Bailout Embedding as a Blowout Bifurcation

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We show that the bailout embedding of a Hamiltonian dynamical system provides an example of blowout bifurcation with conservative dynamics on the invariant manifold. The detachment of the embedding trajectories from the original ones can thus be thought of as transient on-off intermittency.

§1. Introduction

Symmetry plays an important role in physics from fundamental formulations of basic principles to concrete applications. In dynamics, for instance, symmetries usually imply the existence of lower dimensional submanifolds of the full phase space that are dynamically invariant because symmetric states must evolve into symmetric states. Often the motion restricted to these invariant manifolds (IM) is chaotic. This happens for example, in some extended systems with spatial symmetry or in synchronized chaotic oscillators. There, the existence of chaotic attractors in the restricted dynamics has interesting consequences for the behavior of the full system.[1),2)]

One interesting question is whether the attractors on IM's are also attractors of the unrestricted dynamics. Ashwin et al.[1)] examine the situation where a parameter controls the dynamics transversal to the IM while having no effects on the restricted dynamics. Below a critical value of this parameter, the manifold is transversally stable and any attractor there is also a global attractor. Increasing the parameter some invariant set embedded in the attractor become transversally unstable. While most trajectories initially close to the attractor in the IM may still remain close to it, there is now a small set of points in any neighborhood of the attractor that diverge from the IM. When the parameter exceeds a critical value, the full attractor becomes transversally unstable. This scenario, named *blowout bifurcation*, may lead either to the so-called riddled basins or to a strong temporal bursting called on-off intermittency[2)] depending on the nature of the dynamics away from the IM.

We have so far assumed that both the restricted and the unrestricted dynamics is dissipative. We might envision, however, a situation where a global dissipative dynamics has an IM where the restricted dynamics is conservative, i.e. described, for example, by a Hamiltonian flow or a volume-preserving map. The orbits on the IM should now exhibit the typical Hamiltonian phase space structure characterized by the coexistence of Kolmogorov-Arnold-Moser (KAM) tori and chaotic regions instead of attractors. However, since the full dynamics is dissipative it may happen that part or even the whole family of the restricted Hamiltonian orbits be global attractors. Moreover, it would also be possible that a parameter controls the transversal stability of the IM in the same way as in blowout bifurcations but now governing which part of the Hamiltonian dynamics is globally attracting and which part is not.

The main purpose of this paper is to show that this kind of behavior has been actually observed in a natural system — finite size impurities driven by an incompressible flow — and put constructively to work in the different context of the study of chaotic dynamical systems by means of the technique of bailout embedding.

§2. Bailout embedding

In contrast to the symmetry induced restriction of a dynamics to an IM of lower dimensions, dynamical systems can also be embedded into other dynamics on a larger dimensional phase space. For instance the so-called bailout embedding of an arbitrary dynamical system $\dot{x} = f(x)$ is defined as

$$\frac{d}{dt}\left(\dot{x} - f(x)\right) = -k(x)\left(\dot{x} - f(x)\right). \tag{2·1}$$

Making $k(x) < 0$ on a set of unwanted original orbits ensures that these will be unstable orbits of the larger dynamics. The trajectories of the embedding then detach or bail out from the unwanted ones of the original system wandering in the larger phase space until they reach region with $k(x) > 0$, where they converge back to trajectories of the original dynamical system. In this way we can create a larger version of the dynamics in which specific sets of orbits are removed from the asymptotic set, while preserving the dynamics of another set of orbits — the wanted one — as attractors of the enlarged dynamical system.

A remarkable example in nature is the dynamics of passive scalars in incompressible flows which is indeed embedded in the dynamics of finite size neutrally-buoyant spherical particles with the special choice $k(x) = (\gamma + \nabla f)$.[3),4)] It has been shown that the embedding — the finite size particles — detach from the original dynamics — the passive scalars — near saddle points and other unstable regions of the base flow, converging back only in the KAM islands that are now attractors in the embedding. Applying the same form of $k(x)$ to any Hamiltonian or divergence-free flow we can devise a method to target small KAM island. We can also define the bailout embedding of an arbitrary mapping dynamics[5)] $x_{n+1} = f(x_n)$ as

$$x_{n+2} - f(x_{n+1}) = K(x_n)(x_{n+1} - f(x_n)), \tag{2·2}$$

provided that $|K(x)| > 1$ over the unwanted set. The particular choice of the gradient as the bailout function $k(x) = -(\gamma + \nabla f)$ in a flow translates now to $K(x) = e^{-\gamma}\nabla f$.

For example, the gradient bailout embedding of the Hamiltonian standard map

$$\begin{aligned} x_{n+1} &= x_n + \frac{k}{2\pi}\sin(2\pi y_n) \equiv f_x(x_n, y_n), \quad (\text{mod}1) \\ y_{n+1} &= y_n + x_{n+1} \equiv f_y(x_n, y_n). \quad (\text{mod}1) \end{aligned} \tag{2·3}$$

— where k is the parameter controlling integrability — becomes:

$$\begin{aligned} x_{n+1} &= u_n + f_x(x_n, y_n), u_{n+1} = e^{-\gamma}\left(u_n + k\cos(2\pi y_n)\cdot v_n\right), \\ y_{n+1} &= v_n + f_y(x_n, y_n), v_{n+1} = e^{-\gamma}\left(u_n + (k\cos(2\pi y_n) + 1)\cdot v_n\right). \end{aligned} \tag{2·4}$$

The most remarkable feature of this embedding is the fact that while all the original trajectories of (2·3) are also trajectories of the embedding, only the KAM tori of the former are asymptotically approached by the ones of the latter. This phenomenon has been extensively described in Refs. 6), 7).

§3. Bailout embeddings as blowout bifurcations

We now show the connection between the bailout effect and blowout bifurcations, in the example of the standard map. We first notice that by construction $u_n = v_n = 0$ define a two-dimensional IM in the four dimensional phase space of the embedded dynamics, i.e. if $u_n = v_n = 0$ at $t = 0$, Eq. (2·4) implies that $u_n = v_n = 0$ at any $t > 0$). We then consider the evolution of small perturbations $(\delta u_n, \delta v_n)$ transverse to the IM and compute the transversal Lyapunov exponent defined as

$$h_\perp = \lim_{n\to\infty} \frac{1}{n} \cdot \ln[\delta_j/\delta_0], \tag{3·1}$$

with $\delta_j = \{[\delta u_j]^2 + [\delta v_j]^2\}^{\frac{1}{2}}$ as a function of the parameter $\exp(-\gamma)$. Figure 1(a) shows the results of this computation for a specific chaotic trajectories of (2·3). Notice that $h_\perp$ increases with $\exp(-\gamma)$ and changes sign at $\gamma \approx 0.3$ indicating that this trajectory experience there a blowout bifurcation. This value is coincidental with the onset of the bailout effect. Since the dynamics on the invariant subspace is Hamiltonian, $h_\perp$ depends nontrivially on the initial conditions. In Fig. 1(b) we have plotted $h_\perp$ for a fixed γ above the bifurcation and initial conditions along a line that cuts through several chaotic regions and KAM islands. Notice that $h_\perp$ changes sign correspondingly being negative in the KAM islands, in accordance to the fact that the embedding finally settles there.

When the dynamics on the IM is dissipative with a unique global attractor lying on it, a blowout bifurcation is the onset of an instability named on-off intermittency: the system occasionally burst away from the former attractor to later return for relatively long stays. In the presence of other attractors out of the IM, the bifurcation usually riddles the basin of the attractor in the manifold with sections

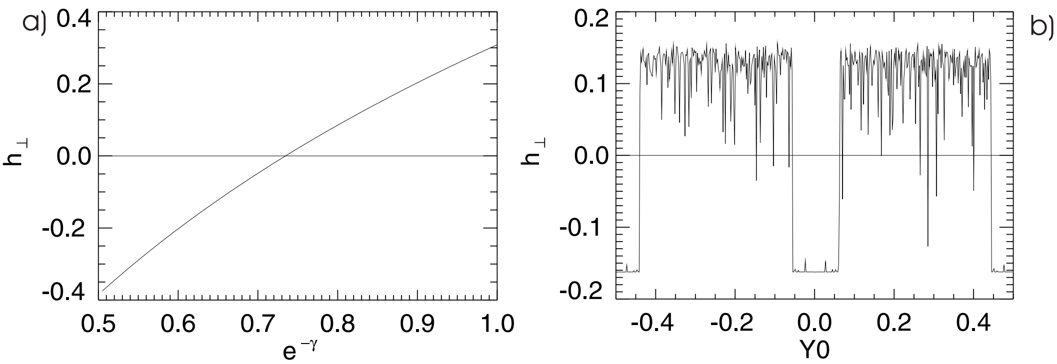

Fig. 1. (a) The transverse Lyapunov exponent $h_\perp$ versus the parameter $\exp(-\gamma)$ for $0.5 \leq \exp(-\gamma) \leq 1$. (b) $h_\perp$ versus the initial condition $y_0 \in [-0.5, 0.5], x_0 = -0.5$ for $\exp(-\gamma) = 0.85$.

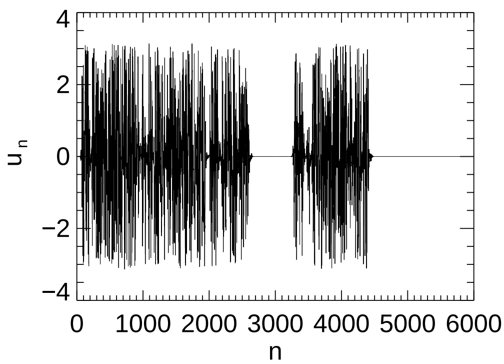

Fig. 2. Time evolution for u_n showing a transient on-off intermittency until the embedding converges to a KAM island.

of the basin of the others letting the system eventually escape to one of them to never come back. The bailout effect is distinctive in that the dynamics on the IM is conservative which technically does not allow for attractors. The embedding is also intended not to have any other global attractors than those lying in the IM. Hence, the blowout bifurcation cannot lead to riddled basins. It leads instead to a transient motion that resembles on-off intermittency produced by the orbits in the IM that are transversally unstable. However, when the dynamics returns to the IM at an orbit that is transversally stable, it remains there for ever. This is illustrated in Fig. 2.

§4. Conclusion

We have shown that the workings of the bailout embedding can be seen as a Hamiltonian instance of blowout bifurcations. In systems such as synchronized chaotic oscillators, blowout bifurcations lead to undesirable phenomena like riddled basins or on-off intermittency that spoil the synchronism.[8)−10)] In contrast, the bailout embedding positively exploits the presence of such a bifurcation and its associated on-off intermittency to force a system to avoid chaotic trajectories and target KAM islands of order for control purposes.

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