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Stochastic resonance and generalized information measures

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Abstract

We use Tsallis' information measure and the recently advanced "escort-Tsallis" one in order to investigate the stochastic resonance (SR) phenomenon. In particular, on the basis of these generalized information measures (GIM), both characterized by a non-extensivity parameter q, we study the effect of modifying the type of noise. The GIM are used to measure SR phenomenon comparing the input and output probability distributions, finding that for the range of noise intensities enhancing SR, $q \to 0$ gives the optimum measure. An important advantage of using GIM instead of Shannon's measure is the possibility of detecting smaller signal amplitudes with the former than with the latter one.

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1. Introduction

Since its discovery, the phenomenon of Stochastic Resonance (SR) has been the subject of growing interest. SR reveals the counterintuitive character of noise in non-linear systems, as fluctuations can be exploited so as to boost the output response of a system subjected to a weak, periodic external signal. The broad range of phenomena—indeed drawn from almost every field of scientific endeavor—for which this mechanism can

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offer an explanation is extensively discussed in many review articles. We cite, for instance, the excellent presentation given in Ref. [1], possibly the most comprehensive one in this respect, for a scanning of the state of the art.

Different manners of characterizing SR have been advanced, namely, (i) signal-to-noise ratio (SNR) [2], (ii) the spectral amplification factor (SAF) [3], (iii) the residence time distribution [4], and, more recently, (iv) information theory based tools [5,6]. SR measurements often involve gathering data over long time intervals so as to attain reliable results. A tendency evidenced in recent papers, and determined by the possible SR technological applications, points towards achieving an enhancement of the system's response (that is, obtaining a larger output SNR) by means of the coupling of several stochastic resonance units in what conforms an *extended medium* [7,8]. Forms of controlling this phenomenon have also been investigated [9].

Many studies on SR have been carried out with the help of a paradigmatic system: a bistable one-dimensional double-well system, the noise being of a Gaussian nature [10]. However, some results associated to the sensory system [11], particularly for a crayfish [12], yield strong indications on the possibility that noise sources could have a non-Gaussian character. Such a point of view is supported by results obtained in a recent contribution [13], which focuses attention upon a particular class of Langevin (and its associated Fokker–Planck) equations whose stationary solutions are non-Gaussian distribution functions [14]. The work in Ref. [14] is based on the generalized thermostatistics advanced by Tsallis [15,16] that has been successfully applied to a wide variety of physical systems [16,17]. It is pertinent to mention here recent works [14,15] connecting information measures [18] with equations of the Langevin type and suggesting that non-Gaussian noises seem to be governed by statistics different from the orthodox Boltzmann–Gibbs one. In such a spirit, the present effort analyzes the effects on the SR phenomenon on the *nature* of the concomitant noise using generalized information measures (GIM). These are characterized by a non-extensivity index q, and we intend to ascertain its optimum value. Such a value will be the one making the information measure the most sensitive one to small changes in the system's parameters.

This paper is organized as follows: the next section briefly introduces the information theory-based treatment used in connection with SR systems, and *generalizes it to a non-extensive Tsallis setting* [15,16]. Section 3 presents the model to be employed together with the results of our numerical simulations. Finally, some conclusions are drawn in Section 4.

2. Information measures

2.1. Information theoretical treatment

One of the first works advancing the idea of quantifying SR using information theory tools was [6]. The ensuing SR characterization compares two probability distributions functions (pdf): one associated to the *input* signal and the other to the *output* signal. A relevant related contribution is in Ref. [19] and references therein.

The classical SR formulation in terms of the output SNR is subjected to some problems, as detailed in Ref. [19]. The mutual information between the input signal and the output one advantageously replaces the classical measure [19]. *SR is now defined as a peak in the mutual information versus noise relation* [7]. It is our intention here that of generalizing this viewpoint. There is an infinite number of mutual-information measures, but previous studies have restricted themselves to just one of them: *Kullback*'s relative information measure [20]. It surely makes sense to consider alternatives and here we will discuss generalized information measures [16].

Consider an external signal of frequency Ω (period $2\pi T^{-1}=\Omega$). We take the temporal interval T, divide it into n parts and examine the concomitant n time-samples of a given system's quantity $\mathscr{V}(t)$. Out of these samples we construct the vector $\vec{V}_n=(v_1,\ldots,v_n)$, with $v_i \in \{+,-\}$, $\forall i=1,\ldots,n$, determined according to

$$v_i = \begin{cases} + & \text{if } \mathscr{V}(t_i) \geqslant 0 \,, \\ - & \text{if } \mathscr{V}(t_i) < 0 \,. \end{cases}$$

We assume that a stationary distribution $P(\vec{V}_n)$ gives the occurrence probability for the binary sequence (of length n) $\vec{V}_n$. We will encounter below a case in which all sequences have the same probability.

We are here concerned with an SR measure that quantifies the degree of synchronization between (i) the input signal (as represented by the vector $\vec{V}_n^i$) and (ii) a system's output, defined for an isolated set of points as indicated above, to be denoted by $\vec{V}_n^o$. One is interested in evaluating the *Kullback*'s relative information measure [20] for these two vectors

$$K_1[P(\vec{V}_n^i)\,P(\vec{V}_n^o),P(\vec{V}_n^{io})] = \sum_{\vec{V}_n^{io}} P(\vec{V}_n^{io}) \ln \frac{P(\vec{V}_n^{io})}{P(\vec{V}_n^i)P(\vec{V}_n^o)} \,. \tag{1}$$

Here $P(\vec{V}_n^{io})$ is the probability for the joint appearance of a given input–output pair, and one normalizes things according to

$$W_1[P(\vec{V}_n^i)P(\vec{V}_n^o),P(\vec{V}_n^{io})] = \frac{K_1[P(\vec{V}_n^i)\cdot P(\vec{V}_n^o),P(\vec{V}_n^{io})]}{H^n[P(\vec{V}_n^i)P(\vec{V}_n^o)]} \leqslant 1 \,, \tag{2}$$

where H^n stands for *Shannon*'s entropy [20], which measures the degree of order in sequences of length n. In this respect one has

$$H^n[P(\vec{V}_n)] = -k \sum_{\vec{V}_n} P(\vec{V}_n) \ln P(\vec{V}_n) \,, \tag{3}$$

with $k=\ln 2$. In [6], it was shown that such an SR characterization results useful only for large signal amplitudes. We will see that using GIM we can overcome this limitation.

2.2. Generalized information measures

We will now *extend the preceding considerations to a non-extensive Tsallis setting* by recourse to the GIM S_q, characterized by a real, positive index $q (q \geqslant 0)$, discussed in Refs. [15–18,21], i.e.,

$$S_q = -k \sum_{\vec{V}_n} P(\vec{V}_n) \times \ln_q P(\vec{V}_n) \; . \tag{4}$$

In this work we will employ the so-called generalized q-logarithmic function [15,16,18,21]

$$\ln_q(x) \equiv \frac{x^{q-1} - 1}{q - 1} \; , \tag{5}$$

with $q \in \mathscr{R}$. Obviously, for $q = 1$ this function becomes the ordinary natural logarithm and, as a consequence, $S_1 \equiv H$. The case $q = 0$ will be seen to be of relevance. Of course, $\ln_{q=0}(x) \equiv 1 - 1/x$. If the sum in Eq. (4) contains N terms, then $S_{q=0} = -k(N-1)$, no matter what the pdf $P(\vec{V}_n)$ might be.

Tsallis' measure is characterized by a special property: pseudo-additivity, i.e., for two independent systems A, B [15,17,18,21],

$$S^q(A \oplus B) = S^q(A) + S^q(B) + (1-q)S^q(A)S^q(B) \; . \tag{6}$$

We have super-additivity if

$$S^q(A \oplus B) \geqslant S^q(A) + S^q(B) \; , \tag{7}$$

and sub-additivity otherwise. The case $q = 0$ yields the maximum possible super-additivity (minimum allowable q-value).

The generalized relative information (counterpart of Eq. (1)) is now

$$\begin{aligned} K_q[P(\vec{V}_n^i)P(\vec{V}_n^o), P(\vec{V}_n^{io})] = -\sum_{\vec{V}_n^{io}} & P(\vec{V}_n^{io})[\ln_q P(\vec{V}_n^i) + \ln_q P(\vec{V}_n^o) \\ & + (1-q)\ln_q P(\vec{V}_n^i)\ln_q P(\vec{V}_n^o) - \ln_q P(\vec{V}_n^{io})] \; . \end{aligned} \tag{8}$$

It is clear that Eq. (8) reduces to Eq. (1) for $q = 1$. For $q = 0$, the contribution to K_q of the term

$$\sum_{\vec{V}_n^{io}} P(\vec{V}_n^{io}) \ln_q P(\vec{V}_n^{io}) \; ,$$

is constant (independent of the nature of the associated pdf), but the other three terms do not necessarily exhibit such a feature. Normalization proceeds according to

$$\begin{aligned} & W_q[P(\vec{V}^i)P(\vec{V}^o), P(\vec{V}^{io})] \\ & = \frac{K_q[P(\vec{V}_n^i).P(\vec{V}_n^o), P(\vec{V}_n^{io})]}{-\sum_{\vec{V}_n^{io}} P(\vec{V}_n^{io})[\ln_q P(\vec{V}_n^i) + \ln_q P(\vec{V}_n^o) + (1-q)\ln_q P(\vec{V}_n^i)\ln_q P(\vec{V}_n^o)]} \; . \end{aligned} \tag{9}$$

Our main objective is that of finding an "optimal q" (denoted by q^*), such that the gradient of W_q is, for $q = q^*$, the largest possible one. The corresponding q-statistics

W_{q^*} is, among all the W_q, the most sensitive one to small changes (due, for example, to variation in the system's internal parameters) that affect the correlation between the random variables $P(\vec{V}_n^i)$ and $P(\vec{V}_n^o)$.

2.3. A new information measure

In Ref. [22], it was shown that a new information measure of the Tsallis kind, originally introduced in Ref. [23], seems to be more sensitive to a variation of relevant parameters of complex systems than the original measure in Eq. (4). Here we extend such a measure (in what follows called escort-Tsallis measure) [22,23] in the sense of introducing its associate relative (à la Kullback) measure, to be abbreviated as the GK one. Here we exploit the ensuing relative measure for the study of SR.

At this point we introduce the useful concept of escort probabilities (see Ref. [24] and references therein). One introduces the following transformation between an original, normalized probability distribution (PD) r and a new one R

$$r \to R\,, \tag{10}$$

with

$$R_i = \frac{r_i^q}{\sum_i r_i^q}\,, \tag{11}$$

q being any real parameter. We reiterate: r is the *original* PD one is concerned with. For $q = 1$ we have $R \equiv r$ and, obviously, R is normalized to unity. General global quantities formed with escort distributions of different order q, such as the different types of information or mean values, will give more revealing information than those formed with the original distribution only. Changing q is indeed a tool for scanning the structure of the original distribution [24]. Given the Tsallis' information measure constructed with some probability distribution r one may think of computing the Tsallis measure that results from replacing r by R. This was investigated in Ref. [23]. If, in terms of r, one has Eq. (4) expressed as

$$S_q = \frac{1}{q-1}\sum_i [r_i - r_i^q]\,, \tag{12}$$

then the associated escort-Tsallis, as a functional of the *original* measure, reads

$$S_q^{esc} = \frac{1}{q-1}\left\{1 - \left[\sum_i r^{1/q}\right]^{-q}\right\}\,. \tag{13}$$

Returning now to the context of our SR discussion, the pertinent escort-Tsallis measure reads

$$\mathscr{H}^n[P(\vec{V}_n)] = \frac{1}{q-1}\left(1 - \sum_{\vec{V}_n^{io}} P(\vec{V}_n)^{1/q}\right)^{-q}\,, \tag{14}$$

while its associated relative measure is

$$\mathscr{K}_q[P(\vec{V}_n^i)P(\vec{V}_n^o),P(\vec{V}_n^{io})]=\frac{1}{q-1}\left[-\left(\sum_{\vec{V}_n^{io}}P(\vec{V}_n^{io})^{1/q}\right)^{-q}\right.$$
$$\left.+\left(\sum_{\vec{V}_n^{io}}P(\vec{V}_n^{io})[P(\vec{V}_n^i)P(\vec{V}_n^o)]^{(1/q)-1}\right)^{-q}\right] . \tag{15}$$

Normalization, as in the preceding Section, yields

$$\mathscr{W}_q[P(\vec{V}_n^i)P(\vec{V}_n^o),P(\vec{V}_n^{io})]$$
$$=\frac{-(\sum_{\vec{V}_n^{io}}P(\vec{V}_n^{io})^{1/q})^{-q}+(\sum_{\vec{V}_n^{io}}P(\vec{V}_n^{io})[P(\vec{V}_n^i)P(\vec{V}_n^o)]^{(1/q)-1})^{-q}}{-(\sum_{\vec{V}_n^{io}}P(\vec{V}_n^{io})^{1/q})^{-q}+(\sum_{\vec{V}_n^{io}}[P(\vec{V}_n^i)]^{(1/q)-1}[P(\vec{V}_n^o)]^{(1/q)})^{-q}} . \tag{16}$$

We show below that the escort-Tsallis measure can be advantageously employed in an SR context.

3. Model and results

We start by studying a particular class of Langevin equations whose stationary distribution functions are of a non-Gaussian nature [25]. We consider the following problem:

$$\dot{x}=f(x,t)+g(x)\xi(t) , \tag{17}$$

$$\dot{\xi}=-\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\xi}V_q(\xi)+\frac{1}{\tau}\eta(t), \tag{18}$$

where $\eta(t)$ is a Gaussian white noise of zero mean and correlation $\langle\eta(t)\eta(t')\rangle = D\delta(t-t')$, and $V_q(\xi)$ is given by [25]

$$V_q(\xi)=\frac{1}{\beta(q-1)}\ln\left[1+\beta(q-1)\frac{\xi^2}{2}\right] , \tag{19}$$

where $\beta=\tau/D$.

The function $f(x,t)$ is derived from $U(x,t)$, a double-well potential, and a periodic signal $S(t)\sim\varepsilon\cos(\Omega t)$, i.e., $f(x,t)=-\partial U/\partial x=-U_0'+S(t)$. In absence of the driving signal ($\varepsilon=0$), this corresponds to the case of diffusion in a potential $U_0(x)$, induced by ξ, a colored non-Gaussian noise.

We have here studied three different noise sources within the context of this model, namely

1. *Gaussian noise*: Clearly, when $q\to 1$ we recover the limit of ξ being a Gaussian colored noise (Ornstein–Uhlenbeck process). For the particular case $\tau\to 0$

we recover $\xi_G(t)$, the usual white noise, with stationary distribution given by

$$P(\xi) = \frac{e^{-\xi^2/D}}{\sqrt{\pi D}} .$$

2. *Lorentz noise*: For $q > \frac{5}{3}$, the distribution's second moment becomes infinite. An interesting case within this range is $q = 2$ in the limit $\tau \to 0$. This corresponds to a noise $\xi_L(t)$ with a Lorentzian distribution given by

$$P(\xi) = \frac{\pi}{\alpha} \frac{1}{\xi^2 + \alpha^2} ,$$

where α is related with the distribution width.

3. *Tsallis' distribution*: As has been shown in Ref. [26], the stationary probability distribution for the random variable ξ_{Ts} is given by

$$P_q^{st}(\xi) = \frac{1}{Z_q} [1 + \beta(q-1)\xi^2]^{-1/(q-1)} .$$

In particular, if the parameter $q \in (-\infty, 1)$ the distribution has a compact support:

$$P_q(\xi) = \begin{cases} \frac{1}{Zq}[1 - (\frac{\xi}{w})^2]^{1/(1-q)} & \text{if} |\eta| < w , \\ 0 & \text{if} |\eta| \geqslant w , \end{cases}$$

where Z_q is the normalization factor and $w = [(1-q)\beta/2]^{-1/2}$. The distribution function is an even function with zero mean and standard deviation given by $\langle\xi\rangle^2 = 2/\beta(5-3q)$.

The possibility of the SR phenomenon in a system with a noise of infinite variance has not been studied in Ref. [26]. It is thus appropriate to start delving into the case of a Lorentzian Noise. In order to obtain our results we (numerically) integrated Eq. (17), using the Heun [27,28] algorithm. The time step is $h = 10^{-5}$, and one averages over 5000 realizations in order to achieve the results depicted below. As we have chosen $\tau < h$, from a practical point of view we can assume $\tau = 0$. A larger τ do not qualitatively changes the results.

Firstly, we show results obtained using two standard techniques, namely, (i) the SNR [2] and (ii) the spectral amplification factor (SAF) [3].

Fig. 1(b)–(d) depict the power spectra for different noise intensities α. The SR phenomenon is clearly seen. In Fig. 1(a) we exhibit both R, the SNR, and η, the spectral amplification factor. The amplification effect on the weak external signal is apparent. Such an effect vanishes if the noise intensity becomes strong enough. In the limit of zero noise the SNR grows without bound (corresponding to the intra-well response [2]) while η vanishes.

3.1. Information measures and results

We consider now the results obtained in the case of a Gaussian noise source. We numerically integrated the equations, with a time discretization of $h = 10^{-3}$.

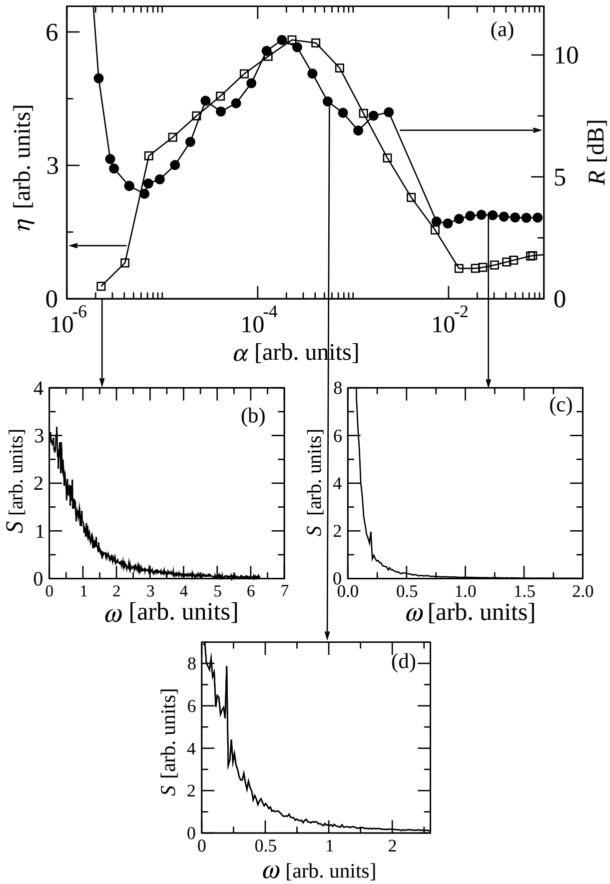

Fig. 1. (a) Signal-to-noise ratio, R and spectral amplification factor η (squares) versus a function of α, with $\xi(t) = \xi_L(t)$. The signal is characterized by the values $\Omega = 1.00/\pi$ and $\varepsilon = 0.15$. Graphs (b), (c), and (d), depict, for different α-values, the power spectrum density of the system when the noise intensity is varied while keeping both the frequency and amplitude of the signal fixed at the values given in (a). The α-values are: (b) $\alpha = 2 \times 10^{-5}$, (c) $\alpha = 10^{-3}$, and (d) $\alpha = 3 \times 10^{-2}$.

We focus our attention upon an input signal of the form $\mathscr{V}^s(t) = \varepsilon \cos(\Omega t) + \xi(t)$ for which we compute the corresponding output signal. It is worthwhile noting here that this is the only measure that can be experimentally measured. Fig. 2(a) and (b) depict W_q as a function of either the noise intensity D (for the Gaussian case) or the parameter α (for the Lorentz one). A clear maximum in the input–output transmitted information becomes apparent. The SR effect can be appreciated even for very small modulation intensities. We took $n = 12$ and the computation proceeded by making 5×10^5 copies of the vectors $\vec{V}_n$.

We emphasize the fact that the SR phenomenon is detected by the two GIMs introduced in this work. *The Shannon measure*, *instead*, *can only be used for large*

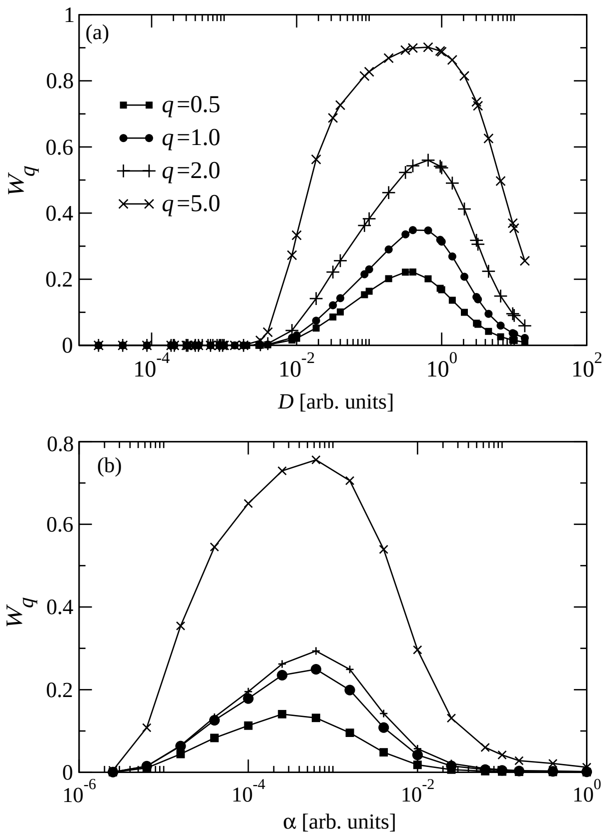

Fig. 2. Kullback–Tsallis generalized information measure as a function of the noise intensity: (a) Gaussian noise, (b) Lorentzian noise. In the two cases one has $\Omega = 1.0/\pi$ and $\varepsilon_0 = 0.7$. The input signal is $\mathscr{V}^s(t) = \varepsilon\cos(\Omega t)$. In both cases the trans-information reaches a maximum for a given (intermediate) value of external noise, which is the typical signature for a stochastic resonance phenomenon.

amplitudes, i.e., $\varepsilon \geqslant 0.18$ for the system under consideration (cf. Ref. [6]), although conventional quantities (like the SNR, SAF or time-residence distribution) detect the existence of the phenomena even for signal amplitudes as small as $\varepsilon = 0.01$. This fact limits the possible use of Shannon-based treatment of SR phenomena. Although the pertinent results are not shown here, extensive simulations show that GIM-based measures detect the peak of SR for the range $\varepsilon \geqslant 0.03$, if the q-index fulfills the condition $q \geqslant 3$. This range is almost the same as that corresponding to the use of the conventional measures indicated above. The GIM feature we are discussing here significantly increases the range of application of information theory-based techniques in an SR context. In addition, it is worth remarking here that information theory approaches offers the possibility of analyzing the SR phenomenon with arbitrary input signals while the SNR and SAF methods deal with periodic ones.

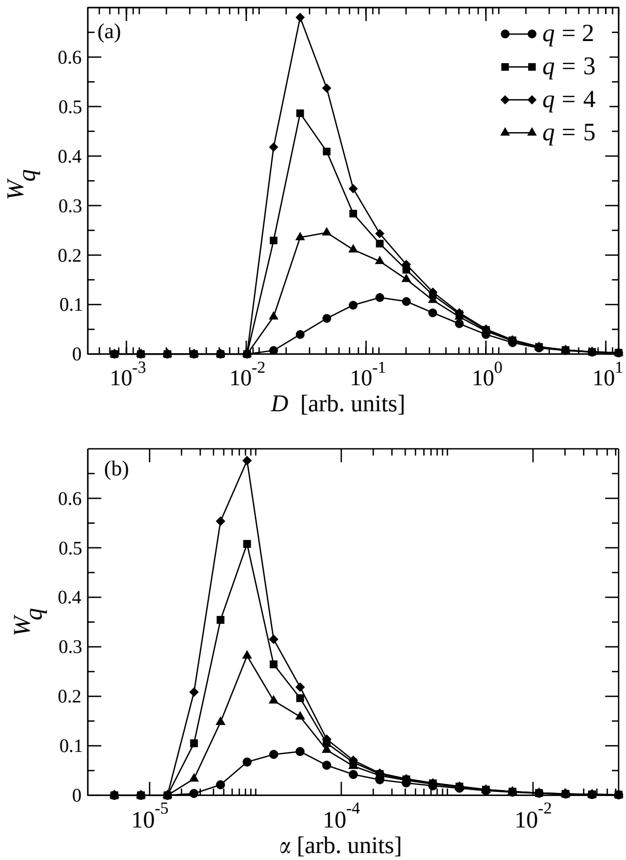

Fig. 3. Normalized escort-Tsallis generalized information measure, $\mathscr{W}_q$, for different values of q: (a) Gaussian noise source, (b) Lorentzian noise. Notice the maximum for an intermediate value of external noise strength. One takes $\Omega = 1.0/\pi$ and $\varepsilon_0 = 0.3$. The input signal is $\mathscr{V}^s(t) = \varepsilon\cos(\Omega t)$.

Fig. 3(a) and (b) depict results for the information measures under consideration. $\mathscr{W}_q$ is plotted as a function of the noise intensities (D or α). Again, the stochastic resonance peak is quite noticeable. These curves were obtained using a time-discretization that employs just $n = 4$. The number of realizations is 10^4.

The different speeds of convergence to the final result for the distinct information measures merit special mention. When Shannon-based information measures are used to determine whether or not an SR phenomenon is present, very long time-runs are required so as to achieve enough statistics in order to build the appropriate histograms. Evaluating the convergence speed of the two generalized information measures introduced in this work requires the prior introduction of the quantity

$$E_r = \frac{\max(M_q) - \min(M_q)}{\langle M_q \rangle}\ , \tag{20}$$

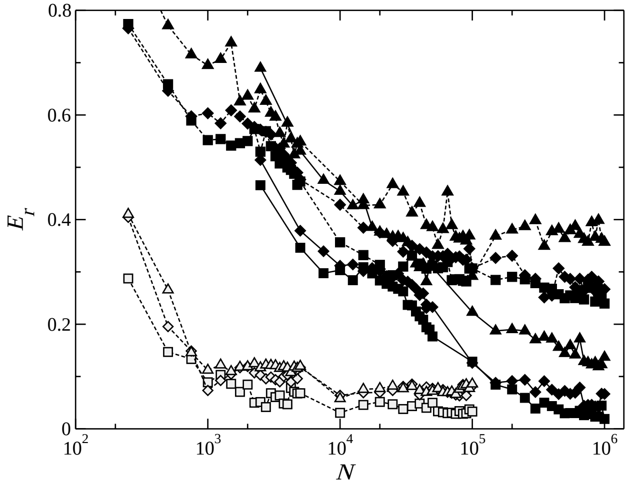

Fig. 4. Errors for the different information measures. The symbols represent: (i) (△) $q=2$; (ii) ◇ $q=4$; and (iii) (□) $q=5$. Grey symbols and dashed lines correspond to the escort-Tsallis' generalized information measure and $n=4$. Finally, white and black symbols are associated to Tsallis' generalized information measure, for $n=4$ and 12, respectively. It is apparent that the escort-Tsallis generalized information measure reaches the steady error regime in a time *two orders of magnitude* shorter than Tsallis' original GIM (see the text).

where M_q is the generalized information measure for which one wishes to determine the measurements' "confidence" and $\langle M_q \rangle$ stands for its average value. In order to evaluate such a degree of confidence we computed 1000 times the pertinent information measure for a given set of system's parameters.

Fig. 4 depicts E_r as a function of the time discretization parameter n. We plotted the escort-Tsallis measure for $n=4$ and different values of q-parameter ($q=2,4,5$). These results are compared with those obtained using the original Tsallis measure for the same set of q-values and two different time discretization parameters: $n=4,12$. Considering $n=4$ for both GIMs, it is apparent that *the escort-Tsallis GIM exhibits a standard deviation for its transmitted information measure that is significantly lower than that of its original-Tsallis counterpart*. We can even assert that Tsallis' GIM cannot give account of the phenomena we are interested in precisely due to the size of the fluctuations that we report here. The same picture arises from simulations for $n=6,8$.

On the other hand, if we compare escort-Tsallis' results ($n=4$) with original-Tsallis' ones ($n=12$), it is apparent that the escort-Tsallis measure yields convergence to a steady error value for the transmitted information measure *two orders of magnitude faster* than in the case of the original-Tsallis measure. Although this fact could be attributed to the difference between the concomitant time discretization parameters n, it is worthwhile remarking here that, for values $n<12$, the Tsallis transmitted information error is always greater than the corresponding escort-Tsallis one (with $n=4$). We conclude that, *in order to reach a given precision in trans-information determination,*

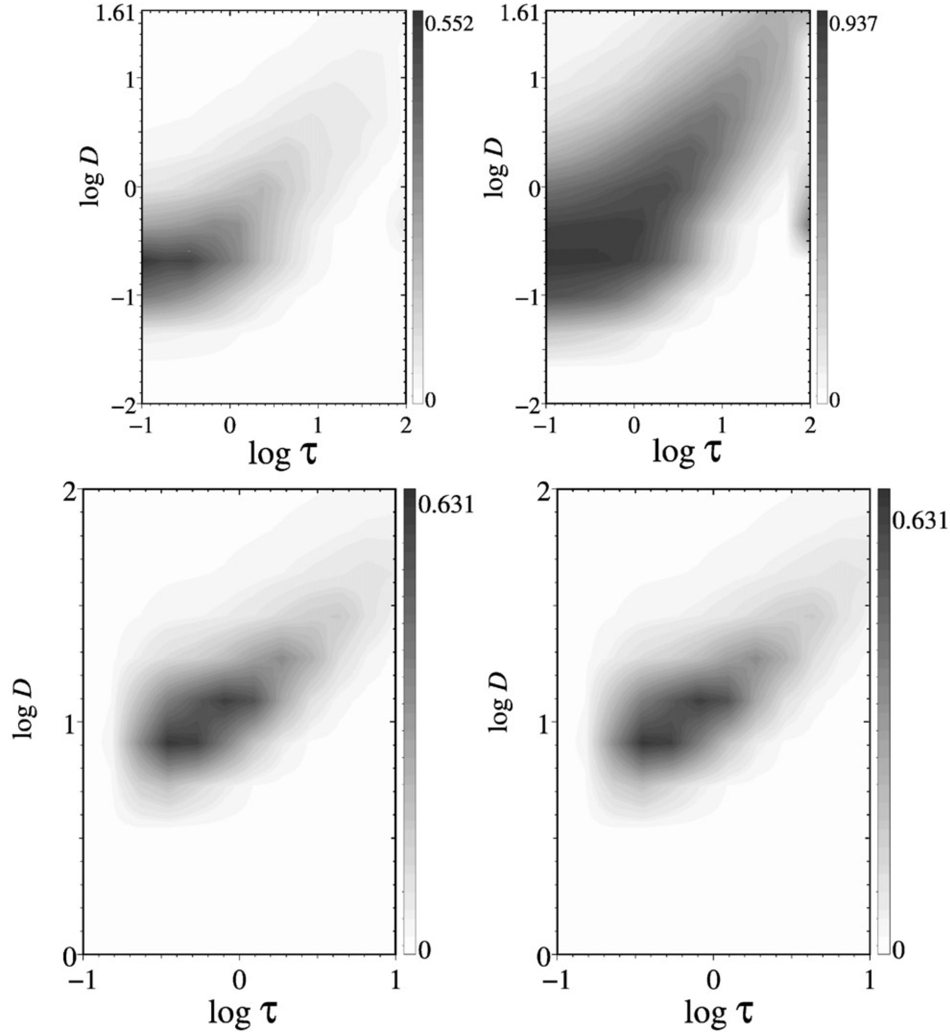

Fig. 5. Contour plots of the escort-Tsallis' measure $\mathcal{W}_q$ as a function of $\tau \in [0.01, 100]$ (horizontal axis) and $D \in [0.01, 50]$ (vertical axis). In each row the noise source q is constant, being in the first $q = 1$ and in the second $q = 0.25$. The value of non-extensivity parameter, used to measure the phenomena is constant in each column, being in the first one $q_m = 0.25$ and in the second $q_m = 3$.

the escort-Tsallis measure requires processing times significantly smaller than the Tsallis one.

We now consider noise sources following the Tsallis' distribution. Fig. 5 shows contour plots for noise sources with $q = 1$ and 0.25 using the escort-Tsallis measure. All these plots were obtained with $n = 6$ and $N = 10^4$. First of all, from these plots it is apparent that (as shown in the previous figures) the higher the q_m-value

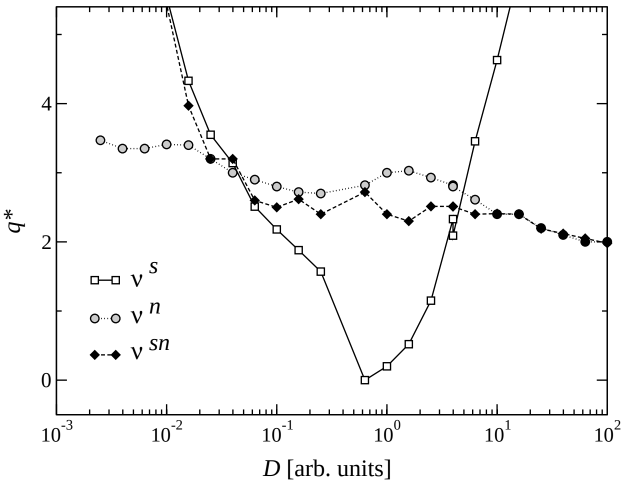

Fig. 6. Optimal value of q, q^*, as a function of D, for a system driven by a Gaussian noise source. The input signals in each plot are $\mathcal{V}^s(t)$, $\mathcal{V}^n(t)$, and $\mathcal{V}^{ns}(t)$.

(the particular parameter q value chosen for the generalized measures), the higher is the value of trans-information. We remark that, for $q \ll 1$ and very low values of τ, the SR phenomenon vanishes. For values of τ large enough (i.e., $\tau > 1$), the SR phenomenon still exists. This is so because such parameter appears in the denominator of Eq. (18). Finally, the response of the system for q and D variable and fixed $\tau = 1$ shows that the lower the noise, (i) the more pronounced the SR peak and (ii) the less dependent on D its localization becomes.

3.2. Optimal q

We have studied the dependence of our W_q results upon the index q, that quantifies the difference between generalized information measures and Shannon's one. We notice that, for fixed system's parameters, W_q grows in monotonic way with q, and that $\lim_{q\to\infty} W_q = 1$. Our interest lies in the optimal value $q = q^*$, such that the gradient of W_q is the largest possible one.

In Figs. 6 and 7, we plot q^* as a function of the noise intensity (D or α). In these graphs we take $\Omega = 1.0$, $\varepsilon = 0.3$, and $n = 12$. In order to compute the information measures we considered as the input signal $\mathcal{V}^{sn}(t) = \varepsilon\cos(\Omega t) + \xi$, the noise realization plus a periodic signal. However, and in order to check limiting behaviors, we also discuss two other cases where the input consists only of (i) pure signal ($\mathcal{V}^s(t)$) or (ii) just noise ($\mathcal{V}^n(t)$). Let us first discuss the results for these two limiting cases.

- Input signal $\mathcal{V}^s(t)$.
    - For both types of noise, $\xi = \xi_G$ and $\xi = \xi_L$, q^* attains a *minimum* for noise intensities such that the SR effect is greater, i.e., for $q \cong 0$. This is the situation anticipated in Section 2.2. It corresponds to maximum super-additivity.

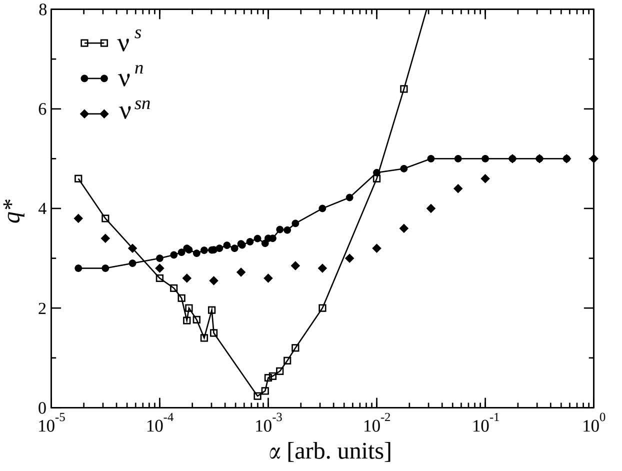

Fig. 7. Optimal q (q^*) as a function of the Lorentzian noise strength α. The input signals considered are $\mathscr{V}^s(t)$, $\mathscr{V}^n(t)$, and $\mathscr{V}^{ns}(t)$.

As we saw in Section 2.2, the GIM become independent of the probability distribution for $q=0$. But that is not necessarily so for the relative measures à la Kullback that are the subject of interest here. As remarked in that section, just one of the terms of, say, the K_q measure, becomes a constant, but not necessarily the other three (cf. Eq. (8)). These read

$$\ln_q P(\vec{V}_n^i) + \ln_q P(\vec{V}_n^o) + (1-q)\ln_q P(\vec{V}_n^i)\ln_q P(\vec{V}_n^o) .$$

What we see here is that the difference between these terms and the one involving the $P(\vec{V}_n^{io})$-probability distribution function becomes a maximum for $q=0$, which is reasonable. In such an instance, a change in the system's parameters affects only the three terms above, but not the constant one, so that the difference between the pertinent terms involved in the evaluation of K_q can grow, and, indeed, it does.

- In the limit in which the noise intensity grows without bound we see that q^* also grows.
- In the limit of no external noise q^* grows as well.

The last two cases involve situations in which the input and output are less correlated than in the instance that precedes them. Indeed, if the noise intensity grows indefinitely, the dynamics of the system will be governed just by noise.

From Eqs. (4) and (6) we gather that for uncorrelated input and output signals

$$\sum_{\vec{V}_n^{io}} P(\vec{V}_n^{io})\ln_q P(\vec{V}_n^{io}) = S_q^i + S_q^o + (1-q)S_q^i S_q^o . \quad (21)$$

which implies, according to (8), that $K_q \equiv 0$. As the input–output correlation decreases, fewer and fewer "input–output pairs"[1] possess probabilities that significantly differ from the product probability that characterizes zero correlation. The relative importance of these special pairs *grows with* q [17]. For both large-noise and vanishing noise intensities, relative measures involving low q values are quite small, and slight modifications in the system's parameters do not affect their value. Conversely, for large enough values of q, the value of W_q is higher, and so are variations due to slight parameters' changes.

- Input signal $\mathcal{V}^n(t)$. In the limit on a diverging noise intensity q^* tends to an asymptotic value q^*_{asint}.
  - For a Gaussian noise $q^*_{asint} = 2$.
  - For a Lorentz noise $q^*_{asint} = 5$.
- Input signal $\mathcal{V}^{sn}(t)$. The relevant limits are
  - For a noise much stronger than the signal, q^* approaches the values obtained when the signal is $\mathcal{V}^n(t)$.
  - For a vanishing noise, q^* behaves as in the $\mathcal{V}^s(t)$-case.

  These limits are easily understood, as the system's dynamics is ruled either by noise (in the case of diverging noise intensity), or by the periodical signal (in the opposite limit).

Finally, Fig. 8 depicts q^* versus (i) D and (ii) the q-parameter of the noise source. Notice that, interestingly enough, for D-values such that the SR phenomenon is greater, as in the previous cases, we obtain $q^* = 0$.

4. Conclusions

The main original contribution of the present work is that of having performed an SR study by exploiting generalized information measures. Two such measures have been employed in this respect: (i) the well known Tsallis one and (ii) the new escort-Tsallis measure [23]. Three (discretized) input signals have been considered: (i) noise, (ii) periodic external modulation, and (iii) the sum of both, while as output we consider the position of a particle, discretized as well. Clearly, the relevant (real) case is the last one, the other two being mainly used for checking limiting behaviors.

The SR effect is seen to be measurable no matter the value of the GIM's characteristic index q. The generalized measures exhibit an important advantage over that of Shannon's: employing them, the SR phenomenon is detected *even if the modulation intensity is very small.* Moreover, of the two GIMs here considered, the escort-Tsallis seems to be the most convenient one. For it, the speed of convergence to acceptable values of transmitted information is much faster than either in Tsallis' or Shannon's cases. Hence, one can anticipate that such an information measure will be the most adequate to an "on-line" control of a signal detection system.

[1] Remember that $P(\vec{V}_n^{io})$ is the probability for the joint appearance of a given input–output pair.

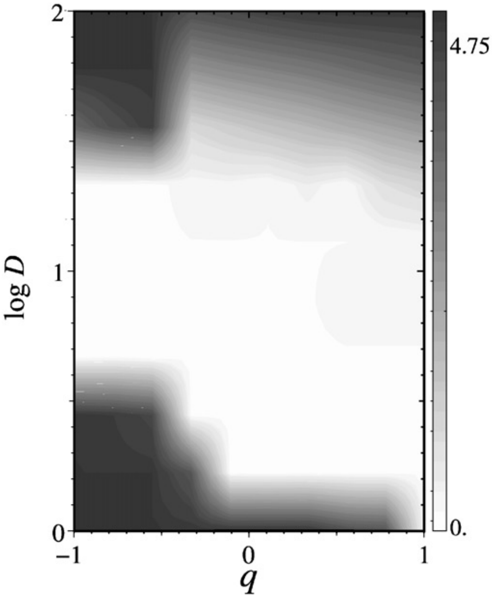

Fig. 8. A contour plot of q^* for $D \in [0.1, 10]$ and noise's $q \in [0.1]$. The noise intensity D is given by the vertical axis, the source of noise q in the horizontal one. White areas corresponds to $q^* = 0$ and black areas to $q^* > 5$.

We have also investigated the existence of an optimal value of q ($q = q^*$), such that the corresponding measure is the most sensitive to variations in the system's external parameters. If the input signal is the periodic modulation we find that, for intensity values that yield the largest SR effect, $q^* \to 0$, no matter what sort of noise is involved. Hence, in this case, super-additivity favors the detection of the SR effect. If the input signal is just noise that grows indefinitely, governing by itself the concomitant dynamics, we find $q^* = 2$ for a Gaussian noise, and $q^* = 5$ for a Lorentz one. Finally, for the relevant case of periodic modulation plus noise, we have found that $q^* = 2$ for a Gaussian noise, while q^* seems to grow as the intensity grows in the case of Lorentz noise. Inspection of Fig. 3a and b, leads to the following conclusion: in the noise intensity (D or α) regime for which a maximum of the information measure is found, the respective values of q^* remain almost constant. This fact underlines the usefulness of determining such optimum q-value (that in both cases is of the order of $q^* = 2.5$). Indeed, processing the relative information with the associated q^*-generalized information measure we find the most efficient manner of detecting the SR phenomenon.

The present study is only a first step towards the analysis of the effect of non Gaussian noises on the SR phenomenon. Clearly, the use of alternative forms of noise opens new possibilities for the SR phenomenon. For instance, if a systematic form of studying non Gaussian noises is developed, it could be possible to determine which is the most convenient degree of departure from "Gaussian-character" as to enhance the system's response. The extension to other (for instance, Tsallis like [15]) probability distribution functions will be the subject of further work.

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