

NOISE-INDUCED ORDER OUT OF CHAOS BY BAILOUT EMBEDDING

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We review the concept of bailout embedding; a general process for obtaining order from chaotic dynamics by embedding the system within another larger one. Such an embedding can target islands of order and hence control chaos. Moreover, a small amount of noise enhances this process.

Keywords: Bailout embedding; emergence in noisy environments; noise-induced patterns; noisy self-organization.

1. Introduction

“But how does it happen,” I said with admiration, “that you were able to solve the mystery of the library looking at it from the outside, and you were unable to solve it when you were inside?”

“Thus God knows the world, because He conceived it in His mind, as if from the outside, before it was created, and we do not know its rule, because we live inside it, having found it already made.”

Dialogue between Adso and William, in Umberto Eco’s *The Name of the Rose*, Third Day: Vespers.

Many times we have a given system, we want to study it or modify its behaviour, but we cannot get inside. We are thus obliged to do so from the outside. The way that this is normally done consists of adding external degrees of freedom to which the system is coupled; we have then a bigger system of which the old one is just one slice. This powerful

idea has many incarnations, all involving immersing or embedding the original system in a larger space. For example, control theory is founded on the idea that we give liberty to a system by providing extra degrees of freedom, only to take it away by stiffening interactions that enforce what is being controlled. In particular, chaos control is usually performed in exactly this fashion [12]: extra degrees of freedom are used as a forcing to the system, which are then subjected to the system's behaviour by making the forcing a functional of the recent history of the behaviour. In this manner, one can preserve desired features of the dynamics, such as the existence of a fixed point, while completely changing the dynamics around it (i.e., by making a formerly stable fixed point unstable). Two things should be noted: (a) that this technique makes modifications on the basis of local observables, and (b) that the dynamics of the original system are largely destroyed.

In this work we show how to embed a system within a larger space, in such a way that the augmented dynamics both accomplishes a global measurement of certain properties of the system, and simultaneously forces it so as to take it away from behaviour we do not want. A fundamental point is that, in order for the measurements to exist, they must be made on an intact system; we achieve this by ensuring that a privileged slice of our embedding leaves the original system completely untouched. We call this method a bailout embedding for reasons that will promptly become clear. Bailout embedding may be considered a general process to create order from disorder. With it chaos may be controlled and islands of order encountered within a chaotic sea. The technique is applicable to both continuous and discrete systems, and to both dissipative and conservative dynamics. The addition of noise to a bailout embedding displays the quite remarkable feature that it enhances the ability of the embedding to attain order. It is possible to define a type of space-dependent temperature for the dynamics; with this, we can see that the chaotic regions are hotter and the nonchaotic regions colder, and that the system evolves into the colder regions.

The plan of the paper is as follows. In Sec. 2 we discuss embeddings and in Sec. 3 define the concept of a bailout embedding, and present an example of its use in finding regular regions in chaotic dynamics. In Sec. 4 we add noise to the bailout embedding, and show with a further example how its addition enhances the ordering effect. In Sec. 5 we discuss the relevance of these findings to fields of investigation in which the emergence of order from disorder is of interest, ranging from our previous work in fluid dynamics, to speculations on other applications.

2. Embeddings

We say an object A is embedded within a larger object B when there is a mapping $T : A \rightarrow B$, called the embedding, such that $T(A)$, the image of A under T , is somehow completely equivalent to A in the appropriate sense. This usually entails at least there being a T^{-1} whose restriction to $T(A)$ is very well behaved, like 1-1 or diffeomorphic. For example, manifolds may or may not be embedded within other larger manifolds: a sphere can be embedded in \mathbb{R}^3 , but a Klein bottle cannot. Dynamical systems can also be so embedded within larger dynamical systems. A differential equation on a space A given by $\dot{x} = f(x)$, $x \in A$, $f \in T(A)$ is embedded within a differential equation in a larger space B given by $\dot{y} = g(y)$, $y \in B$, $g \in T(B)$ through a function $T : A \rightarrow B$ if: (a) A is properly embedded in B by T : $T(A) \equiv A$ and $T^{-1}|_{T(A)}$ is a diffeomorphism; and (b) the Jacobian of T maps f to g : $g = dT f$. Thus, if an initial condition in B is chosen

exactly on $T(A)$, the subsequent evolution will remain forever on $T(A)$, and the orbits can be pulled back diffeomorphically through T^{-1} to A . We call the embedding stable if a perturbation normal to $T(A)$ decays back to $T(A)$; i.e., if $T(A)$ is a stable attractor within B .

Somewhat less trivially, the ordinary differential equation $\dot{x} = f(x)$, $x \in R$ can be embedded in a larger differential equation by taking a derivative: $\ddot{x} = f'(x)\dot{x}$ or also $\ddot{x} = f'(x)f(x)$. These two embeddings are inequivalent, though both contain the original dynamical system; there are infinitely many such derivative embeddings that double phase-space dimensionality by placing a first-order ordinary differential equation within a second order one, and they can be quite inequivalent to one another. The derivative embedding $\dot{x} = f(x) \rightarrow \ddot{x} = ff'$ embeds the original differential equation within a conservative system: $\ddot{x} = -\partial_x H$, where $H = -f^2/2$. How can a dissipative dynamical system be embedded within a Hamiltonian system? It turns out that the fixed points of f are mapped by the embedding to the maxima of H , which is negative definite. Thus the isopotentials of $H + \dot{x}^2/2 \equiv 0$ are the separatrices of these maxima; since they are defined through the two ordinary differential equations $\dot{x} = \pm f(x)$, we see that our original dissipative system has been embedded into the separatrices of the larger Hamiltonian system. This embedding is intrinsically unstable since the former fixed points $f(x) = 0$, both stable and unstable, are mapped to saddle points at the maxima of H . This transformation can be extended to higher dimensions when f is curl free:

$$\begin{aligned} \dot{x}_i = f_i(x) \quad \Rightarrow \quad \ddot{x}_i &= \frac{d}{dt}f_i(x) = \sum_{j=1}^n \frac{\partial}{\partial x_j}f_i(x)\dot{x}_j \\ &= \sum_{j=1}^n f_j \frac{\partial f_i}{\partial x_j} = \sum_{j=1}^n \left(f_j \frac{\partial f_j}{\partial x_i} + f_j \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \right) \right) \\ &= \sum_{j=1}^n \frac{1}{2} \frac{\partial}{\partial x_i} (f_j f_j) \end{aligned} \tag{1}$$

when $\partial f_i/\partial x_j - \partial f_j/\partial x_i = 0$. This condition is satisfied automatically by any $f \equiv \nabla V$, though it only requires $f \equiv \nabla V$ if the space is simply connected. So any gradient flow of the form $\dot{x} = \nabla V$ can be embedded within the conservative flow $\ddot{x} = \frac{1}{2}\nabla |\nabla V|^2$. The other derivative embedding, $\ddot{x} = f'(x)\dot{x}$ is inequivalent since it is nonconservative.

Of course, embedding a system changes notions of stability, because stability refers to perturbations, and in a larger system there are all of the old perturbations plus a batch of new ones. So, even though all of the solutions of the original system are preserved, by adding new directions away from the old solutions we may transform formerly stable solutions into unstable ones in the larger setting. The trivial way to embed a system is through a cross product; for instance, $\dot{x} = f(x)$, $x \in \mathbb{M}$ is embedded within $\mathbb{M} \times \mathbb{R}$ as

$$\begin{aligned} \dot{x} &= f(x) + g(x, y), \\ \dot{y} &= \alpha y, \end{aligned} \tag{2}$$

where $g(x, y)$ is arbitrary except for requiring that $g(x, 0) \equiv 0$, which guarantees that for $y = 0$ we have the original system. If $\alpha < 0$ then y always dies out, and so we always recover the original object; in this case, we can call the embedding itself stable, in the

sense that any motion away from the embedded object takes us back. Stable derivative embeddings can be constructed rather simply. Take, for instance,

$$\begin{aligned} \frac{d}{dt}(u - f(x)) &= -k(u - f(x)), \\ \frac{dx}{dt} &= u, \end{aligned} \tag{3}$$

of which the previous examples were the $k = 0$ limit. This embedding ensures that the distance between the actual trajectory and the embedding diminishes exponentially with time for any initial condition.

3. Bailout Embeddings

We define a bailout embedding as one of the form

$$\begin{aligned} \frac{d}{dt}(u - f(x)) &= -k(x)(u - f(x)), \\ \frac{dx}{dt} &= u, \end{aligned} \tag{4}$$

where $k(x) < 0$ on a set of orbits that are unwanted, and $k(x) > 0$ otherwise. Thus the natural behaviour of a bailout embedding is that the trajectories in the full system tend to detach or bail out from the embedding into the larger space, where they bounce around. If, after bouncing around for a while, these orbits reach a stable region of the embedding, $k(x) > 0$, they will once again collapse onto the embedding, and so onto the original dynamical system. In this way we can create a larger version of the dynamics in which specific sets of orbits are removed from the asymptotic set, while preserving the dynamics of another set of orbits — the wanted one — as attractors of the enlarged dynamical system. For the special choice of $k(x) = -(\gamma + \nabla f)$, these dynamics have been shown to detach from saddle points and other unstable regions in conservative systems [1].

A striking application of Eq. (4) is to divergence-free flows, of which Hamiltonian systems are the most important class. A classical problem in Hamiltonian dynamics is locating Kolmogorov–Arnold–Moser (KAM) tori. Hamiltonian systems live between two opposite extremes, of fully integrable systems and fully ergodic ones. Fully integrable systems are characterized by dynamics unfolding on invariant tori. The KAM theorem asserts that as a parameter taking the system away from integrability is increased, these tori break and give rise to chaotic regions in a precise sequence; for any particular value of this parameter in a neighbourhood of the integrable case, there are surviving tori. The problem with finding them is that, the dynamics being volume-preserving, merely evolving trajectories either forwards or backwards does not give us convergence onto tori.

It is not hard to extend flow bailout, Eq. (4), to maps in the obvious fashion [3]. Given a map $x_{n+1} = f(x_n)$ the bailout embedding is given by

$$x_{n+2} - f(x_{n+1}) = K(x_n)(x_{n+1} - f(x_n)), \tag{5}$$

provided that $|K(x)| > 1$ over the unwanted set. (In the map system, almost any expression written for the ordinary differential equation translates to something close to an exponential; in particular, stability eigenvalues have to be negative in the ordinary differential equation case to represent stability, while they have to be smaller than one in absolute value in the map case). The particular choice of the gradient as the bailout function

$k(x) = -(\gamma + \nabla f)$ in a flow translates in the map setting to $K(x) = e^{-\gamma} \nabla f$. A classical testbed of Hamiltonian systems is the standard map, an area-preserving map introduced by Chirikov and Taylor. The standard map is given by

$$\begin{aligned} x_{n+1} &= x_n + \frac{k}{2\pi} \sin(2\pi y_n) \pmod{1}, \\ y_{n+1} &= y_n + x_{n+1} \pmod{1}, \end{aligned} \tag{6}$$

where k is the parameter controlling integrability.

In general, the dynamics defined by this map present a mixture of quasiperiodic motions occurring on KAM tori and chaotic ones, depending on where we choose the initial conditions. As the value of k is increased the region dominated by chaotic trajectories pervades most of the phase space except for increasingly small islands of KAM quasiperiodicity. Since the only factor that decides whether we are in one of these islands or in the surrounding chaotic sea is the initial condition of the trajectory, locating them may become an extremely difficult problem for large values of the nonlinearity. Let us show, however, that the bailout embedding may come to help by transforming the KAM trajectories into global attractors of the embedded system.

In order to embed the standard map, we only need to replace f and $K(x)$ in Eq. (5) with the appropriate expressions. f stems directly from Eq. (6) and, in accordance with the previous definitions, $K(x)$ becomes

$$K(x) = e^{-\gamma} \begin{pmatrix} 1 & k \cos 2\pi y_n \\ 1 & k \cos 2\pi y_n + 1 \end{pmatrix}. \tag{7}$$

Therefore, the bailout embedding of the standard map is given by the coupled second-order iterative system

$$\begin{aligned} x_{n+1} &= u_n, \\ y_{n+1} &= v_n, \\ u_{n+1} - f_x(u_n, v_n) &= e^{-\gamma} (u_n - f_x(x_n, y_n) + k \cos 2\pi y_n (v_n - f_y(x_n, y_n))), \\ v_{n+1} - f_y(u_n, v_n) &= e^{-\gamma} (u_n - f_x(x_n, y_n) + (k \cos 2\pi y_n + 1)(v_n - f_y(x_n, y_n))), \end{aligned} \tag{8}$$

where f_x and f_y are the components of the function $f(x_n, y_n) = (f_x(x_n, y_n), f_y(x_n, y_n))$.

Notice that due to the area preserving property of the standard map, the two eigenvalues of the derivative matrix must multiply to one. If they are complex, this means that both have an absolute value of one, while if they are real, generically one of them will be larger than one and the other smaller. We can then separate the phase space into elliptic and hyperbolic regions corresponding to each of these two cases. If a trajectory of the original map lies entirely on the elliptic regions, the overall factor $\exp(-\gamma)$ damps any small perturbation away from it in the embedded system. But for chaotic trajectories that inevitably visit some hyperbolic regions, there exists a threshold value of γ such that perturbations away from a legal standard map trajectory are amplified instead of dying out in the embedding. As a consequence, trajectories are expelled from the chaotic regions to finally settle in the safely elliptic KAM islands. This process can be seen clearly in Fig. 1. As the value of γ is decreased, the number of trajectories starting from random initial conditions that eventually settle into the KAM tori increases.

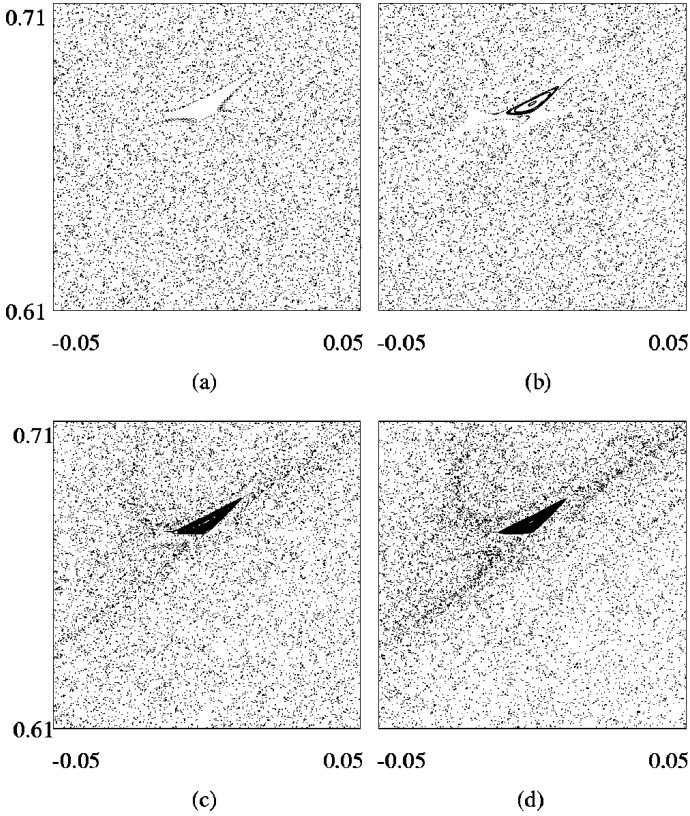


Fig. 1. The bailout process can find extremely small KAM islands. The standard map for $k = 7$ has a chaotic sea covering almost the entire torus, except for a tiny period-2 KAM torus near position $(0, 0.6774)$. 1000 random initial conditions were chosen and iterated for 20000 steps; the next 1000 iterations are shown. (a) Original map, (b) $\gamma = 1.4$, (c) $\gamma = 1.3$, (d) $\gamma = 1.2$.

A Hamiltonian system does not usually just satisfy volume conservation, but also will conserve the Hamiltonian itself. Given a flow $\dot{x} = f(x)$ with a conserved quantity $E \equiv 0$, then $f \cdot \nabla E = 0$. However, building a bailout embedding by the procedure above does not lead to dynamics that satisfy $\dot{E} \equiv 0$, because the bailout embedding should be $2n - 2$ dimensional. This is clearly undesirable in the case of Hamiltonian systems, so we may derive a bailout embedding that will obey a conservation law. The bailout equation can be written

$$\ddot{x} = (\nabla f - \gamma) \cdot (\dot{x} - f) + \nabla f \cdot \dot{x} \equiv u. \tag{9}$$

We need to correct this acceleration so that it stays on the second tangent space of the $E \equiv 0$ surface. Let us call the raw bailout acceleration u . The second derivative \ddot{x} has to

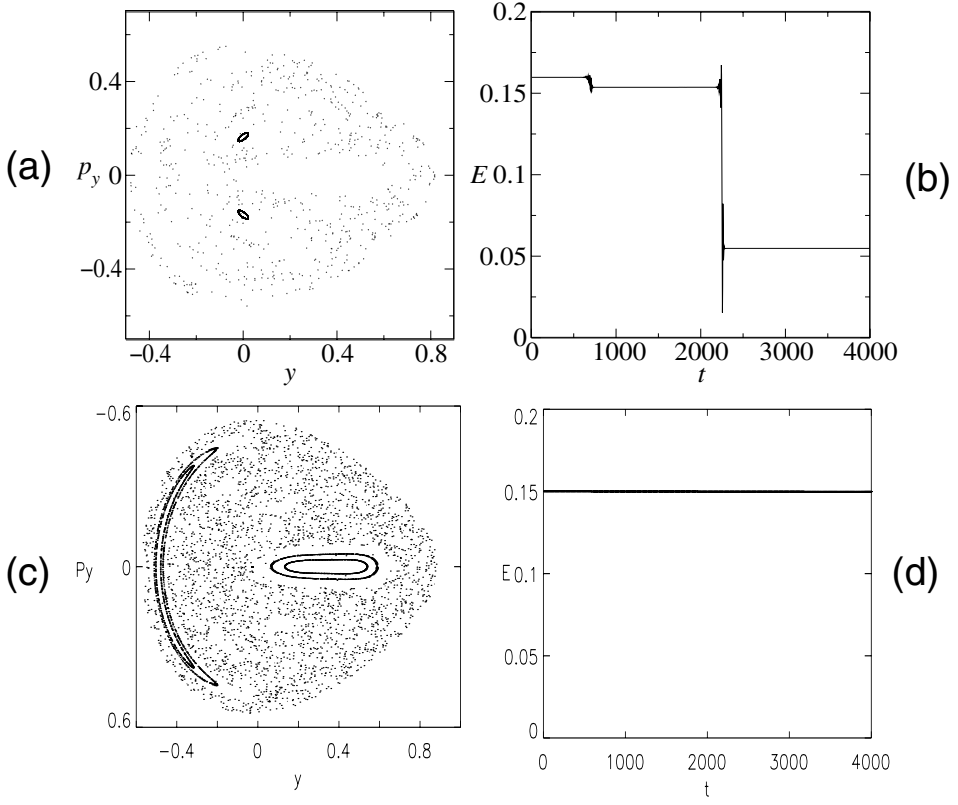


Fig. 2. (a) and (c), A bailout embedding of the Hénon–Heiles Hamiltonian finds a KAM torus in the flow, without energy conservation and with energy conservation in the process, respectively. (b) and (d), the Hénon–Heiles energy E against time in each case. In (a) and (b) there is one trajectory plotted, while in (c) and (d), two initial conditions with the same energy are used.

satisfy $\ddot{x} \cdot \nabla E + \dot{x} \cdot \nabla \nabla E \cdot \dot{x} = 0$, so we can modify u to

$$\ddot{x} = u - \frac{u \cdot \nabla E}{|\nabla E|^2} \nabla E - \frac{\nabla E}{|\nabla E|^2} \dot{x} \cdot \nabla \nabla E \cdot \dot{x}. \quad (10)$$

This equation, given that we start on $\dot{x} \cdot \nabla E = 0$, will then preserve this property. In Fig. 2 we illustrate the effect with the Hénon–Heiles Hamiltonian $H = 1/2(x^2 + y^2 + p_x^2 + p_y^2) + x^2 y - y^3/3$, which displays a conservative dynamics given by the Hamilton equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p}_x &= -\frac{\partial H}{\partial x}, \\ \dot{y} &= \frac{\partial H}{\partial p_y}, & \dot{p}_y &= -\frac{\partial H}{\partial y}. \end{aligned} \quad (11)$$

The bailout embedding can be written as:

$$\dot{X} = A + f(X),$$

$$\dot{A} = \ddot{X} - \nabla f \cdot \dot{X}. \tag{12}$$

where $X = (x, y, p_x, p_y)$, and the acceleration \ddot{X} is given by Eq. (10) or Eq. (9), with and without the constant energy modification respectively. Figures 2a and 2c show with a Poincaré section how a bailout embedding of a Hénon–Heiles flow ends up on a KAM torus, in each case. Figures 2b and 2d show energy against time for the same trajectories. Figure 2b shows a series of plateaux punctuated by rapid energy jumps, while in Fig. 2d, which includes the constant-energy modification, there are no jumps.

In contrast to the two-dimensional case, in three-dimensional volume-preserving (Liouvillean) maps the incompressibility condition only implies that the sum of the three independent eigenvalues must be zero. This less restrictive condition allows for many more combinations, and a richer range of dynamical situations may be expected. As an example, consider the bailout embedding of the ABC map [5], a non-Hamiltonian system given by

$$f = f_{ABC} : (x_n, y_n, z_n) \longrightarrow (x_{n+1}, y_{n+1}, z_{n+1}), \tag{13}$$

where

$$\begin{aligned} x_{n+1} &= x_n + A \sin z_n + C \cos y_n \pmod{2\pi}, \\ y_{n+1} &= y_n + B \sin x_{n+1} + A \cos z_n \pmod{2\pi}, \\ z_{n+1} &= z_n + C \sin y_{n+1} + B \cos x_{n+1} \pmod{2\pi}. \end{aligned} \tag{14}$$

Depending on the parameter values, this map possesses two quasi-integrable behaviours: the one-action type, in which a KAM-type theorem exists, and with it invariant surfaces shaped as tubes or sheets; and the two-action type displaying the phenomenon of resonance-induced diffusion leading to global transport throughout phase space [7]. In an application of bailout embedding to the first case we find an interesting generalization of the behaviour already found in two dimensions; particle trajectories are expelled from the chaotic regions to finally settle in the regular KAM tubes, as shown in Fig. 3. In the two-action case, no regular regions separated by barriers exist, and the map dynamics lead to global diffusion over all the phase space. In this case there exists a resonant structure that controls the dynamics. This structure can be easily recovered with the use of noise in a bailout embedding, as we shall see below.

4. Noisy Bailout Embeddings

In this Section we study the properties of bailout embeddings in the presence of an extremely small amount of white noise [4]. We show that there are two stages to the modulation of the invariant density in the small-noise limit. At first the bailout is everywhere stable, but fluctuations around the stable embedding may be restored towards the stable manifold at different rates and thus acquire different expectation values. These fluctuations leave a mark on the invariant density through a mechanism similar to spatially modulated temperature [2, 10], namely, the dynamics prefer to escape the hot regions. This is balanced in a nontrivial fashion by mixing in the map to create interesting scars in the invariant density. As the bailout parameter is changed, the noise prefactor can diverge, and the embedding loses stability at some points, so honest-to-goodness detachment ensues.

The study of Hamiltonian systems is hampered by the uninteresting features of the ergodic measures: if the system is ergodic, then it is Lebesgue automatically; if it is not, then

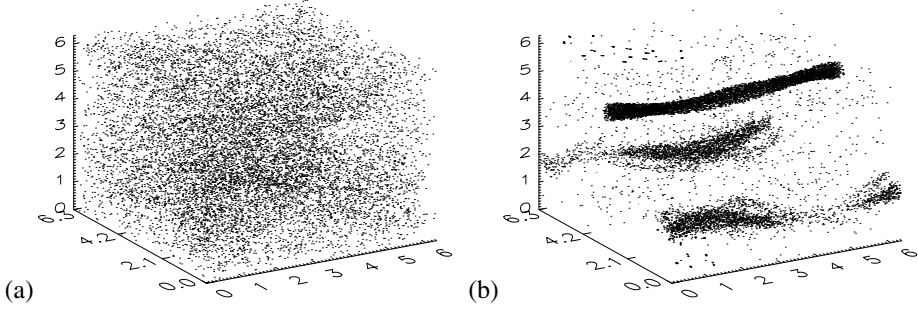


Fig. 3. A bailout embedding of the ABC map in the one-action regime demonstrates how it seeks out the regular regions of KAM tubes. For a homogeneous distribution of initial conditions we plot only the last 1000 steps of the map evolution for different values of γ : (a) $\gamma = 2$, (b) $\gamma = 0.5$. The images represent the $[0, 2\pi]$ cube in the phase space.

it does not have a unique invariant measure to begin with: in the case of KAM systems, the measure disintegrates into a millefeuille of KAM tori and ergodic regions. The addition of a small amount of white noise — i.e., coupling to a thermal bath — renders the system automatically ergodic, and a unique stable invariant measure (in the Sinai–Ruelle–Bowen sense [6, 11]) appears; but it is still the Lebesgue measure, and thus all properties to be studied are necessarily higher order. This is even a problem for plotting what the phase space looks like. Bailout embeddings allow the use of measure-theoretical studies in Hamiltonian systems, by permitting the dynamics to leave nontrivial shadows upon an invariant measure.

We study the bailout embedding

$$x_{n+2} - f(x_{n+1}) = e^{-\gamma} \nabla f|_{x_n} (x_{n+1} - f(x_n)) + \xi_n \quad (15)$$

of a dynamical system $x_{n+1} = f(x_n)$, in which as before we have used the gradient of the map as the bailout function. New here is the noise term ξ_n , with statistics

$$\begin{aligned} \langle \xi_n \rangle &= 0, \\ \langle \xi_n \xi_m \rangle &= \varepsilon (1 - e^{-2\gamma}) \delta_{mn} \mathbf{I}. \end{aligned} \quad (16)$$

We can separate this two-step recurrence into two one-step recurrences

$$x_{n+1} = f(x_n) + \delta_n, \quad (17)$$

$$\delta_{n+1} = e^{-\gamma} \nabla f|_{x_n} \delta_n + \xi_n. \quad (18)$$

Note that now the second equation is affine, being linear in the δ plus a homogeneously added noise process, so it could be solved analytically for δ if we knew what the x were in the past. Under the assumption that the δ are infinitesimally small, we get the classical orbits $x_{n+1} = f(x_n)$, and we can explicitly write down the solution for the δ

$$\begin{aligned} \delta_{n+1} &= \xi_n + e^{-\gamma} \nabla f|_{x_n} \\ &\quad \times \left(\xi_{n-1} + e^{-\gamma} \nabla f|_{x_{n-1}} \right. \\ &\quad \left. \times \left(\xi_{n-2} + e^{-\gamma} \nabla f|_{x_{n-2}} (\xi_{n-3} + \dots) \right) \right), \end{aligned} \quad (19)$$

or, after unwrapping,

$$\begin{aligned} \delta_{n+1} = & \xi_n + e^{-\gamma} \nabla f|_{x_n} \xi_{n-1} \\ & + e^{-2\gamma} \nabla f|_{x_n} \nabla f|_{x_{n-1}} \xi_{n-2} \\ & + e^{-3\gamma} \nabla f|_{x_n} \nabla f|_{x_{n-1}} \nabla f|_{x_{n-2}} \xi_{n-3} + \dots, \end{aligned} \tag{20}$$

which may be written more compactly as

$$\delta_{n+1} = \sum_{j=0}^n \left(\xi_{n-j} e^{-j\gamma} \prod_{k=0}^j \nabla f|_{x_{n-k}} \right). \tag{21}$$

Then, given that the ξ are uncorrelated, the expectation value of δ^2 is given as the sum of the squares of the terms, or

$$\frac{\langle \delta^2 \rangle}{\langle \xi^2 \rangle} = \sum_{j=0}^{\infty} \left(e^{-j\gamma} \prod_{k=0}^j \nabla f|_{x_{n-k}} \right)^2, \tag{22}$$

where the $\langle \cdot \rangle$ are averages over the ξ process. Clearly, as $\gamma \rightarrow \infty$ this expression tends to 1.

In the regime in which $\gamma \gg 0$ and $\langle \xi^2 \rangle \ll 1$, the $\langle \delta_{n+1}^2 \rangle \approx \langle \xi^2 \rangle \ll 1$ and hence the trajectories collapse upon the classical orbits: $x_{n+1} = f(x_n) + \delta_n \approx f(x_n)$. Under these circumstances, the embedding is always stable, and there is no detachment. In this regime we can compute explicitly the above expression Eq. 22 which depends only on the current value of the position:

$$T(x) = \frac{\langle \delta^2 \rangle}{\langle \xi^2 \rangle} = \sum_{j=0}^{\infty} \left(e^{-j\gamma} \prod_{k=0}^j \nabla f|_{f^{-k}(x)} \right)^2. \tag{23}$$

Thus $T(x)$ defines a sort of temperature for the fluctuations δ .

As long as the δ are infinitesimally small, they do not — and cannot — affect the x dynamics, which has collapsed unto the classical trajectories; thus they do not affect the invariant density $P(x)$ either, and hence $P(x)$ is asymptotic to the Lebesgue measure. For infinitesimally small $\langle \xi^2 \rangle$, as γ is made smaller, the sum acquires more and more terms because the prefactor $e^{-j\gamma}$ decays more and more slowly. For any value of γ , the products of the gradients grow or shrink roughly as the exponential of the Lyapunov exponent times j . Thus, when γ equals the local Lyapunov exponent at x , the series defining $T(x)$ stops being absolutely convergent at x and may blow up. As γ is lowered further, more and more points x have local Lyapunov exponents greater than γ and so $T(x)$ formally diverges at more and more points x .

Where $T(x) = \infty$ it means that $\langle \delta_n^2 \rangle$ is finite even if $\langle \xi^2 \rangle$ is infinitesimally small. Thus the embedding trajectories have detached from the actual trajectories, and the approximations given above break down. Detachment is the process that was first envisioned as being characteristic of bailout embeddings [1, 3]. However, by employing noise in the embedding and carefully controlling its use, we can see the process that occurs before detachment. If $T(x)$ is finite and smaller than $1/\langle \xi^2 \rangle$, then we have a regime in which the δ s behave as a noise term added to the classical trajectories: $x_{n+1} = f(x_n) + \delta_n$ with $\langle \delta_n^2 \rangle = \langle \xi^2 \rangle T(x)$.

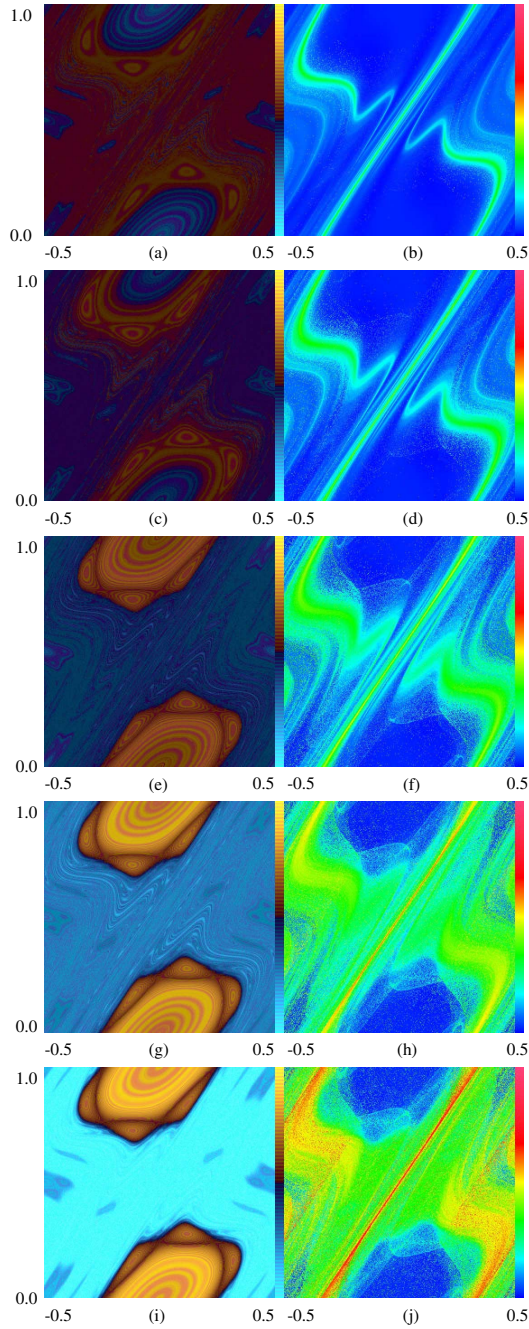


Fig. 4. Noisy bailout embedding histograms (left) and temperature plots (right) for standard map nonlinearity parameter $k = 1.5$, noise parameter $\varepsilon = 10^{-8}$, and bailout parameter (a), (b), $\gamma = 0.7$; (c), (d), $\gamma = 0.65$; (e), (f), $\gamma = 0.6$; (g), (h), $\gamma = 0.55$; and (i), (j), $\gamma = 0.5$. The striped histogram colour scale runs from yellow at high densities to cyan at low densities, while the temperature colour scale runs from red (high) to blue (low).

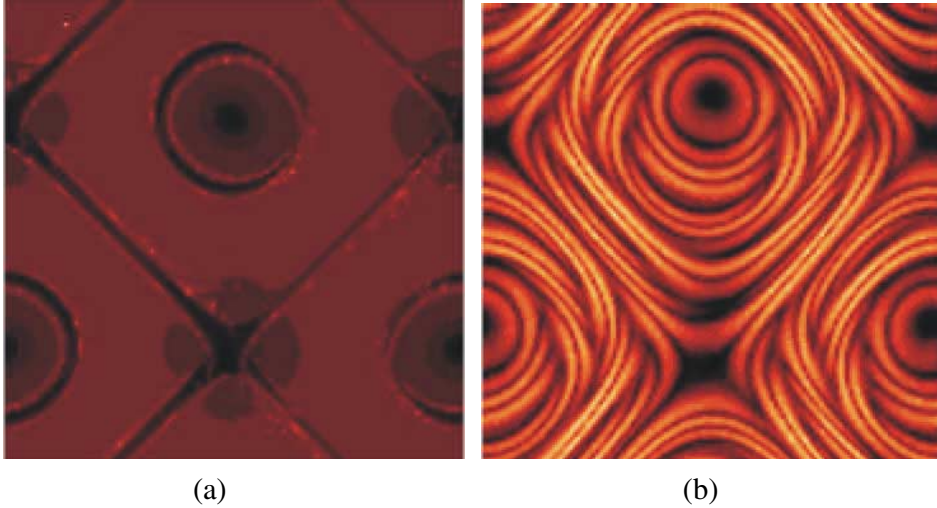


Fig 5. A noisy bailout embedding displays the temperature amplitude for the (a) two-action and (b) most chaotic cases in the ABC map. The images are the $[0,2\pi] \times [0,2\pi]$ region in the xy axis, for a slice in the z direction corresponding to the values $z \in [0,0.49]$.

We have, alas, lost the whiteness of the process, since δ_{n+1} and δ_n are not any longer statistically independent. However, this is in this case a second-order effect compared with the fact that the noise process amplitude, being modulated as a function of position, will immediately lead to inhomogeneous coverage by the dynamics: hot regions will be avoided while cold regions will preserve the dynamics. All of this is in a context in which the embedding is essentially stable throughout. Thus this process is not detachment per se, but rather avoidance.

We can illustrate this best in the context of the standard map acting as before as the base flow. Figure 4 shows side by side the visit histogram — the invariant measure — (left-hand side) together with the corresponding space-dependent temperature (right-hand side) for a decreasing sequence of the bailout parameter γ and fixed values of the standard-map nonlinearity and noise parameters. While, for γ larger than 0.55, the temperature is a well-defined function of the space coordinates, it shows signs of divergence — the red regions, which become larger as γ decreases — for γ smaller than 0.55. On the other hand, however, the invariant measure displays features related to the structure of T on both sides of this transition, i.e., even before detachment occurs.

As we anticipated at the end of the previous Section, the noisy bailout can give us useful information about the dynamics in a non-Hamiltonian context as well. In Fig. 5 we show how it works for the ABC map in those instances in which the parameters values lead to a two-action type map or even to a completely chaotic one. In the first case, Fig. 5a, the noisy bailout allows us to recover the resonant structure of the two-action ABC map; the main property of its dynamics. In Fig. 5b we apply noisy bailout to a generic chaotic case in which we do not have any information about the phase space structure. We obtain a

representative picture of how the phase space looks; note, for example, that the invariant manifolds are clearly very twisted.

5. Discussion

Embedding is a technique with the potential to be applied to many fields of research. One instance is that in which global or long-range measurements are made by use of auxiliary variables. In this case our original system X is used as a forcing for another system Y ; it is the internal dynamics of Y that keep a memory of what is being measured. The most elementary and best studied instance of this are infinite impulse response filters (IIRFs) [9, 13]. A trivial example is given by $y_{i+1} = \alpha y_i + (1 - \alpha)x_i$, where the x_i are observations of our system; the y_i then keep running averages with an exponential memory of the input, i.e., an arbitrarily long memory has been achieved on the basis of the extra degrees of freedom added. Another example is the calculation of a Lyapunov exponent, the decay constant that measures how perturbations grow or shrink around a given trajectory of a dynamical system. If the system X is given by the ordinary differential equation $\dot{x} = f(x)$, then perturbations δx around a given solution $x(t)$ of this equation will evolve as $\dot{\delta x} = \nabla f|_{x(t)}\delta x$, an inhomogeneous linear equation. This calculation is numerically accomplished by keeping track of the compound matrix; hence, by adding extra degrees of freedom to a system, we can compute all Lyapunov exponents of the trajectory, a dynamically important global measure of a system's behaviour. Note that the dynamical behaviour of the system Y permits in general extremely complex calculations.

Bailout embeddings constitute a subclass of embeddings with much utility for obtaining order from chaotic systems. Here, as in previous work, we have provided examples of the use of bailout embeddings for finding regularity — quasiperiodic orbits — in both Hamiltonian maps [3, 4] and flows [1], and in the more general setting of volume-preserving maps [5]. We have also demonstrated that the addition of a small amount of noise enhances the bailout process in a nontrivial fashion [4]. With noisy bailout we can define an effective space-dependent temperature, with which we can appreciate that the system evolves into the colder regions of the flow. We have shown in other works the applicability of bailout embeddings to fluid dynamics, where they describe the dynamics of small neutrally buoyant particles in both two [1], and three-dimensional [5] incompressible fluid flows. Such flows are conservative systems, and learning in many evolutionary games can also be put into a conservative form [8]. In particular, the rock–paper–scissors game, which is the simplest non-trivial model for a network of interacting entities — which may be species, economies, people, etc — none of which is an outright winner in competition between the three (scissors beat paper, rock beats scissors, but paper beats rock), displays Hamiltonian chaos [14]. Could bailout embeddings contribute to learning in evolutionary games? Certainly, as we have seen, a bailout embedding allows chaos to be controlled by targeting zones of order within the phase space of the system. An entity using a bailout embedding to conduct its strategy of competition could take advantage of this dynamical mechanism; it could perturb the system to control it to attain a learning capability. We may speculate that such a process could occur in nature; it could be useful in evolutionary game theory, in Lotka–Volterra population dynamics, in voter dynamics, or possibly even in human learning. While the latter ideas are purely speculative, bailout embeddings provide a general mechanism capable of creating order from chaotic systems, which is even more effective in the presence of noise.

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