza, from the Meteorology Group of the Physics Department of the UIB during the academic courses 1998/1999 and 1999/2000.

Specialization Courses

Ph.D. courses in the University of the Balearic Islands

- *Mètodes de simulació numérica en física*. R. Toral. UIB, 2003 [3 credits]
- *Sistemes dinàmics*. O. Piro and E. Hernández. UIB, 2003 [3 credits]
- *Mecànica estadística: transicions de fase i sistemes de polímers*. T. Sintes. UIB, 2003 [3 credits]
- *Models de cooperació basats en agents*. M. San Miguel. UIB, 2003 [3 credits]
- *Dinàmica d’estructures espaciotemporals*. E. Hernández, O. Piro, and M. San Miguel. UIB, 2004 [5 credits]
- *Tècniques de supercomputació*. J. Masso. UIB 2004 [3 credits]

Other specialization courses

- *Models for Complex Systems in Human and Social Sciences*, 17/07/2004 to 31/07/2004, Summer School organized by the Ecole Normale Superior, Lyon.

Stays at foreign research centers

- 15/07/2003 to 07/09/2003 at the Department of Electrical and Electronic Engineering, University of Adelaide, Australia, under the supervision of Professor Derek Abbott.

- 02/05/2005 to 08/05/2005 at the Interdisziplinäres Zentrum für Bioinformatik, Universität Leizig, Germany, for a collaboration with Dr. Konstantin Klemm.

Seminars

- *"Parrondo's games and information theory"*,21/04/2004, Interdisciplinary Physics Department (IMEDEA), Palma de Mallorca, Spain.

- *"Parrondo's games as a discrete ratchet"*,04/05/2005, Interdisziplinäres Zentrum für Bioinformatik. Universität Leizig. Germany.

Congresses attended

- *Parrondo's Paradox and the Fokker-Planck equation. Congreso de Física Estadística FisEs 2003* . Pamplona (Spain) October 23rd to 25th 2003. Poster.

- *Localidad en la Adopción Social de una Opinión Minoritaria. Congreso de Física Estadística FisEs 2003* . Pamplona (Spain) October 23rd to 25th 2003. Poster.

- *III Jornades de la Xarxa Temàtica : Non-linear Dynamics of Spatio-temporal Self-organization*. Barcelona (Spain) February 5th to 7th 2004.

- *Parrondo's games and the Zipping algorithm. Control in Games and Ratchets*. Workshop held in Toledo, April 29th to May 1st, 2004. Oral.

- *An exact description of Parrondo's games as a ratchet. SPIE 'Fluctuations and Noise'* . Maspalomas, Gran Canaria (Spain). May 25th-28th 2004. Oral

- *Parrondo's games and the Zipping algorithm. SPIE 'Fluctuations and Noise'* . Maspalomas, Gran Canaria (Spain). May 25th-28th 2004. Oral.

- Discrete ratchets and Parrondo's games. *Non-Equilibrium Phenomena and Phase Transitions in Complex Systems*. Bayreuth (Germany). September 28th to October 2nd 2004. Poster.

- *Cost P10 Physics of Risk. WG2-2nd WORKSHOP*. Mallorca (Spain). November 24th to 28th 2004.
- *Discrete ratchets and Parrondo's games*, *8th Granada Seminal on Computational and Statistical Physics*. Granada (Spain). February from 7th to 11th 2005. Poster.

Publications

Articles

- R. Toral, P. Amengual & S.Mangioni, *Parrondo's Games as a discrete ratchet* , *Physica* A 327, 105-110 (2003).
- P. Amengual, A. Allison, R. Toral & D. Abbott. *Discrete–time ratchets, the Fokker–Planck equation and Parrondo's paradox*. *Proceedings of the Royal Society of London* A 460, 2269-2284 (2004).
- C.J. Tessone, R. Toral, P. Amengual, H.S. Wio & M. San Miguel. *Neighborhood models of minority opinion spreading*. *European Physical Journal* B 39, 535-544 (2004).
- P. Amengual & R. Toral. *Transfer of information in Parrondo's games*. *Fluctuation and Noise Letters* 5, L63–L71 (2005).

Proceedings

- R. Toral, P. Amengual & S.Mangioni. *A Fokker-Planck description for Parrondo's Games* , *Noise in Complex Systems and Stochastic Dynamics*. Proceedings of SPIE Vol. 5114 (2003).
- P. Amengual & R. Toral . *Parrondo's games and the zipping algorithm*. *Noise in Complex Systems and Stochastic Dynamics II*. Proceedings of SPIE Vol. 5471 (2004).
- P.Amengual & R. Toral . *Exact Ratchet description of Parrondo's games with self-transitions*. *Noise in Complex Systems and Stochastic Dynamics II*. Proceedings of SPIE Vol. 5471 (2004).
- R. Toral & P.Amengual. *Distribution of winners in truel games*. Submitted to *AIP* Proceedings. (2005).

Computer experience and languages

- Programming: Fortran, C, Java, Matlab, Maple.
- Operative systems: Linux, Windows, DOS.
- Languages: Spanish (native), Catalan (native), English (good).