Bubbling and on-off intermittency in bailout embeddings

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We establish and investigate the conceptual connection between the dynamics of the bailout embedding of a Hamiltonian system and the dynamical regimes associated with the occurrence of bubbling and blowout bifurcations. The roles of the invariant manifold and the dynamics restricted to it, required in bubbling and blowout bifurcating systems, are played in the bailout embedding by the embedded Hamiltonian dynamical system. The Hamiltonian nature of the dynamics is precisely the distinctive feature of this instance of a bubbling or blowout bifurcation. The detachment of the embedding trajectories from the original ones can thus be thought of as transient on-off intermittency, and noise-induced avoidance of some regions of the embedded phase space can be recognized as Hamiltonian bubbling.

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I. INTRODUCTION

Symmetry is a fundamental concept in science, both at the level of physical laws and when applied to specific applications. In dynamics, the presence of symmetries may imply the existence of lower-dimensional submanifolds of the full phase space, which are dynamically invariant, because symmetric states must evolve into symmetric states. The motion restricted to these invariant manifolds is often chaotic. This occurs, for instance, in synchronized chaotic oscillators and in extended systems with spatial symmetry. The existence of chaotic attractors in the restricted dynamics can have interesting consequences for the behavior of the full system [1,2].

An interesting question is whether the attractors on invariant manifolds are also attractors of the unrestricted dynamics. Consider the case in which a parameter controls the dynamics transversal to the invariant manifold while leaving unaffected the dynamics restricted to the invariant manifold [2]. Below a critical value of this parameter, the manifold is transversally stable and any attractor within it is also a global attractor. When the parameter exceeds the critical value, some invariant set embedded in the attractor becomes transversally unstable. Although most trajectories initially close to the attractor in the invariant manifold may still remain close to it, there is now a small set of points, in any neighborhood of the attractor, which diverges from the invariant manifold. Increasing the parameter still further, the full attractor becomes transversally unstable. These two scenarios, named bubbling and blowout bifurcations, may lead either to the so-called riddled basins [3] or to a strong temporal bursting termed on-off intermittency [1,4–6], depending on the nature of the dynamics away from the invariant manifold.

In the foregoing, we have supposed that both the restricted and the unrestricted dynamics are dissipative. But we might envision a case in which the global dissipative dynamics have an invariant manifold on which the restricted dynamics is conservative; described, for example, by a Hamiltonian flow or a volume-preserving (Liouvillian) map. The orbits on the invariant manifold then exhibit, instead of attractors, the typical Hamiltonian phase-space structure characterized by the coexistence of chaotic regions and Kolmogorov-Arnold-Moser (KAM) tori. However, since the full dynamics are dissipative, a part or even the whole family of the restricted Hamiltonian orbits may be global attractors. Furthermore, a parameter can now control the transversal stability of the invariant manifold in the same way as in blowout bifurcations; but now governing which part of the Hamiltonian dynamics is globally attracting, and which is not.

The aim of this work is to show that these dynamics are observed in a physical system of finite-sized particles driven by an incompressible fluid flow [7,8], and have been harnessed to work in the different context of the study of chaotic dynamical systems in the technique of bailout embedding, first introduced by us in Ref. [9]. In Sec. II we recall the necessary background on bailout embeddings. In Sec. III we consider bubbling and blowout bifurcations and in Sec. IV we relate bailout embeddings with blowout bifurcations. Section V follows with a discussion of the addition of noise to a bailout embedding, and the occurrence of bubbling. We conclude in Sec. VI.

II. BAILOUT EMBEDDINGS

Given an arbitrary dynamical system $\dot{x}=f(x)$, we define a bailout embedding as one of the form

$$\frac{d}{dt}[v-f(x)]=-k(x)[v-f(x)],$$
$$\frac{dx}{dt}=v. \tag{1}$$

This is a dynamical system defined in the space of variables x and v. Notice, though, that the original dynamics, satisfy-

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ing $\dot{x}=f(x)=v$, also solves Eqs. (1). This implies that $v=f(x)$ defines an invariant submanifold for the bailout embedding. However, we can design an embedding to satisfy the condition that $k(x)<0$ on a set of unwanted orbits of the original dynamics while $k(x)>0$ on the others. We thus force the unwanted orbits to be unstable in the larger dynamics even though they are stable in the original system. In this way, trajectories in the embedding tend to detach or bail out from the unwanted set of the original system, bouncing around in the larger space until eventually reaching a stable region with $k(x)>0$, where they may converge back onto trajectories of the original dynamical system. In other words, by means of a bailout embedding we can create a larger version of the dynamics in which specific sets of orbits are removed from the asymptotic set, while preserving the dynamics of another set of orbits, the wanted one, as attractors of the enlarged dynamical system.

There is a remarkable example in nature where a bailout embedding describes the dynamics of a physical system. The motion of finite-sized neutrally buoyant spherical particles suspended in an incompressible fluid flow embeds the Lagrangian dynamics of a perfect tracer satisfying $\dot{\mathbf{x}}=\mathbf{u}(\mathbf{x})$, where $\mathbf{u}(\mathbf{x})$ is the Eulerian velocity field describing the flow [10]. In the full model of the drag on a sphere [7], this is a highly nontrivial embedding. For example, the contribution to the drag, known as a Basset-Boussinesq force, contains the difference between the fluid and particle accelerations integrated over the whole trajectory. However, in the limit where the particles are sufficiently small and the Basset-Boussinesq term can be disregarded, the dynamics reduce to a bailout embedding of the form given in Eq. (1) with $\mathbf{u}(\mathbf{x})$ playing the role of f and $k(\mathbf{x})=-(\gamma+\boldsymbol{\nabla} f)$, where γ turns out to be the Stokes coefficient [8]. In this case, it appears that the embedding (the finite-sized particle dynamics) detaches from the original dynamics (the passive scalar one) near saddle points and other unstable regions of the basic volume-preserving flow, converging back only in the KAM islands that act now as attractors of the embedding. This result generalizes immediately to any Hamiltonian or divergence-free flow using a similar form of $k(x)$, and can be used as a method to target small KAM islands in these types of systems [9].

The notion of bailout embedding Eq. (1) can be extended to maps in a rather obvious fashion [9,11]. Given a map $x_{n+1}=f(x_n)$, the bailout embedding is given by

$$x_{n+2}-f(x_{n+1})=K(x_n)[x_{n+1}-f(x_n)]. \tag{2}$$

Condition $k(x)<0$ over the unwanted orbits has to be replaced here by $|K(x)|>1$, because the deviations from the original dynamics are multiplied by K in each iteration. The particular choice of the gradient as bailout function $k(x)=-(\gamma+\boldsymbol{\nabla} f)$ in a flow translates in the map setting to $K(x)=e^{-\gamma}\boldsymbol{\nabla} f$.

Let us illustrate the functioning of a map bailout embedding in the Hamiltonian framework by means of the prototypical area-preserving standard map introduced by Chirikov and Taylor [12]. This map is defined on the two-dimensional torus by

$$\begin{aligned} x_{n+1}&=x_n+\frac{k}{2\pi}\sin(2\pi y_n)\equiv f_x(x_n,y_n) \quad (\mathrm{mod}1),\\ y_{n+1}&=y_n+x_{n+1}\equiv f_y(x_n,y_n) \quad (\mathrm{mod}1). \end{aligned} \tag{3}$$

Parameter k, controlling the nonlinearity, takes the standard map from the integrable limit at $k=0$ to a highly chaotic regime when $k\gg 1$. At intermediate k this map shows a mixture of quasiperiodic trajectories on the KAM tori together with chaotic ones, depending on where we set the initial conditions. As the value of k is increased, the region dominated by chaotic trajectories pervades most of the phase space except for increasingly small islands of KAM quasiperiodicity. Notice that the only factor that decides whether we are in one of these islands or in the surrounding chaotic sea is the initial condition of the trajectory. Therefore, in order to locate one of these islands, it would be necessary to make the initial condition scan the whole phase space while watching for quasiperiodicity in the resulting dynamics. In a bailout embedding, on the other hand, the KAM trajectories of the embedded system can be transformed into effective global attractors in the extended phase space. The search for KAM islands then becomes a matter of simply iterating the bailout map forwards until its trajectories converge to those of the embedded one.

The bailout embedding of the standard map is obtained by simply replacing f in Eq. (2) with the definitions from Eq. (3). Accordingly, $K(x)$ in Eq. (2) becomes

$$K(x)=e^{-\gamma}\begin{pmatrix} 1 & k\cos(2\pi y_n) \\ 1 & k\cos(2\pi y_n)+1 \end{pmatrix}. \tag{4}$$

These replacements then lead to the coupled second-order iterative system

$$\begin{aligned} x_{n+1}&=u_n+f_x(x_n,y_n),\\ y_{n+1}&=v_n+f_y(x_n,y_n),\\ u_{n+1}&=e^{-\gamma}(u_n+k\cos(2\pi y_n)v_n),\\ v_{n+1}&=e^{-\gamma}(u_n+[k\cos(2\pi y_n)+1]v_n). \end{aligned} \tag{5}$$

Notice that due to the area-preserving property of the standard map, the two eigenvalues of the derivative matrix must multiply to one. If they are complex, this means that both have an absolute value of one, while if they are real, generically one of them will be larger than one and the other smaller. We can then separate the phase space into elliptic and hyperbolic regions, corresponding to each of these two cases. If a trajectory of the original map lies entirely in the elliptic regions, overall factor $e^{-\gamma}$ damps any small perturbation away from it in the embedded system. But, for chaotic trajectories that inevitably visit some hyperbolic regions, there exists a threshold value of γ such that perturbations away from a standard-map trajectory are amplified instead of dying out in the embedding. As a consequence, trajectories

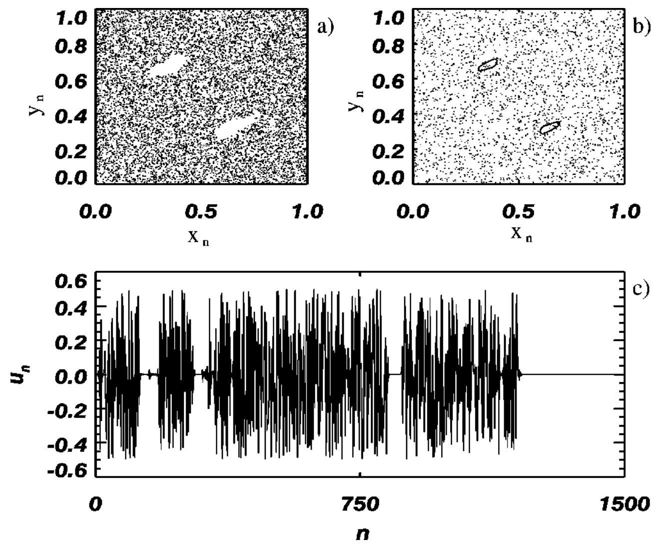

FIG. 1. (a) Chaotic trajectory of the standard map for $k=5$, covering practically the whole phase space except for a small period-2 KAM island. (b) Trajectory of a bailout embedding of the standard map with $e^{-\gamma}=0.85$. After wandering around the phase space for a while, the trajectory finally settles inside the islands. (c) Deviation of the bailout embedding trajectory from the original dynamics showing transiently intermittent behavior as a function of time. The parameters here are the same as in (a) and (b).

are expelled from the chaotic regions to finally settle in the safely elliptic KAM islands. This process can be seen clearly in Figs. 1(a,b).

The temporal behavior of the departure of the bailout-embedding trajectory from the embedded dynamics is interesting by itself. This departure is measured by components u_n and v_n of Eq. (5). Over the evolution, the embedding visits, in turn, areas where convergence and divergence from the original dynamics is reinforced. The result is typically a highly intermittent behavior where periods of exponentially small values of u and v alternate with periods with very large values of these components, as depicted in Fig. 1(c). Except for the fact that finally the fluctuations are bound to die out at some stage, this behavior is reminiscent of the so-called on-off intermittency taking place after a blowout bifurcation. In the following sections we investigate further the connection between these two dynamical phenomena.

III. BUBBLING AND BLOWOUT BIFURCATIONS

The behavior of a bailout embedding can be analyzed in connection with bubbling and blowout bifurcations. These bifurcations have been studied in dissipative dynamical systems which, due to symmetries or other constraints, have an invariant submanifold of the whole phase space that contains an attractor of the global dynamics. In such situations, one may wish to study the stability of the invariant manifold with respect to small departures in a transversal direction. This stability is indicated by the transversal Lyapunov exponent, which is growth rate $h_\perp$ of a transversal perturbation averaged over a trajectory on the manifold. Specifically, we may follow the evolution of an infinitesimal perturbation δ_n, transversal to the invariant manifold, along attractor trajectory $\mathbf{X}_n$. The transversal Lyapunov exponent is then defined as

$$h_\perp = \lim_{n\to\infty} \frac{1}{n} \ln[\delta_n/\delta_0]. \quad (6)$$

If $h_\perp < 0$ for all orbits in the manifold then any transversal perturbation will eventually die out. In this case, if there is a unique topological attractor [13] for the dynamics restricted to the manifold, it will also be a topological attractor for the full dynamics.

The situation is far more complex if the attractor itself is transversally stable while one or more periodic orbits embedded in the attractor are not, so that $h_\perp > 0$ for these orbits. When, as a consequence of changing a parameter, the system passes from the fully stable situation of the previous paragraph to one in which some transversally unstable orbits coexist with the stable attractor, it is said that a bubbling bifurcation has taken place.

In the new-born regime, named bubbling, when a trajectory visits the neighborhood of an orbit with positive $h_\perp$, a transversal perturbation temporarily grows. If the attractor in the manifold is the only attractor of the system, this local relative instability is inconsequential; at most it spoils temporarily the asymptotically safe convergence to the attractor. But, on the other hand, if the attractor is not unique, the local instability provides a gateway for a trajectory apparently converging to the stable attractor to escape and end up on another attractor. In other words, the basin of attraction of one attractor is riddled by filaments of the basin of attraction of the other. Notice that this implies that the attractor is no longer of the topological type. It is, instead, a so-called Milnor attractor because it attracts trajectories with initial conditions in a set of positive Lebesgue measure, but there is no neighborhood of the attractor from which all trajectories are attracted.

Even in the case where the attractor in the invariant manifold is unique, the bubbling regime has another interesting property that manifests itself when a small amount of noise is added to the deterministic dynamics. In the absence of noise, the overall negative transversal Lyapunov exponent implies that fluctuations away from the invariant manifold are bound to asymptotically die out. A finite noise term, however, may be considered as a permanent source of finite-time fluctuations that now are amplified on the occasions when the dynamics passes near a transversally unstable orbit. The result is a kind of noise-sustained intermittency, where short intervals of relatively highly fluctuating transversal motion alternate randomly with intervals where the motion occurs very close to the invariant manifold.

As the same parameter responsible for the bubbling bifurcation is increased further, a second threshold is commonly reached: a value at which the full attractor of the invariant manifold becomes transversally unstable. This transition is called a blowout bifurcation. Now, intermittent bursts of motion, away from the manifold, are unavoidable unless a very close matching of the initial condition of the trajectories is made to situate the trajectory on that manifold. It is enough

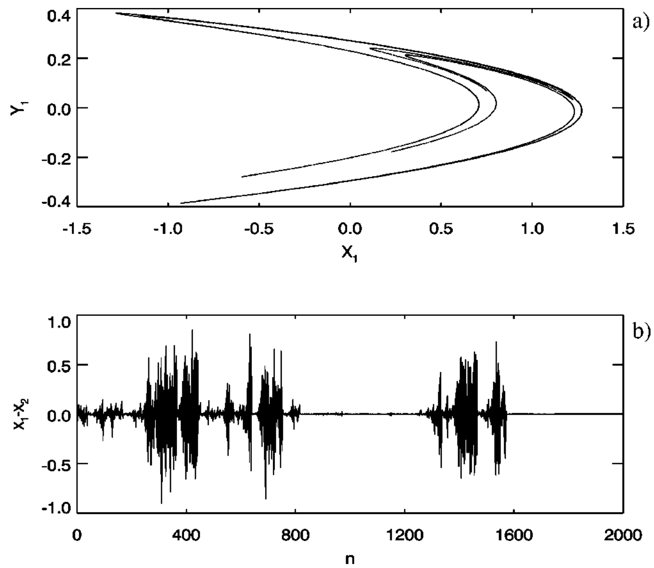

FIG. 2. (a) The Hénon attractor for the individual uncoupled maps with $a=1.4$ and $b=0.3$. (b) Time evolution for x_1-x_2 showing on-off intermittency.

to have a very small uncertainty in the initial condition to sustain the fluctuations. The regime, called on-off intermittency, has now been reached.

This behavior is illustrated by the well studied example of a system of two identical dissipative Hénon maps

$$\mathbf{x}(n+1)=\mathbf{T}\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} \equiv \begin{pmatrix} y(n)+1-ax^2(n) \\ bx(n) \end{pmatrix}, \tag{7}$$

coupled through diffusive type of interaction [14]:

$$\mathbf{x}_1(n+1)=\mathbf{T}(\mathbf{x}_1(n))+\varepsilon[\mathbf{T}(\mathbf{x}_2(n))-\mathbf{T}(\mathbf{x}_1(n))],$$
$$\mathbf{x}_2(n+1)=\mathbf{T}(\mathbf{x}_2(n))+\varepsilon[\mathbf{T}(\mathbf{x}_1(n))-\mathbf{T}(\mathbf{x}_2(n))]. \tag{8}$$

Obviously, manifold $(x_1,y_1)=(x_2,y_2)$ is dynamically invariant and hosts the same attractor as the individual uncoupled maps; Fig. 2(a) displays the attractor. In Fig. 2(b) we show the temporal behavior of the mismatch between the x coordinates of each map for an ε value for which the computed transversal Lyapunov exponent is positive, i.e., in the regime of on-off intermittency. The relevance of this example here is to show the strong similarity with the behavior reported in Fig. 1(c). However, it should be remarked that while the wild fluctuations of x_1-x_2 in the present case never cease to occur, the fluctuations of u_n in the former only last for a finite period of time, until the trajectory finally settles within a KAM torus.

IV. BAILOUT EFFECT AND BLOWOUT BIFURCATION

The discussion in Sec. III concerns dissipative systems. This is important in the sense that the invariant submanifold under consideration may contain an attracting set for trajectories starting either in the manifold itself or in the rest of the phase space.

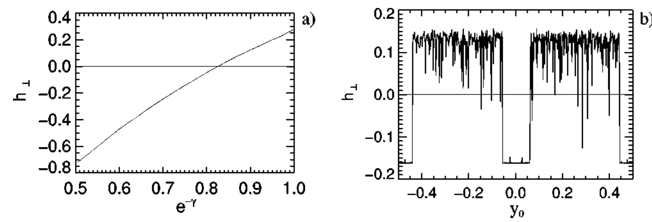

FIG. 3. For the bailout embedding of the standard map with $k=1.5$, the transverse Lyapunov exponent $h_\perp$ vs. (a) parameter $e^{-\gamma}$ for $0.5\leqslant e^{-\gamma}\leqslant 1$ and (b) the initial condition $y_0\in[-0.5,0.5], x_0=-0.5$ for $e^{-\gamma}=0.85$.

In order to show that the bailout effect corresponds to the occurrence of a blowout bifurcation, we again consider the specific example of the standard map. The phase space for the bailout embedding of this problem is four dimensional, with coordinates (x_n, y_n, u_n, v_n), and its dynamics was defined in Sec. II. It is clear from the construction of the embedding that $u_n=v_n=0$ is an invariant two-dimensional submanifold of the four-dimensional phase space; if $u_n=v_n=0$ at $n=0$, Eqs. (5) imply that this is so for all $n>0$.

We now consider again the dynamics of infinitesimal perturbations $(\delta u_n, \delta v_n)$ transverse to the invariant manifold. In order to compute the transversal Lyapunov exponent, one has to take a trajectory on the invariant manifold and plug it into the linearized equations for the perturbations. Notice that the last two Eqs. (5) are linear in u and v, and therefore represent also the evolution of δu_n and δv_n. Note also that the trajectories on the invariant manifold are the trajectories of the original standard map obtained by setting u and v to zero in the first two Eqs. (5). Plugging a typical chaotic solution (x_n, y_n) of the standard map into the last two we compute its transversal Lyapunov exponent $h_\perp$ by setting $\delta_n=\{[\delta u_n]^2+[\delta v_n]^2\}^{1/2}=\{u_n^2+v_n^2\}^{1/2}$ in Eq. (6).

Figure 3(a) shows a plot of $h_\perp$ versus parameter $e^{-\gamma}$ for a specific chaotic trajectory of the standard map. We see that $h_\perp$ increases with increasing $e^{-\gamma}$, changing sign at the critical value of $\gamma=\gamma_c\approx 0.3$. One can say that at this value, a Hamiltonian version of a blowout bifurcation occurs. Let us clarify the special characteristics imposed by the Hamiltonian dynamics. By definition, $h_\perp$ is an average over the whole chosen chaotic trajectory, and therefore the change in its sign has the same implications as in the case of the dissipative dynamics in so far as that particular chaotic region is concerned. There are, however, two main differences. First, in the Hamiltonian case, no trajectory, and in particular, no chaotic trajectory, is an attractor. Therefore, the positiveness of the transversal Lyapunov exponent only has an effect on the trajectories starting in the corresponding chaotic region on the invariant manifold. On the other hand, in Hamiltonian systems one typically encounters a very complex coexistence of chaotic regions with nearly integrable ones, the KAM islands, and the corresponding $h_\perp$ are completely unrelated since no trajectory can visit both regions. As a consequence, one can typically find that while trajectories are forced to

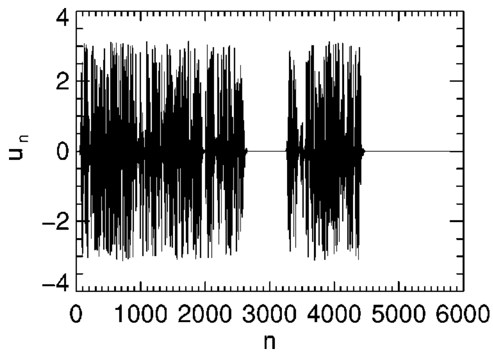

FIG. 4. Time evolution of u_n for a randomly selected initial condition, for $\gamma \leq \gamma_c$. Transient on-off intermittency is clearly seen before the embedding finally collapses inside the KAM tori.

diverge from the invariant manifold in some regions they may reconverge onto it in some others.

To demonstrate that this is the situation with the bailout embedding of Hamiltonian systems, we study the behavior of the transversal Lyapunov exponent as a function of the initial conditions in the bailout-embedded standard map. We compute $h_\perp$ for a given value of γ above the bifurcation and initial conditions uniformly distributed on the one-dimensional line defined by $y_0 \in [-0.5,0.5], x_0 = -0.5$. Since the dynamics in the invariant subspace are Hamiltonian, the line of initial conditions cuts both chaotic areas and KAM surfaces. Figure 3(b) shows the transversal Lyapunov exponent along this line, making it evident that it is positive for most chaotic trajectories, but negative in the regular regions where the embedding finally converges.

By construction, the bailout embedding of any dynamical system does not possess global attractors other than those in the invariant manifold. This is crucial to ensure that the blowout bifurcation leads to on-off intermittency. However, the Hamiltonian nature of the dynamics restricted to the invariant manifold implies the coexistence of sets of orbits that are transversally unstable with other sets that are not. Embedding trajectories starting in the neighborhood of the invariant manifold are repelled from it and bounce around the phase space, coming back to the invariant manifold and diverging away from it in an intermittent fashion, until they eventually arrive close enough to one of the transversally stable sets to become trapped. The on-off intermittency displayed in this case is therefore only transient, as we can appreciate in Fig. 4.

Lastly, we note that on decreasing γ, progressively more orbits on the invariant manifold become transversally unstable, and the number of trajectories, starting from random initial conditions, that eventually settle into the KAM tori in this way increases, as we anticipated in Sec. II.

V. BLOWOUT BIFURCATIONS IN THE PRESENCE OF NOISE: BUBBLING AND AVOIDANCE

The alteration of the dynamical behavior around bubbling and blowout bifurcations in the presence of imperfect symmetry and noise has been studied by several authors [2,4–6,15]. Among the many observed effects, it is particularly important for our purposes to recall that the addition of noise to a system that experiences a blowout bifurcation may lead to dynamics qualitatively similar to on-off intermittency, but appearing before the actual blowout bifurcation has occurred. This regime has been dubbed bubbling [2] because its onset is associated with the occurrence of a bubbling bifurcation.

In turn, the properties of bailout embeddings in the presence of a small amount of white noise have been studied [16,17]. The noisy bailout embedding of a map is defined as

$$x_{n+1} = f(x_n),$$
$$x_{n+2} - f(x_{n+1}) = e^{-\gamma} \nabla f|_{x_n} [x_{n+1} - f(x_n)] + \xi_n, \tag{9}$$

where, as before, the gradient of the map has been taken as the bailout function. The additional noise term ξ_n here is supposed to have the statistics

$$\langle \xi_n \rangle = 0,$$
$$\langle \xi_n \, \xi_m \rangle = \varepsilon (1 - e^{-2\gamma}) \delta_{mn} I. \tag{10}$$

It has been shown that as the bailout parameter is changed while keeping the noise intensity fixed, two regimes displaying increasingly strong modulations of the invariant density appear [16]. At first the bailout is globally stable, but fluctuations around the stable embedding are restored towards the stable manifold at different rates, thereby acquiring different expectation values at different points on the manifold. This behavior leaves a mark on the invariant density that can be described by means of a mechanism similar to spatially modulated temperature [18,19]. Indeed, the transversal fluctuations can be shown to be locally proportional to the noise amplitude with a space-dependent prefactor that only depends on dynamical quantities. The dynamics thus prefer to escape the hot regions of the invariant manifold (the phase space of the embedded system) and to freeze onto the cold ones. This is balanced in a nontrivial fashion by mixing in the map, to create interesting scars in the invariant density. This regime has been called avoidance. As the bailout parameter is changed, the noise prefactor eventually diverges and the embedding loses stability at some points. This is the stage described in Sec. II, in which the embedding trajectories detach from those of the original system; to distinguish it from the avoidance regime it has been termed detachment [16].

Let us now show that the avoidance regime is a Hamiltonian manifestation of bubbling. For this purpose, we first investigate the behavior of the finite-time transversal Lyapunov exponent $h_\perp$ for a bailout parameter below the onset of detachment. In Fig. 5 we have plotted a histogram showing the values of such an exponent computed for a thousand different initial conditions for the bailout embedding of the standard map in a regime where relatively large

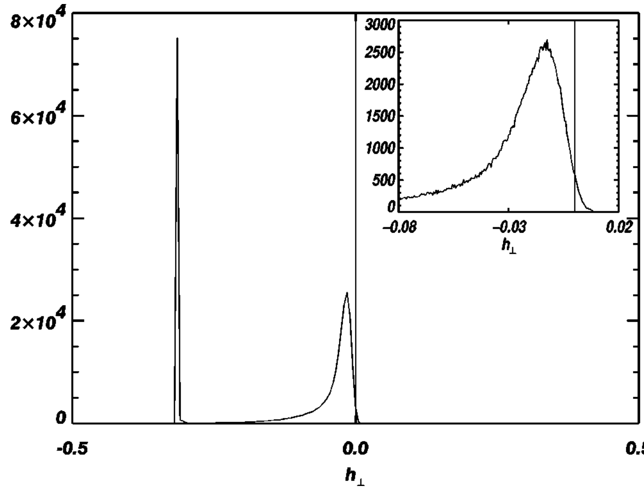

FIG. 5. For a parameter setting below the blowout bifurcation, $e^{-\gamma}=0.65$, a histogram of the finite-time transverse Lyapunov exponent $h_\perp$ computed from 1000 initial conditions randomly chosen in the phase space of the standard map with $k=1.5$. Although the asymptotic value of $h_\perp$ for trajectories on the chaotic sea is slightly negative, at finite times there are many trajectories that experience repulsion from the invariant subspace, reflected by the spread of this histogram into the positive region.

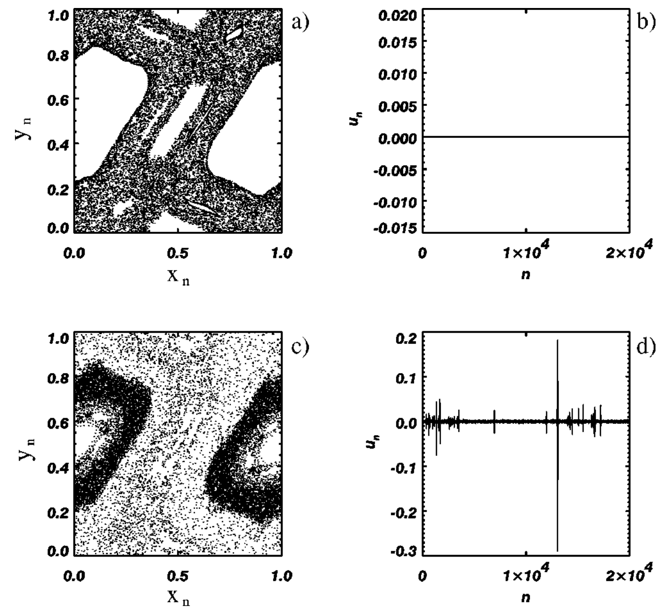

FIG. 6. For the same parameters as in Fig. 5 [(a) and (c), respectively], a trajectory of the standard map with and without the added noise term; (b) and (d) the time evolution of the mismatch u_n in both cases.

KAM islands coexist with broad chaotic regions. The calculations were carried out over a large number of iterations. Notice first that the histogram is composed of two peaks centered around negative values of $h_\perp$. The most negative peak corresponds to initial conditions lying within KAM islands where the embedding is known to be stable even for larger values of the bailout parameter. The second peak, closer to zero but still negative, corresponds to initial conditions within the chaotic sea. Notice that this peak has a tail that includes positive values of $h_\perp$. On an average, however, $h_\perp$ is clearly negative even if it is restricted to the individual peaks. Moreover, both peaks converge to Dirac distribution functions supported at negative values as the computation time for $h_\perp$ increases. This behavior is the signature of a bubbling type of bifurcation within the chaotic region: a few individual unstable orbits acquire positive transversal exponents while the whole chaotic trajectory is still transversally stable.

The similarity of the bubbling and avoidance regimes is illustrated in Fig. 6 with the same embedding of the standard map. The bailout parameter is the same as in Fig. 5, i.e., below the onset of detachment. Figures 6(a,b) represent the noise-free situation: the embedding trajectory coincides with a chaotic trajectory of the embedded system, and this coincidence is stable. In this situation, any initial mismatch transversal to the invariant manifold decays irreversibly to zero. In Figs. 6(c,d), on the other hand, a small amount of noise has been added to the embedding in the way indicated in Eq. (9). We can see that as a consequence of this noise term, on one hand the boundaries of the chaotic region have became fuzzier and on the other, a bursting mismatch between the embedding and the original system is now sustained over time. This noise-driven intermittent behavior is typical of systems in a bubbling regime.

VI. CONCLUSIONS

We have demonstrated here that two distinct dynamical behaviors, previously studied as unrelated phenomena, are different manifestations of a common situation. The blowout bifurcation, which arises naturally in the synchronization of chaotic oscillators, or in continuum systems with spatial symmetry, is usually accompanied by undesirable phenomena, such as riddled basins or on-off intermittency, that spoil the synchronism [20–22]. The difference between these scenarios and bailout embedding is the Hamiltonian nature of the dynamics restricted to the invariant manifold in the latter case. The nonexistence of an attractor in the invariant manifold in a bailout embedding permits one to avoid these undesirable phenomena, and at the same time allows one to exploit them. This is done by transforming a selected set of orbits into global attractors for the dynamics, allowing the embedding to target islands of order within the chaos.

One interesting outstanding issue is whether the scaling behavior, found in the intermittent dynamics of attractors experiencing bubbling and blowout bifurcations, has a counterpart in the Hamiltonian case. This is a nontrivial question for two reasons. One is the transient nature of Hamiltonian on-off intermittency (the bailout effect) and the other is the fact that the statistics of the intermittent behavior of these systems under the action of noise is strongly sensitive to the statistics of KAM island sizes, which is highly nonuniversal. This strongly affects the typical time scale of transient intermittency in the absence of noise. We shall explore these problems in future.

ACKNOWLEDGMENTS

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