Coupled Brownian motors: anomalous-to-normal hysteresis transition and noise induced limit cycle

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ABSTRACT

We study a model consisting of N nonlinear oscillators with *global periodic* coupling and *local multiplicative* and additive noises. The model was shown to undergo a nonequilibrium phase transition towards a broken-symmetry phase exhibiting noise-induced "ratchet" behavior. Here we review some aspects leading to an "anomalous–to–normal" transition in the ratchet's hysteretic behavior and also show –as suggested by the absence of stable solutions when the load force is beyond a critical value– the existence of a limit cycle induced by both: multiplicative noise and *global periodic* coupling.

Keywords: Noise induced phenomena, noise induced phase transitions, Brownian motors, anomalous hysteresis, limit cycle

1. INTRODUCTION

The field of noise-induced transport or "Brownian motors" is now about one decade old.[1] In the early works, a requisite for these devices to operate (besides their obvious built-in, ratchet-like, bias) seemed to be that the fluctuations be correlated. That requirement was relaxed when "pulsating" ratchets were discovered: in these it is the random *switching* between uncorrelated noise sources which is responsible of the rectifying effect.[1] A recent new twist has been to relax also the requirement of a built-in bias[2]: a system of periodically coupled nonlinear phase oscillators in a symmetric "pulsating" environment has been shown to undergo a noise-induced nonequilibrium phase transition (NIPT), wherein the spontaneous symmetry breakdown of the stationary probability distribution gives rise to an *effective* ratchet-like potential. The aforementioned mechanism has striking consequences, such as the appearance of *negative zero-bias conductance* and *anomalous hysteresis*. By anomalous hysteresis we refer to the case where the cycle runs clockwise, in opposition to the normal one (as typified by a ferromagnet) that runs counterclockwise.

The study of dynamical systems has shown that limit cycles are ubiquitous in a wide range of physical applications.[3,4] From a physicist's point of view, limit cycles are thought of as a way to balance the in- and out- energy flows. Even when those flows are not oscillatory in time, a system's oscillatory motion can occur equalizing such flows over one period. Usually, limit cycles arise in dynamical systems described by ordinary differential equations (ODE),[3,4] but there are several examples where such kind of cycles also arise in partial differential equations (PDE) or "extended systems", as for instance, in the "brusselator" model for the so called "chemical clocks".[5,6]

Limit cycles arise also in systems with noise. Noise or fluctuations have been generally considered as a factor that destroys order. However, a wealth of investigations on nonlinear physics during the last decades have shown numerous examples of nonequilibrium systems where noise plays an "ordering" role. Some examples of such nonequilibrium phenomena are: noise induced unimodal-bimodal transitions in some zero dimensional models,[7]

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shifts in critical points, stochastic resonance in zero-dimensional and extended systems,[8,9] noise-delayed decay of unstable states, noise-induced spatial patterns,[10] noise induced phase transitions in extended systems,[11] etc.

Here, we discuss an extended system described by PDE's, where noise plays a key role in both, inducing a noise induced phase transition and controlling and inducing a limit cycle. The model that we analyze here is the one used in[2,12] to study a ratchet-like transport mechanism arising through a symmetry breaking, noise–induced, nonequilibrium phase transition. In a recent paper[13] a system showing a NIPT, based on a model that is a variant of Kuramoto's model for coupled phase oscillators[14]; was analyzed. In addition to the phenomenon of anomalous hysteresis, evidence of the existence of a limit cycle for a given parameter region was also shown.

The model we analyze consists of a system of periodically coupled nonlinear phase oscillators with a multiplicative white noise. Coupled oscillators have been used to model systems with collective dynamics exhibiting plenty of interesting properties like equilibrium and nonequilibrium phase transitions, coherence, synchronization, segregation and clustering phenomena. In this particular model a ratchet-like transport mechanism arises through a symmetry breaking, noise–induced, nonequilibrium phase transition,[2] produced by the simultaneous effect of coupling between the oscillators and a multiplicative noise. The symmetry breaking does not arise in the absence of any of these two ingredients. In[2] it was also shown that the current, as a function of a load force F, produces an anomalous (clockwise) hysteresis cycle. Also, by changing the multiplicative noise intensity Q and/or the coupled constant K_0, a transition from anomalous to normal (counter-clockwise) hysteresis is produced.[12] The results were obtained exploiting a mean field approximation.

Here, in addition to a brief review of results on the phase diagram and the character of the hysteresis cycle, we focus on the time behavior. We exploit a method for detecting the existence of a limit cycle based on the evaluation of the distance between two solutions separated by a (fixed) time interval.[15] In this way, we not only show the existence of a limit cycle for the loading force overcoming some threshold $F > F_c$ (with F_c the threshold value), but also determine its period. We also found the time dependence of the probability distribution function along the cycle and calculate the order parameter of the model vs. t, clearly showing the limit cycle. Next, we gain insight into its origin through the study of the large coupling limit. Finally, we draw some conclusions.

2. THE MODEL AND MEAN FIELD APPROXIMATION

Here we present a brief description of our model and some useful results. The model is similar to the one used in Refs.[2] and.[12] We consider a set of globally coupled stochastic differential equations (to be interpreted in the sense of Stratonovich) for N degrees of freedom (phases) $X_i(t)$

$$\dot{X}_i = -\frac{\partial U_i}{\partial X_i} + \sqrt{2T}\,\xi_i(t) - \frac{1}{N}\sum_{j=1}^{N} K(X_i - X_j). \tag{1}$$

This model can be visualized (at least for some parameter values) as a set of overdamped interacting pendulums. The second term in Eq. (1) considers the effect of thermal fluctuations: T is the temperature of the environment and the $\xi_i(t)$ are additive Gaussian white noises with

$$\langle \xi_i(t)\rangle = 0, \qquad \langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t'). \tag{2}$$

The last term in Eq. (1) represents the interaction force between the oscillators. It is assumed to fulfill $K(x-y) = -K(y-x)$ and to be a periodic function of $x-y$ with period $L = 2\pi$. We adopt[2,12]

$$K(x) = K_0 \sin x, \qquad K_0 > 0. \tag{3}$$

The potential $U_i(x,t)$ consists in a static part $V(x)$ and a fluctuating one. Gaussian white noises $\eta_i(t)$, with zero mean and variance 1, are introduced in a multiplicative way (with intensity Q) through a function $W(x)$. In addition; a load force F, producing an additional bias, is considered

$$U_i(x,t) = V(x) + W(x)\sqrt{2Q}\,\eta_i(t) - Fx. \tag{4}$$

In addition to the interaction $K(x-y)$, $V(x)$ and $W(x)$ are also assumed to be periodic and, furthermore, to be symmetric: $V(x) = V(-x)$ and $W(x) = W(-x)$. This last aspect indicates that there is no built-in ratchet effect. The form we choose is[2, 12]

$$V(x) = W(x) = -\cos x - A\cos 2x. \tag{5}$$

We introduce a mean-field approximation (MFA) similar to the one used in Ref.[12] The interparticle interaction term in Eq. (1) can be cast in the form

$$\frac{1}{N}\sum_{j=1}^{N} K(X_i - X_j) = K_0\left[C_i(t)\sin X_i - S_i(t)\cos X_i\right]. \tag{6}$$

For $N \to \infty$, we may approximate Eq. (6) in the Curie-Weiss form, replacing $C_i(t) \equiv N^{-1}\sum_j \cos X_j(t)$ and $S_i(t) \equiv N^{-1}\sum_j \sin X_j(t)$ by $C_m \equiv \langle \cos X_j\rangle$ and $S_m \equiv \langle \sin X_j\rangle$, respectively. As usual, both C_m and S_m should be determined by self-consistency. This decouples the system of stochastic differential equations (SDE) in Eq. (1) which reduces to essentially one Markovian SDE for the single stochastic process $X(t)$

$$\dot{X} = R(X) + S(X)\eta(t), \tag{7}$$

with (hereafter, the primes will indicate derivatives with respect to x)

$$\begin{aligned} R(x) &= -V'(x) + F - K_m(x) \\ &= -\sin x(1 + K_0 C_m + 4A\cos x) + K_0 S_m \cos x + F, \end{aligned} \tag{8}$$

(where $K_m(x) = K_0[C_m \sin x - S_m \cos x]$) and

$$S(x) = \sqrt{2\{T + Q[W'(x)]^2\}} = \sqrt{2\{T + Q[\sin x + 2A\sin 2x]^2\}}. \tag{9}$$

The Fokker-Planck equation (FPE) associated with the SDE in Eq. (7) (in Stratonovich's sense) is

$$\partial_t P(x,t) = \partial_x\left(-[R(x) + \frac{1}{2}S(x)\,S'(x)]P(x,t)\right) + \frac{1}{2}\partial_{xx}\left[S^2(x)P(x,t)\right] \tag{10}$$

where $P(x,t)$ is the probability distribution function (PDF).

Since $\sin x$ is an antisymmetric function, as was indicated in Ref.[12] , and in order to find the curve that separates the ordered phase from the disordered one, given that on that curve S_m is still zero, we should solve the following system:

$$\int_{-L/2}^{L/2} dx\, \cos x\, P^{st}(x, C_m, 0) = C_m, \tag{11}$$

$$\int_{-L/2}^{L/2} dx\, \sin x\, \left.\frac{\partial P^{st}}{\partial_{S_m}}\right|_{S_m=0} = 1. \tag{12}$$

Figure 1 shows (on the same scale as Fig. 1b of Ref.[2] , with which it fully coincides) the phase–like diagram in the plane (K_0, Q), obtained by solving Eqs. (11) and (12). The transition curve, separating the region where the hysteresis cycle is anomalous from the one where it is normal, is also indicated. To the left of the dashed line we have the "interaction driving regime" (idr) while the "noise driving regime" is located to the right of the line.

In the region above the full line ("ordered region") the stable solution has $S_m \neq 0$. Notice that this noise-induced phase transition is *reentrant*: as Q increases for $K_0 =$ const., the "disordered phase" ($S_m = 0$) is met again. The multiplicity of mean-field solutions in the ordered region, together with the fact that some of them may suddenly disappear as either K_0 or Q are varied (a fact that is closely related to the occurrence of anomalous hysteresis) could hinder the pick of the right solution in this region. A detailed analysis could be found in.[12]

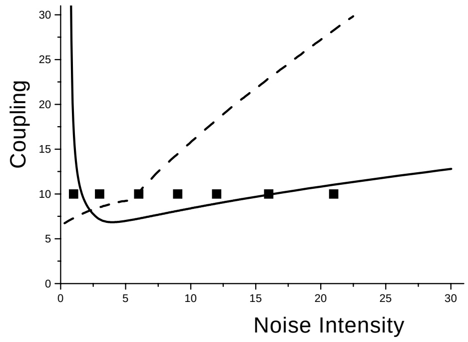

Figure 1. Phase diagram of the model for $T = 2$, $A = 0.15$ and $F = 0$. The ordered region lies above the full line. Above the dashed line there is anomalous hysteresis, while below it there is normal hysteretic behavior. The squares represent states that have been investigated with $K_0 = 10$ and Q =1, 3, 6, 9, 12, 16 and 21 respectively.

The appearance of a ratchet effect amounts to the existence of a non zero drift term $\langle \dot{X} \rangle$ in the stationary state, in the absence of any forcing ($F = 0$). As discussed in[2] , the cause of this spontaneous particle current is the noise-induced asymmetry in $P^{st}(x)$.

As was shown in[12] , for the particle current we have

$$\langle \dot{X} \rangle = \int_{-L/2}^{L/2} dx \left[R(x) + \frac{1}{2} S(x) S'(x) \right] P^{st}(x, C_m, S_m), \tag{13}$$

with the final result

$$\langle \dot{X} \rangle = J\,L = \left\{ \frac{1 - e^{\phi(L)}}{2\mathcal{N}} \right\} L. \tag{14}$$

Hence $\langle \dot{X} \rangle$ has the sign of J and can be also regarded as an order parameter.

Figures 2(a) to 2(d) present a sequence of $\langle \dot{X} \rangle$ vs F plots, varying Q across the dashed line of Fig. 1. For $Q = 5.97$ (Fig. 2(a)) two (unstable) solutions meet at $\langle \dot{X} \rangle = 0$ for $F = 0$. The progressive withdrawal of one of them out of the $F \approx 0$ region with increasing Q until its complete disappearance (Figs. 2(b) to 2(d)) can be traced back (through their corresponding branches) to the disappearance of solutions for $S_m = 0$. Moreover, it is only after this solution has completely disappeared that the stable solution begins to exist for larger values of F and thus normal hysteresis sets in (Fig. 2(d)).

3. LIMIT CYCLE

In[12] we have shown that in the so called "interaction driven regime" (IDR) –where the hysteretic cycle is anomalous– and for each F value, in addition to the two stationary stable solutions with the corresponding values of current there are other three unstable ones. Two of them merge with the two stable, yielding a closed curve of current vs. F. Beyond a critical (absolute) value of the load force F, indicated by F_c, those stable solutions disappear. This does not happen for the "noise driven regime" (NDR) –where the hysteretic cycle is normal–, where for each F value, one stationary stable solution exists (for small $|F|$ even two stationary stable solutions and an unstable one exist).

It is worth remarking here that the absence of a stationary stable solution, beyond the critical value F_c in the IDR, suggest the possibility that a limit cycle exists. Already in[2] , in a strong coupling analysis (that is considering the limit $K_0 \to \infty$), it was indicated that for very large $|F|$ the probability distribution function approaches a periodic long time behavior.

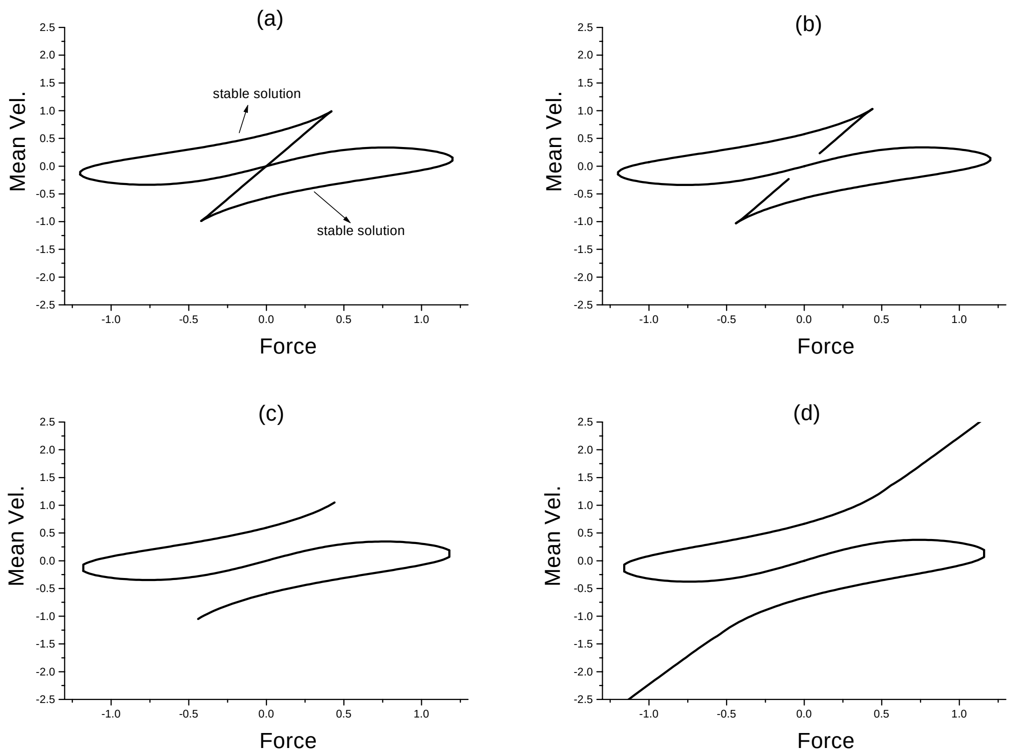

Figure 2. (a) $V_m = \langle \dot{X} \rangle$ vs F for $K_0 = 10$ and $Q = 5.97$ (just on the left of the dashed line of Fig. 1). (b) Same as for $Q = 6.0$: one of the unstable solutions has receded from the $F \approx 0$ region. (c) Same as for $Q = 6.1$, showing a complete recession from the $F \approx 0$ region. (d) Same as for $Q = 6.5$: not until the dotted line has completely disappeared do solutions in the stable branch appear for $|F| > 0.5$ and normal hysteresis sets in.

In order to analyze the existence of a limit cycle, we exploit a novel method used in Ref..[15] It is based on the measurement of the distance between different solutions of a system and evaluating its evolution in time. The approach applied in Ref.[15] uses a generalization of the known Kullback-Leibler information function,[16] which is based on the nonextensive thermostatistics. Within such a formalism, the exponential and logarithmic functions are generalized according to the following definitions[15]

$$\begin{aligned} \exp_q(x) &= [1 + (1-q)x]^{1/(1-q)} \\ \ln_q(x) &= \frac{x^{1-q} - 1}{1-q}. \end{aligned} \tag{15}$$

The distance can be measured between an evolved initial condition and a known stable stationary solution, or between two solutions at different times (separated by a time interval $\Delta\tau$ which is fixed along the whole calculation). In this work we choose the later. In Ref.[15] the following definition for the distance between two solutions of a reaction-diffusion equation was adopted (valid for both indicated criteria)

$$I_q(P_{t+\Delta\tau}, P_t) = -\int P_{t+\Delta\tau}(x, t+\Delta\tau) \ln_q \left[\frac{P_\tau(x,t)}{P_{t+\Delta\tau}(x, t+\Delta\tau)} \right] dx, \tag{16}$$

where P represent a (probability-like) distribution (necessary to use the information theory formalism), evaluated at t and $t+\Delta\tau$, according to the criterion that we have chosen. We used this definition of distance, and evaluated $I_q(P_{t+\Delta\tau}, P_t)$, using for P the PDF obtained solving the FPE Eq. (10). We adopted $q = 2$, as it is the value for which the sensibility of the method seems to be a maximum.[15] The FPE was numerically solved with a

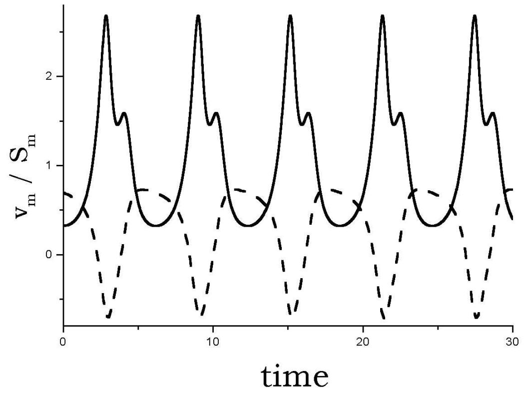

Figure 3. v_m and S_m vs. time (t) for $A = 0.15$, $T = 2$, $K_0 = 10$, $Q = 3$ and $F = 1.5$ (for this set of parameters there is no stationary stable solution). Thick line for v_m and thin line for S_m.

Runge-Kuta method, using a time step $\delta t = 6.25\,10^{-7}$ and a space interval $\delta x = 0.02944$. We have tested that variations in both steps, δt and δx, produce no changes in our results. Remembering that C_m and S_m should be determined self-consistently, at each time step both were calculated with the modified PDF. As our initial condition we adopted one stationary solution for $F < F_c$ calculated as in Ref..[12] The integral in Eq. (16) was calculated simultaneously. Furthermore, we also obtained v_m –the particle mean velocity– according to Eq. (13), which is adopted as the order parameter like in.[12]

3.1. NUMERICAL RESULTS AND STRONG COUPLING ANALYSIS

Figure 3 shows v_m and S_m vs. t for $A = 0.15$, $T = 2$, $K_0 = 10$, $Q = 3$ and $F = 1.5$ (a set of parameters for which a stationary stable solution does not exists: see Fig 6 in Ref.[12]). They have a time periodic behavior, at variance to the case $F \leq F_c$, where v_m ($\neq 0$) and S_m ($\neq 0$) are both constants in time. We have also verified that the transition to the limit cycle occurs just at F_c (in this case $F_c = 1.2$).

In order to understand the origin of the periodic behavior and gain some insight, we have performed an asymptotic strong coupling analysis. That is, we consider $K_0 \to \infty$, $P \to \delta(x - x_m)$, hence Eq. (16) transforms into

$$\dot{x}_m = R(x_m) + \frac{1}{2} S(x_m) S'(x_m). \tag{17}$$

A simple calculation shows

$$\dot{x}_m = -\sin x_m [1 + 4A\cos x_m][1 - Q\cos x_m - 4AQ(1 - 2\sin^2 x_m)] + F. \tag{18}$$

This equation can be analyzed considering an effective potential $U(x_m)$, given by

$$U(x_m) = V(x_m) - QW'^2(x_m)/2 - F x_m, \tag{19}$$

that allows us to rewrite Eq. (14) as

$$\dot{x}_m = -\frac{\partial U(x_m)}{\partial x}. \tag{20}$$

It is possible to analyze the solution of Eq. (18), $\dot{x}_m$ vs. t, for both situations: just below and above F_c, and observe that while for $F < F_c$, after a transient, the solution becomes stationary, for $F > F_c$ it becomes oscillatory. In the first case x_m is constant in time but it does not imply $v_m = 0$ because, it should be calculated with $S_m = \sin(x_m) \neq 0$, not as in the case with $\dot{x}_m$. Figure 4 shows the effective potential U vs. x_m for the same cases, and also for $F = 0$. It is apparent that in the first case ($F < F_c$) the potential has only one minimum while for the second one, both possible minima are washed out. The latter happens just when the transition to the oscillating regime occurs. It is worth remarking here that, if $K_0 \to \infty$, the hysteresis cycle is anomalous and closed, and a critical load force establishing a threshold for a limit cycle transition always exists.

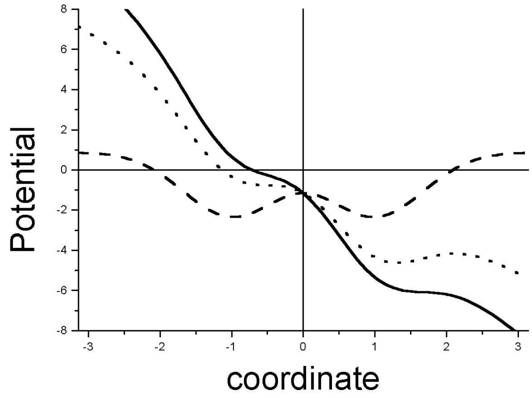

Figure 4. U vs. x_m just below and above of $F_c = 1.2$. Also the case $F = 0$ is shown. The parameters are $A = 0.15$, $K_0 = 10$, and $Q = 3$. It is observed that in the first case ($F < F_c$) the potential has at least a minimum, while for the second one both possible minima are washed out. The solid line indicates the case just above F_c ($F_c = 1.2$), the dotted one indicates the case $F < F_c$ and the dashed one the case $F = 0$.

4. CONCLUSIONS

A wealth of papers have reported on research where, by changing a control parameter, a transition to a limit cycle occurs.[17] However, studies on the existence of limit cycles under (or induced by) the influence of noise are scarce.[13,18,19] Such an aspect was analyzed here, where we have studied a system of periodically coupled nonlinear oscillators with multiplicative white noises, yielding a ratchet-like transport mechanism through a symmetry-breaking, noise–induced, nonequilibrium phase transition.[2,12] The model includes a load force F, used as a control parameter, so that the picture of the current vs. F shows hysteretic behavior.

We have shown, as discussed in detail in,[12] that in the IDR the cycle is anomalous, yielding a closed curve current vs. F when the stationary stable solutions merge with two of the three unstable ones. For $F > F_c$ (force value at which a stable solution merges with an unstable one) there are no stationary stable solutions. Here we have shown, by analyzing the time evolution of the distance between different solutions, that at $F = F_c$ a transition to a limit cycle occurs. Such a distance shows, for $F > F_c$, a typical periodic behavior evidencing a limit cycle.[15] Focusing on the analysis of the time behavior, the evolution of both the PDF and the current can be studied, showing in both cases the time periodicity (a time evolution of the PDF resembling a wave). In order to understand the origin of this transition, we have made a "strong coupling" limit analysis, indicating that the minima of the effective potential are "washed out" as F is increased and all the stationary stable solution are removed with them. The existence of such limit cycle is a new feature of those systems showing a ratchet-like transport mechanism arising through a NIPT. Also, it is another example where the presence of a multiplicative noise contributes to build up some form of order.

As indicated in the introduction, limit cycles balance the in– and out– energy flows –even when those flows are not oscillatory in time– through a system's oscillatory motion that equalize such flows over one period. In the present case we have found a limit cycle in a dynamical system described by PDE's, where the energy inflow is provided by both the load force F and the noise terms, while energy is lost (as the system is an overdamped one) proportionally to the particle's velocity. A remarkable aspect is the fact that it is the multiplicative noise intensity the parameter controlling the bifurcation towards the limit cycle.

It is worth here remarking again that it is the simultaneous effect of multiplicative noise and coupling that yields the NIPT. If we have only coupling without fluctuations, or the opposite situation, that is fluctuations without coupling, there is no macroscopic effect: neither noise–induced transition[7] nor NIPT. The dynamical mechanism –a short time instability– originating the NIPT that we have discussed so far was, till recently, though to be the only, paradigmatic, one. However, a recent work has shown that noise–induced nonequilibrium phase transitions can also arise through a different mechanism, more akin to noise–induced transitions.[20] One can rise the question wether such a new mechanism can also induce a ratchet and/or a limit cycle. These problems are currently under study.

The authors thank R.Toral, R.Deza and G.Izús for helpful discussions. HSW acknowledges the partial support from CONICET and ANPCyT, both Argentine agencies, and thanks the MECyD, Spain, for an award within the *Programa de Sabáticos para Profesores Visitantes*, and the Universitat de les Illes Balears for the kind hospitality extended to him.

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