# Phase Instabilities in the Laser Vector Complex Ginzburg-Landau Equation

Maxi San Miguel*
*Department of Mathematics–Arizona Center for Mathematical Sciences, University of Arizona, Tucson, Arizona 85721*
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The interplay between the polarization state of light and transverse effects in lasers is analyzed through an amplitude equation description of an atomic transition between spin sublevels. Linearly polarized traveling waves are found, whose stability is restricted by a phase instability associated with the direction of polarization. The instability persists for polarization stabilized lasers. Novel states of laser light such as standing waves with a spatially periodic linear polarization or coexisting traveling waves with different wave numbers and circular polarizations are also found.

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The vector complex Ginzburg-Landau equation (VCGLE) [1,2] gives a generic description of spatiotemporal phenomena in systems described by a complex vector field. Lasers give very appealing opportunities for the study of such phenomena due to the vector nature of the complex amplitude of the electric field. The vector degrees of freedom are associated with the polarization of laser light. Polarization instabilities are currently studied in a variety of lasers [fiber, microchip Nd:YAG, and vertical cavity surface emitting lasers (VCSELs)] [3]. In particular, the lack of stability of transverse and polarization modes in VCSELs is known to produce a rich spatiotemporal phenomenology. In spite of the peculiarities of different lasers, a classification and understanding of possible states close to threshold is provided by a VCGLE. In this Letter I use such a framework to discuss pattern forming instabilities in wide aperture lasers [4,5]. The combination of spatial transverse dependence and polarization state is shown to lead to states of laser light so far unexplored. It is also shown that the stability of these states is determined by polarization phase instabilities.

The study of phase instabilities is linked to the concept of spontaneous symmetry breaking. Generally speaking, a neutral (zero energy) mode is associated with a state that breaks a continuous symmetry of the system. In equilibrium, thermal fluctuations excite nearby low energy modes and destroy long range order in low dimensional systems. On the other hand, the stability of nonequilibrium states is often restricted by phase instabilities in which the zero energy mode becomes unstable with respect to long-wavelength fluctuations. For example, in pattern forming systems such as Rayleigh-Bénard, the range of stable spatially periodic states is limited by an Eckhaus phase instability [6]. The consequences of rotational invariance for a vectorial order parameter are also well understood in equilibrium; for example, spin waves can destroy ferromagnetic order. However, phase instabilities associated with rotational symmetry have not yet been considered at length in nonequilibrium systems such as the laser. The complex vector field describing laser light has two neutral modes: a global phase $\theta$ and a rotational phase $\psi$. Spontaneous emission noise causes fluctuations of $\theta$ destroying frequency coherence and giving a finite linewidth. This is the relevant laser phase when the vector direction is fixed by external symmetry breaking such as Brewster windows (scalar case). Otherwise, the additional phase $\psi$, associated with the state of polarization of laser light, needs to be considered. For an isotropic laser cavity emitting linearly polarized light, $\psi$ determines the orientation of the linearly polarized emission on the transverse plane. The selection of $\psi$ breaks the rotational symmetry.

Laser transverse pattern formation has been associated with an instability of the global phase $\theta$ [7]: For a detuning $\Omega < 0$ between atomic and cavity frequencies, the amplitude equation description of the scalar laser instability features a CGLE [8,9]. The stability range of traveling wave solutions of the CGLE around the spatially homogeneous state is limited by a $\theta$ instability of the Eckhaus type. For $\Omega > 0$ [5,9] an amplitude equation description requires two coupled CGLE's and the spatially homogeneous lasing solution is $\theta$ unstable [10]. When the polarizarion degree of freedom is not frozen, $\psi$ instabilities associated with polarization patterns occur. Polarization phase instabilities restrict the stability range of the laser states allowed in the scalar case.

A VCGLE can be written on symmetry grounds, but the determination of the parameters in the equation requires a specific physical model. The intrinsic polarization of laser light is of quantum nature, and it originates in the spin sublevels of the atomic lasing transition [11]. Purely temporal polarization instabilities have been studied using an homogeneously broadened $J = 1 \longrightarrow J = 0$ atomic transition as a prototype situation [12]. I will consider here transverse effects in this same transition [13]. Specifically, I consider a wide aperture single longitudinal mode laser with transverse flat end reflectors. The upper $J = 1$ level is triply degenerate but, neglecting small longitudinal components of the electric field, dipole transitions are only allowed from the states $J_z = \pm 1$ to the lower $J = 0$ level. The complex slowly varying amplitude $\mathbf{E} = (E_x, E_y)$ of the vector electric field can be decomposed into its right and left circularly polarized components,

$$E_\pm = \frac{1}{\sqrt{2}}(E_x \pm iE_y). \quad (1)$$

The components $E_\pm$ originate in the transitions from the states $J_z = \pm 1$, respectively. The Maxwell-Bloch equations for the system can be written as

$$\partial_t E_\pm = i\frac{c^2}{2\nu}\partial_x^2 E_\pm - (\kappa + i\nu)E_\pm - ig^* P_\pm, \quad (2)$$

$$\partial_t P_\pm = -(\gamma_\perp + i\omega_0)P_\pm + i\frac{g}{2}(D_1 \pm D_2)E_\pm + igC_\pm E_\mp, \quad (3)$$

$$\partial_t D_1 = -\gamma_\parallel(D_1 - 2\sigma) + 3i(g^* E_+^* P_+ - gE_- P_-^* - \text{c.c.}), \quad (4)$$

$$\partial_t D_2 = -\gamma_J D_2 + i(g^* E_+^* P_+ + gE_- P_-^* - \text{c.c.}), \quad (5)$$

$$\partial_t C_+ = -\gamma_c C_+ + i(g^* E_-^* P_+ - \text{c.c.}), \quad (6)$$

where $P_\pm$ are the complex dipole polarizations for each allowed transition, $D_1$ is the sum of the population differences between the upper $J_z = \pm 1$ levels and the lower $J = 0$ level, $D_2$ is the population difference between the two upper levels, $C_+$ is the density matrix coherence between the two $J_z = \pm 1$ levels, and $C_- = C_+^*$. The $\nu$ parameter is the cavity frequency, the detuning $\Omega = \omega_0 - \nu$, $g$ is the coupling parameter, $\sigma$ the pump parameter, and $\kappa, \gamma_\perp, \gamma_\parallel, \gamma_J, \gamma_c$ are relaxation constants. Possible different relaxation mechanisms lead in general to different states of polarization [14]. While in general $\gamma_J, \gamma_c > \gamma_\parallel$, only the case $\gamma_\parallel = \gamma_J$ was considered in Ref. [12]. The introduction of $\gamma_J$ [14,15] allows for circularly or linearly polarized light depending on the ratio $\gamma_J/\gamma_c$.

Close to threshold, the set of equations (2)–(6) can be reduced, for $\Omega < 0$ [16], to two coupled equations for the amplitudes $A_\pm$ of the dominant $k = 0$ Fourier mode of $E_\pm$. In appropriate rescaled units one obtains

$$\partial_t A_\pm = \mu A_\pm + (1 + i\alpha)\partial_x^2 A_\pm - (1 + i\beta)(|A_\pm|^2 + \gamma|A_\mp|^2)A_\pm \quad (7)$$

These equations are equivalent to the following VCGLE for the vectorial amplitude $\mathbf{A}$:

$$\partial_t \mathbf{A} = \mu\mathbf{A} + (1 + i\alpha)\partial_x^2\mathbf{A} - (1 + i\beta) \times \{(\mathbf{A}\cdot\mathbf{A}^*)\mathbf{A} + [(\gamma - 1)/2](\mathbf{A}\cdot\mathbf{A})\mathbf{A}^*\}. \quad (8)$$

The parameters $\mu, \alpha, \beta$ have exactly the same expression in terms of the original physical parameters as for a scalar two level model [9]: $\mu$ measures the distance to threshold, $\alpha$ originates in the diffraction, and $\beta$ is associated with detuning. The coupling parameter $\gamma$ between right and left circularly polarized components can be obtained by direct adiabatic elimination of material variables in the absence of transverse effects,

$$\gamma - 1 = \frac{2\gamma_\parallel}{3\gamma_J + \gamma_\parallel}\left(\frac{\gamma_J}{\gamma_c} - 1\right). \quad (9)$$

The novel features associated with a vectorial description of the laser instability originate in the last term in (8), which has no counterpart in a scalar description. The fact that $\gamma$ is a real number, together with $1 + \alpha\beta > 0$ [9], are the two main peculiarities of the version of the VCGLE appropriate to describe laser systems.

A family of solutions of Eq. (7) is

$$A_\pm = Q_\pm \exp[-ik_\pm x + i\omega_\pm t + i(\theta_0 \pm \psi_0)], \quad (10)$$

where the real amplitudes $Q_\pm^2 = [\mu(1 - \gamma) + \gamma k_\pm^2 - k_\mp^2]/(1 - \gamma^2)$, the frequencies $\omega_\pm = -\alpha k \pm^2 - \beta(Q_\pm^2 + \gamma Q_\mp^2)$, and $\theta_0$ and $\psi_0$ are arbitrary phases. Solutions with either $Q_+ = 0$ or $Q_- = 0$ correspond to circularly polarized light. They are only stable for $\gamma > 1$ and will not be considered further in this paper where I will focus on the case $\gamma < 1$. The simplest particular case of (10) occurs for $k_\pm = 0$. It corresponds to linearly polarized light with $A_\pm$ having the same amplitude and frequency. The global phase $\theta_0$ is the usual arbitrary field phase found in the scalar case and $\psi_0$ defines the arbitrary direction of linear polarization: $A_x \propto \cos\psi_0, A_y \propto \sin\psi_0$. For $k_\pm \neq 0$ and because of the pure intensity coupling in (7), in general $A_\pm$ do not have a common frequency and there is no well defined polarization (depolarized solutions). However, for wave numbers $k_- = \pm k_+$, $A_\pm$ have the same amplitudes $Q_\pm = Q$ and frequencies $\omega_\pm = \omega$, so that a linearly polarized state can be defined as explained in cases (a) and (b) below. A special case occurs in the limit $\gamma = 1$. For such marginal coupling, solutions of the family (10) only exist for $k_- = \pm k_+$. They correspond to elliptically polarized light with a common frequency for $A_\pm$ and a third arbitrary phase $\Sigma = \tan(Q_+/Q_-)$ associated with the ellipticity.

The linear stability analysis of the family of solutions (10) identifies two vanishing eigenvalues at zero wave number of the perturbation ($q = 0$). They are associated with the arbitrary phases $\theta_0$ and $\psi_0$. The stability with respect to long-wavelength fluctuations is characterized by phase equations for slowly varying $\theta(x,t)$ and $\psi(x,t)$. Long-wavelength instabilities are characterized in the following for the different solutions mentioned above:

*(a) Linearly polarized traveling waves (TW).*—These solutions occur for $k_\pm = K$. They are linearly polarized with an arbitrary direction $\psi_0$. They are the natural generalization of the traveling waves previously considered in the scalar case [5,9]. The phase equations turn out to be decoupled:

$$\partial_t \theta = 2K(\alpha - \beta)\partial_x\theta + D_\theta \partial_x^2\theta, \quad (11)$$

$$\partial_t \psi = 2K(\alpha - \beta)\partial_x\psi + D_\psi \partial_x^2\psi, \quad (12)$$

$$D_\theta(K) = 1 + \alpha\beta - \frac{2(1+\beta^2)K^2}{\mu - K^2}, \quad (13)$$

$$D_\psi(K) = 1 + \alpha\beta - \frac{2(1+\gamma)(1+\beta^2)K^2}{(1-\gamma)(\mu - K^2)}. \quad (14)$$

The equation for the global phase $\theta$ is the same as the one obtained for the scalar case: The $K = 0$ solution is phase stable, and TW solutions are $\theta$ stable for wave numbers $K < K_t$, where $K_t$ is independent of $\gamma$ and determined by the Eckhaus instability: $D_\theta(K_t) = 0$. A new phase instability associated with the direction of polarization appears for $K > K_p$, where $K_p$ is determined by $D_\psi(K_p) = 0$. Given that $K_p < K_t$ the stability range of TW solutions is determined by a polarization instability which shrinks the range of stable wave numbers obtained for the scalar case in the $(\mu, K)$ plane ("Busse balloon"). In the limit $\gamma \to 0$, $K_p \to K_t$, but (9) restricts $\gamma$ to take values $\gamma > 1/2$. The smallest value of $\gamma$ sets the limit of strongest possible linear polarization and broadest stability range. In the opposite limit $\gamma \to 1$, $K_p \to 0$, the phase polarization instability merges with an amplitude instability to circularly polarized light and all TW solutions of finite wave number $K$ are unstable, with the $K = 0$ solution being marginally stable.

*(b) Polarized standing waves (SW).*—These solutions occur for $k_+ = -k_- = K$. They correspond to counterpropagating traveling waves of opposite circular polarization with a common amplitude, wave number, and frequency. Alternatively, they can be visualized as linearly polarized solutions in which the direction of polarization is periodic in space, with each Cartesian component of the field being a standing wave for the intensity of the electric field:

$$A_x \propto \cos(Kx + \psi_0), \qquad A_y \propto \sin(Kx + \psi_0). \quad (15)$$

For these solutions coupled phase equations are obtained,

$$\partial_t \theta = 2K(\alpha - \beta)\,\partial_x \psi + D_\psi \partial_x^2 \theta, \quad (16)$$

$$\partial_t \psi = 2K(\alpha - \beta)\,\partial_x \theta + D_\theta \partial_x^2 \psi, \quad (17)$$

where $D_\psi$ and $D_\theta$ have now exchanged their role. These equations have a single complex eigenvalue,

$$\lambda_q = \pm 2i(\alpha - \beta)Kq - D_s q^2 + O(q^3), \quad (18)$$

where $D_s = (D_\theta + D_\psi)/2$. A single phase instability of the standing waves occurs when $D_s(K_s) = 0$,

$$K_s^2 = \frac{\mu(1-\gamma)(1+\alpha\beta)}{(1-\gamma)(1+\alpha\beta) + 2(1+\beta^2)}. \quad (19)$$

It is interesting to note that no stable SW are found when neglecting the polarization degree of freedom, and that the polarized SW have a broader range of stable wave numbers than the polarized TW waves. Indeed, the characteristic wave number $K_s$ is such that $K_p < K_s < K_t$. In the unattainable limit $\gamma \to 0$, $K_s \to K_t$, while in the opposite limit $\gamma \to 1$, $K_s \to 0$, and the range of stable SW also shrinks to zero. It is also important to realize that the stability properties of the SW solutions depend critically on the nonvariational character of (7). The calculation of (18) is based on a long-wavelength limit which requires having a finite value of $\alpha - \beta$. This cannot be fulfilled in the variational limit [2] $\alpha = \beta = 0$. In such a limit the phase equations decouple, the eigenvalues become real, and the role of $\theta$ and $\psi$ are just interchanged with respect to the case of polarized TW. Note also that the phase equation description breaks down for $\gamma \to 1^-$ since the eigenvalue associated with the amplitude difference between $A_+$ and $A_-$ becomes positive at finite $q$, while it is negative at $q = 0$.

*(c) Depolarized solutions.*—The solution (10) for arbitrary $k_+ \neq k_-$ corresponds to spatiotemporal states of the laser field without a simple polarization description. These solutions can be parametrized by $K = (k_+ + k_-)/2$ and $d = k_+ - k_-$. The range of $(K, d)$ values for which they are stable is limited by coupled phase equations which in general have two independent complex eigenvalues. A broad range of stable solutions exist in general. A very small value of $d$ stabilizes a range of $K$ values which are $\psi$ unstable for $d = 0$ (Fig. 1). A TW or SW solution could naturally evolve, after becoming unstable, into a depolarized solution. More interesting is that when the laser is switched on a variety of these solutions can locally grow from spontaneous emission noise. By analogy with the spatiotemporal intermittency regime found in the Benjamin-Feir stable region of the CGLE [17] one can envisage rich disordered states of the laser VCGLE. In such states these local and linearly stable depolarized solutions are connected by localized objects.

*Phase anisotropies.*—It is interesting to consider the effect of small anisotropies of the laser cavity usually recognized by a detuning splitting. Such anisotropies can be modeled replacing $\mu$ in (8) by a general matrix $\mathbf{\Gamma}$. The Hermitian part of $\mathbf{\Gamma}$ is associated with amplitude losses

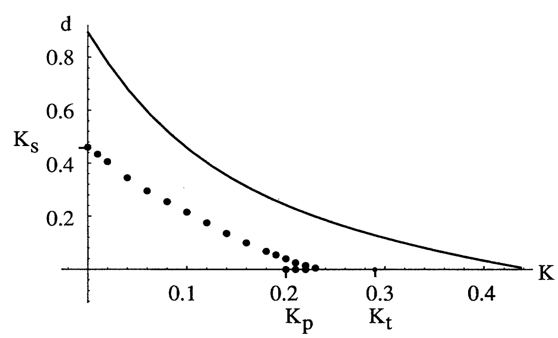


FIG. 1. Stability diagram for the family of solutions (10). Linearly polarized TW's are along the $x$ axis. They are stable for $K < K_p$. Polarized SW are along the $y$ axis. They are stable for $d < 2K_s$. Depolarized solutions exist below the continuous line. They are stable below the dotted line. $\mu = 0.2, \alpha = 2.6, \beta = 0.2, \gamma = 0.5$ ($K_t = 0.291, K_p = 0.198, K_s = 0.231$).

and the anti-Hermitian part with phase anisotropies. I consider here the case of a linear phase anisotropy, which amounts to add a term $i\gamma_p A_\pm$ to the right-hand side of (7). The parameter $\gamma_p$ is arbitrarily taken such that $\beta\gamma_p > 0$. Solutions of the modified laser VCGLE with the form (10) only exist now for $k_+ = k_- = K$. The phase $\psi_0$ is no longer arbitrary but fixed by $\sin\{2\psi_0 = 0\}$, giving rise to states linearly polarized either in the $x$ or $y$ direction. These states have a common amplitude independent of $\gamma_p$, $Q^2 = (\mu - K^2)/(1 + \gamma)$, and different frequencies $\omega_{x,y} = -\alpha K^2 - \beta(1 + \gamma)Q^2 \pm \gamma_p$. The global phase $\theta$ is independent of $\gamma_p$ and obeys the phase equation (11). The long-wavelength $\psi$ stability of the $x$- and $y$-polarized solutions is described by a damped phase equation of the general form,

$$\partial_t \psi = l_0 + v\partial_x \psi + \bar{D}_\psi \partial_x^2 \psi \,. \tag{20}$$

Simple particular cases of (20) illustrate well the effect of $\gamma_p$ on the phase dynamics: When considering the stability of the $K = 0$ solutions, one finds $v = 0$ and $l_0^{x,y}$ is such that the $x$-polarized solution is always stable at $q = 0$, while the $y$-polarized solution is unstable for $\gamma_p < \gamma_p^c = \beta Q^2(1 - \gamma)$. In addition, $\bar{D}_\psi^{x,y}$ is such that the $x$-polarized solution is further stabilized for $\alpha\gamma_p > 0$ by the phase anisotropy at finite $q$. From the point of view of polarization discrimination it is then convenient to take a value of $\gamma_p \ll \gamma_p^c$ for which only a stable solution exists at $q = 0$. In this limit the diffusion coefficient for the $x$-polarized solution of arbitrary $K$ becomes

$$\bar{D}_\psi^x = D_\psi + \gamma_p \frac{2(\beta^2 + 1)}{Q^2(1 - \gamma)}\left(\alpha - \frac{6K^2\beta}{Q^2(1 - \gamma)}\right). \tag{21}$$

For finite $K$ and $q$ the damped phase equation describes a competition between the stabilizing effect of the phase anisotropy and the remanent of the $\psi$ instability: $\bar{D}_\psi^x$ in (21) vanishes at a modified $K_p(\gamma_p) > K_p(\gamma_p = 0)$, so that for $K > K_c$ an amplitude-type instability associated with $\psi$ appears at finite $q$. The wave number $K_p(\gamma_p)$ sets a lower bound for $K_c$. A similar mechanism of stabilization of a modulational instability by external forcing has been invoked to explain pattern formation in passive optical systems [18]. In summary, small phase anisotropies fix a polarization direction of the spatially homogenous lasing solution, but the range of stability of polarized TW is still limited by a polarization instability at finite wave number $q$.

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*Permanent adress: Department de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain.

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