Control of chaotic transients: Yorke's Game of Survival.

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We consider the tent map as the prototype of a chaotic system with escapes. We show analytically that a small, bounded, but carefully chosen perturbation added to the system can trap forever an orbit close to the chaotic saddle, even in presence of noise of larger, although bounded, amplitude. This problem is focused as a two-person, mathematical game between two players called "the protagonist" and "the adversary". The protagonist's goal is to survive. He can lose but cannot win; the best he can do is survive to play another round, struggling ad infinitum. In absence of actions by either player, the dynamics diverge, leaving a relatively safe region, and we say the protagonist loses. What makes survival difficult is that the adversary is allowed stronger "actions" than the protagonist. What makes survival possible is (i) the background dynamics (the tent map here) are chaotic; and (ii) the protagonist knows the action of the adversary in choosing his response and is permitted to choose the initial point x_0 of the game. We use the "slope 3" tent map in an example of this problem. We show that it is possible for the protagonist to survive.

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Transient chaos [1] is an interesting physical phenomenon which occurs in systems where trajectories bounce chaotically for a certain time in a bounded region until they reach a final state, usually nonchaotic. Varied manifestations of transient chaos are present in chaotic scattering [2], chaotic advection in fluid dynamics [3], species competition in ecology [4, 5] or voltage collapse in electric power systems [4, 6], to cite just a few. From the point of view of Nonlinear Dynamics, the phenomenon of transient chaos is associated to the existence of a certain type of sets called chaotic saddles, also known as nonattracting chaotic invariant sets, formed by a bounded set of unstable periodic and aperiodic orbits, for which almost all trajectories diverge. Typical orbits in the system will approach the chaotic saddle following its stable manifold, spend some time bouncing in its vicinity and then escape from it following its unstable manifold. Therefore, a compelling challenge might be to find a simple method to maintain an orbit in the neighborhood of the invariant set for all times, respecting the original dynamics of the system. Since the seminal paper of Ott, Grebogi and Yorke [7], the theory of chaos control in Nonlinear Dynamics has been thoroughly developed, both for Hamiltonian and dissipative systems. Nevertheless, most of the work has been focused into systems with chaotic attractors, both in noiseless and noisy environments [8], while little attention has been paid to controlling chaotic transients [4, 9].

While for a linear system the perturbation needed to change its nature is of the same order of the dynamics of the motion, the extreme sensitivity to initial conditions makes control with very little perturbations a possible task. In this sense, diminishing the amplitude of control is an important goal in this field. Obviously, if the system is embedded in a noisy environment controlling orbits is even harder, and typically stronger amplitudes than in the noiseless case are needed.

Since Akiyama and Kaneko presented the "dynamical systems game theory" [10–12], there has been a growing interest for modelling increasingly more complex game strategies with concepts borrowed from Nonlinear Dynamics. In their work it is shown that Game Theory has resulted to be deeply related to several problems involving dynamical phenomena, and for many cases it is possible to switch from the point of view of Game Theory to that of Nonlinear Dynamics. In fact, the nature of these games can be described as a dynamical system. Our work points in this direction, and we face our problem as a mathematical game between two players called "the protagonist" and "the adversary", being the protagonist's goal to survive inside a bounded region, that is, the vicinity of the chaotic saddle. We describe an idea which we apply here to a very simple nonlinear dynamical system, but can be conveniently adapted for a wide variety of maps with a chaotic saddle, in which some kind of noise and control is present. In a system with attractors, the natural tendency of a particle is to reach one of these attractors, and therefore it is plausible for the protagonist to maintain itself close to one attractor even when the adversary is allowed slightly stronger actions. However, it is important to remark that without any external control, the probability of the protagonist to survive in the vicinity of a chaotic saddle is zero, even

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in the absence of noise, and this fact makes the survival of the protagonist a remarkable achievement.

The simplest form of this game involves a one-dimensional map, the tent map, that is defined as: $T(x) = m(1 - |x|) - 1$. For cases of interest such as $m = 3$, almost all initial points x_0 yield trajectories of $x_{n+1} = T(x_n)$ that go to $-\infty$ as $n \to \infty$. And in this case we say the protagonist does not survive. To survive he must act. The equation of the game is:

$$x_{n+1} = T(x_n) + u_{n+1} + r_{n+1} \qquad (1)$$

where the adversary chooses the perturbation u_{n+1} (knowing x_n and T) and the protagonist then chooses the "response" r_{n+1} (knowing u_{n+1} and x_n and T). The perturbation u_{n+1} might be chosen at random or using an effective strategy. In the long run there is little difference between these two as to whether the protagonist can survive forever. The protagonist faces what appears to be an impossible task because we permit only $|u_n| \le u_0$ and $|r_n| \le r_0$ where r_0 and u_0 are specified with $r_0 < u_0$. If r_n is viewed as the control and u_n is viewed as some kind of noise (or interference), the usual requirement is that the control is stronger than the noise. However, the main goal of this paper is to show that in the context of transient chaos it is possible to control a noisy orbit, even in the case in which noise is stronger than control. The smaller bound on r_n than on u_n might lead us to call r_n an "influence" rather than a "control" since the protagonist cannot control the details of the trajectory. For this problem, we let the "relatively safe" region be the interval $S = [-1, +1]$ and terminate the game if some x_n is outside S. Certainly if x_n is outside S, it is possible for the adversary to choose the sequence u_n that causes the sequence x_n to diverge, and there is a slightly larger interval depending on u_0 and r_0 such that if x_n is outside that, the trajectory must diverge even if the adversary tries to help. To keep formulas simple, we state our results for $m = 3$, though analogous results are available for all $m > 2$. (If $m \le 2$, there is a chaotic attractor and if u_0 is sufficiently small, survival is guaranteed even if the response size is 0.) We begin with an example.

For $u_0 = 4/9$ and $r_0 = 2/9$, there exists a strategy guaranteeing survival.

If $u_0 > 2r_0$ then there is no strategy guaranteeing survival.

The best strategy for survival depends on r_0 as is made clear in the following Theorem. There are different strategies for $r_0 \ge 2/3$, and each integer k where r_0 is in $[2/3^k, 2/3^{k-1})$. Recall $m = 3$.

Theorem. There is a strategy guaranteeing survival for a given r_0 and u_0 if and only if there is an integer $k \ge 1$ for which $2/3^k \le r_0$ and $u_0 \le r_0 + 2/3^k$. (The cross-hatched part of Fig. 1 shows where there are strategies for survival).

This type of problem is quite different from standard control in which the goal is to drive the trajectory to a point. In controlling chaos [7, 9] for example, if noise

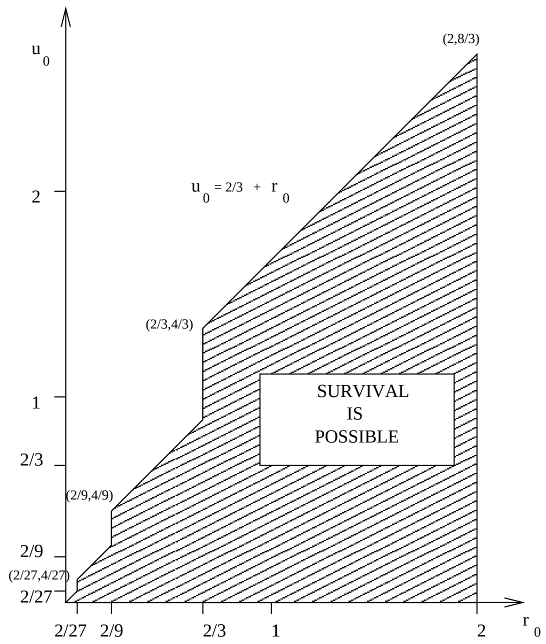

FIG. 1: Parameter region of survival. Survival is possible in the cross-hatched region if the protagonist chooses optimally. Above the cross-hatched region, the adversary can always win.

is present (i.e., u_n chosen at random), the control r_n must dominate u_n so as to be able to drive the trajectory to a specified fixed point and keep it close to the fixed point. In the game of survival for the tent map, there are several "safety points" and r_0 must be large enough that the protagonist can reach one of them, but the choice of which is really determined by what u_n happens to be. The protagonist is bounced between these safety points in an order determined by the sequence of u_n.

The Example. Before analyzing the theorem in detail, we examine the case mentioned above, $u_0 = 4/9$ and $r_0 = 2/9$ and show the protagonist can survive. We designate four points as "safety point", $z_1 = -2/3 - 2/9$, $z_2 = -2/3 + 2/9$, $z_3 = +2/3 - 2/9$ and $z_4 = +2/3 + 2/9$. It is easy to check that $T(z_i) = \pm 2/3$, and $T(\pm 2/3) = 0$. A graph of the tent map appears in Fig. 2 showing all these points, and Fig. 3 shows the evolution of an orbit in this situation. The protagonist's strategy must be to make sure every x_n in Eq. (1) is a safe point if it is to guarantee that he can survive. In particular, the protagonist must choose x_0 equal to one of the safety points to make sure he succeeds (although in fact most points in $S = [-1, 1]$ would also be valid as x_0.) If x_n is a safety point for any integer $n \ge 0$, then we show he can choose r_{n+1} so that x_{n+1} is a safety point, and so he survives another day. Since x_n is a safety point, we may suppose for example $T(x_n)$ is $+2/3$. (The case $-2/3$ is virtually the same.) Then after u_{n+1} is chosen, the point $T(x_n)+u_{n+1}$ must be in the interval $[2/3 - 4/9, 2/3 + 4/9]$ and so is at most $2/9$ from either z_3 or z_4. Hence r_{n+1} can be chosen

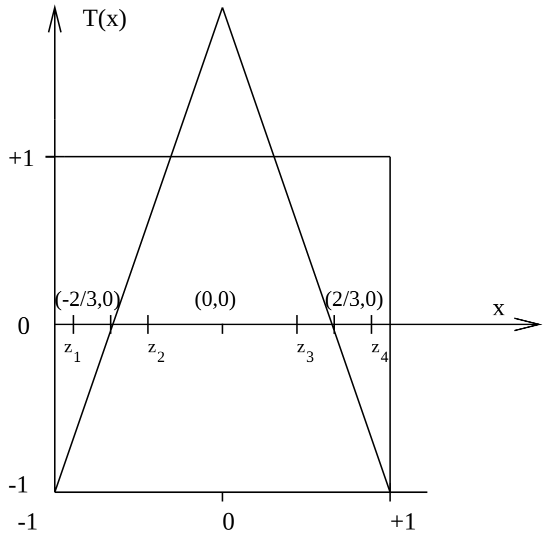

FIG. 2: Graph of the tent map $T(x) = m(1-|x|)-1$ defined in the interval $[-1,+1]$ for $m = 3$. The four points z_i designate "safety points" and also $T(z_i) = \pm\ 2/3$.

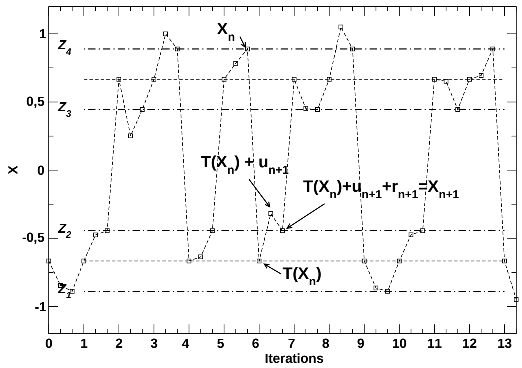

FIG. 3: Evolution of an orbit, for $k = 2$, $m = 3$, $r_0 = 2/9$ and $u_0 = 4/9$. The four dotted-dashed lines represent the "safety points" z_i, and the dashed lines represent their images $T(z_i) = \pm\ 2/3$. The points that do not lie over any of these lines represent the steps of the orbit after the influence of the noise u_n.

with $|r_{n+1}| \leq r_0$ so that x_{n+1} is a safety point. This case may be generalized by noting that this strategy works whenever $u_0 - r_0 \leq 2/9$.

This example illustrates why we call this problem a game of "survival" rather than of "control", since the protagonist is buffeted from safety point to safety point without being able to choose between these points (as it is shown in Fig. 3.) There is typically only one that can be reached with $|r_{n+1}| \leq r_0$ for each n. In the above example calculation, notice that $T(x_{n+1})$ is either $-2/3$ if x_{n+1} is z_4 or $+2/3$ if z_3. The protagonist cannot choose whether $T(x_{n+1})$ is to be positive or negative (unless u_{n+1} was 0 so that z_3 and z_4 were equally close).

The general strategy (called R) for choosing r_{n+1} is to identify a collection of safety points and choose x_0 to be one of them and from then on choose r_{n+1} so that x_{n+1} is a safety point. In the case where $2/3 \leq r_0$ and $u_0 \leq r_0 + 2/3$, $(k = 1)$, there are 2 safety points namely $z_1 = -2/3$ and $z_2 = 2/3$. Then if x_n is a safety point, $T(x_n) = 0$, and the point $T(x_n) + u_{n+1}$ must be in the interval $[-u_0, u_0]$. Since $u_0 \leq r_0 + 2/3$, each point of the interval is within r_0 of a safety point. Hence the strategy can be carried out.

In the general case where $2/3^k \leq r_0$ and $u_0 \leq r_0 + 2/3^k$, there are 2^k safety points, namely $T^{-k}(0)$ which consists of

$$\pm 2/3^1 \pm 2/3^2 \pm\ ... \pm 2/3^k \text{ for } k \geq 1 \tag{2}$$

Note that $T(\pm 2/3^1 \pm 2/3^2 \pm\ ... \pm 2/3^k)$ is a point of the form $\pm 2/3^1 \pm 2/3^2 \pm\ ... \pm 2/3^{k-1}$ (which is the single point 0 if $k = 1$). The argument showing that the strategy can be implemented proceeds as in the special cases discussed above.

We now argue that a guaranteed strategy exists only for the above cases. Hence if k is chosen so that $2/3^k \leq r_0 < 2/3^{k-1}$ for some $k \geq 1$, and $u_0 = r_0 + 2/3^k + \delta$ where $\delta > 0$, then no guaranteed strategy exists; in other words, there is a strategy U for choosing the points u_n so that the protagonist loses.

Let S_k be the set of safe points. The strategy U is to choose u_n so that $T(x_{n-1}) + u_n$ is as far as possible. Let Y_k be the set $\{x : |x - y| \leq r_0$ for some y in $S_k\}$. Hence Y_k is the set of points that are no more than r_0 from some safe points. For any point x_0, there is a u_1 with $|u_1| \leq u_0$ such that $T(x_0) + u_1$ is not in Y_k. Hence $x_1 = T(x_0) + u_1 + r_1$ (with $|r_1| \leq r_0$) is not a safe point. Let J_k be the smallest interval containing S_k.

If x_n is not in J_k, it is easy to check that strategy U results in x_{n+1} also outside J_k, but further from S_k. If x_n is in J_k, let J' denote the smallest interval containing x_n whose ends are safe points. Strategy U results in x_{n+1} which is in $T(J')$, which has no points of S_{k-1} in its interior and x_{n+1} is further from S_k. Furthermore the length of $T(J')$ is greater than that of J'. As the process evolves, the trajectory eventually is outside J_k, a case which is discussed above.

We have carried out several computer experiments to clarify the applicability of our results. A uniform distributed noise with zero mean value has been used as u_n, since its only requisite is to be bounded. Obviously, the same results would have been obtained for any other kind of bounded noise. Note that, for this reason, Gaussian noise does not guarantee the survival of the protagonist. For very different values of k, m, maximum response r_0 and maximum perturbation u_0, being $r_0 \leq u_0$, we have

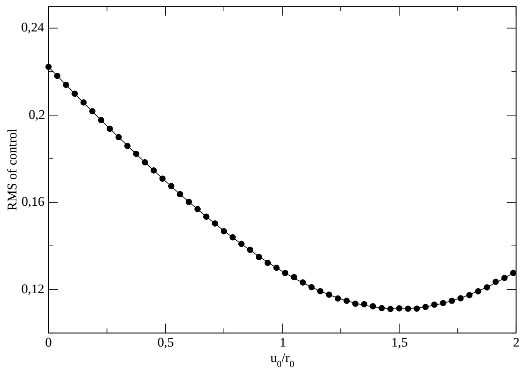

FIG. 4: The control needed *decreases* in presence of (weak) noise. The picture shows the root mean square of applied control for different noise-control ratios when $m = 3$, $r_0 = 2/9$. The dots were calculated numerically, while the straight line represents the analytical curve.

iterated the game up to several million steps. As our theorem asserts, the protagonist survives inside the safe region $[-1, 1]$ if and only if $u_0 \leq 2r_0$.

An interesting property of the system appears when we analyze the root mean square (RMS) of the control r_n, which is expressed as

$$\text{RMS} = \sqrt{\frac{\sum {r_n}^2}{n}}. \tag{3}$$

Figure 4 shows the evolution of the RMS of control when the maximum noise to maximum control ratio is varied, both computationally calculated and analytically derived. We have fixed the control to $r_0 = 2/9$, and u_0 is varied from 0 to $u_0 = 2 \cdot r_0 = 4/9$. For $u_0 = 0$, that is, in the absence of noise, the control strategy is to push repeatedly the system back to a safety point after that the dynamics has displaced it. The strength of control is thus constant and equal to the distance to go from the image of a safety point back to any of the safety points. Calling such a distance d_k, we have $d_k = \max_j\{\min_i\{|z_i - T(z_j)|\}\}$ and $\text{RMS} = d_k$. When noise is switched on, the RMS of control *decreases*, since in this case the orbit is pushed by the noise r_n from the image of a safety point towards one of the 2^k safety points. This result is in contrast with standard algorithms of chaos control, that aim at stabilizing unstable orbits instead of preimages of the escaping region. For these techniques, a *stronger* control is needed if noise increases. Finally, for high values of u_0/r_0, the RMS of control shows a minimum and starts to increase again, as there is a value of the noise for which on average the noise places the orbit optimally close to one of the safety points.

The analytical derivation of the curve for RMS is as follows. Looking at Fig. 2, and noticing that the positions of the safety points z_i are symmetric, the control needed after a noise displacement $u \leq u_0$ can be simply written as:

$$|r(u)| = ||u| - d_k|. \tag{4}$$

Indicating with $\sqrt{\langle r^2 \rangle}$ the RMS, with $r(u)$ the control needed after a noise displacement u, and with $f(u)$ the noise distribution, we obtain the following:

$$\langle r^2 \rangle = \int_{-u_0}^{u_0} r(u)^2 f(u) du = \int_{-u_0}^{u_0} (|u| - d_k)^2 f(u) du. \tag{5}$$

Expanding the expression and distributing the integral, we have:

$$\langle r^2 \rangle = d_k^2 + \textstyle\int_{-u_0}^{u_0} u^2 f(u) du - 2d_k \int_{-u_0}^{u_0} |u| f(u) du = \tag{6}$$

$$d_k^2 + \langle u^2 \rangle - 2d_k \langle |u| \rangle. \tag{7}$$

To give an example, we can evaluate this expression for the case of uniform noise, that is,

$$\begin{cases} f(u) = \frac{1}{2u_0} & -u_0 < u < u_0, \\ f(u) = 0 & \text{otherwise.} \end{cases} \tag{8}$$

A straightforward calculation gives:

$$\langle u^2 \rangle = \frac{1}{2u_0} \int_{-u_0}^{u_0} u^2 du = \frac{1}{3} u_0^2, \tag{9}$$

and:

$$\langle |u| \rangle = \frac{1}{2u_0} \int_{-u_0}^{u_0} |u| du = \frac{1}{2} u_0. \tag{10}$$

Finally, we obtain that the RMS of control for such distribution is:

$$\sqrt{\langle r^2 \rangle} = \sqrt{d_k^2 + \tfrac{1}{3} u_0^2 - d_k u_0}. \tag{11}$$

If maximum control r_0 is set to d_k, this function has a minimum when $u_0/r_0 = 3/2$. Figure 4 confirms this result.

The results of this work can be easily generalized to any unimodal one-dimensional map, showing that it is always possible to survive with less control than noise. The relation $\frac{u_0}{r_0}$, as well as the structure of safety points, will depend on the properties of each map, its symmetry or asymmetry, etc.. In order to point this fact, we have developed a similar analytical study for the asymmetric tent map, and the same strategy yields a noise to control ratio of $\frac{u_0}{r_0} = 1 + \left(\frac{m}{l}\right)^k$, where $m < l$ are the left and right slopes respectively. It is easy to see that this ratio has a maximum equal to 2 for the symmetric case $m = l$ and a minimum equal to 1 when the right slope is infinitely larger than the left one.

In summary, in this paper we are describing an idea which potentially can be applied to a wide variety of maps with a chaotic saddle (i.e., an invariant set in any dimension for which almost all trajectories diverge), embedded in noisy environments, for an appropriate choice of r_0 and u_0. Such an analysis could be far more complex than for the symmetric and asymmetric tent map, for which the problem can be fully explained analytically. Unlike traditional control theory that tries to steer the state of a system to a precise state, there are situations in which we only have influence in a chaotic environment. The difference between *influence* and *control* is roughly speaking $r_0 < u_0$ vs. $r_0 > u_0$.

Finally, the information that is needed in order to apply our method is just the approximate position of the safety points. This information might be obtained from time series analysis, suggesting the applicability of this control to real systems.

We acknowledge James A. Yorke for suggesting us this problem, which we believe may be viewed, in some sense, as an allegory of daily life.

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