Anticipating the dynamics of chaotic maps

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Abstract

We study the regime of anticipated synchronization in unidirectionally coupled chaotic maps such that the slave map has its own output reinjected after a certain delay. For a class of simple maps, we give analytic conditions for the stability of the synchronized solution, and present results of numerical simulations of coupled 1D Bernoulli-like maps and 2D Baker maps, that agree well with the analytic predictions.

Key words: Chaos synchronization, Anticipated synchronization
PACS: 05.45.Xt, 05.45.Gg

The synchronization of chaotic systems is a subject that has attracted a lot of attention in the past years. Since the pioneering works [1] several different regimes of synchronization have been found: complete synchronization, phase synchronization [2], lag synchronization [3], generalized synchronization [4,5], synchronization by common noise force [6,7], among others.

Anticipated synchronization is a recently discovered synchronization regime that occurs in unidirectionally coupled systems [8,9]. In this regime counterintuitive phenomena occur, since the slave system anticipates the chaotic evolution of the master system, despite the fact that chaotic behavior implies long-term unpredictability.

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In the case of coupled time-delayed differential equations the anticipation time is related to the difference between the intrinsic delay time of the systems and the delay time of the coupling [10]. In coupled ordinary differential equations (as the Lorenz and Rossler systems) the anticipation time must be small to have a stable synchronization manifold [8]. However, by using a chain of slave systems, anticipation times that are multiples of the coupling delay time and that exceed characteristic time scales of the chaotic dynamics can be obtained [9,11]. Many additional numerical [12,13] and experimental [14,15] studies of anticipated synchronization have been performed.

Recently, analytic conditions for the synchronization of coupled maps with delays were given by Masoller and Zanette [16]. In the maps considered in [16] the chaotic behavior is induced by the delay term in the map (in other words, without delay the master map is not chaotic). In Ref. [16] the master map (x_n) and the slave map (y_n) are of the form

$$x_{n+1} = bx_n + f(x_{n-N}), \tag{1}$$

$$y_{n+1} = by_n + (1-\eta)f(y_{n-N}) + \eta f(x_{n-M}). \tag{2}$$

with $|b| < 1$ and f a nonlinear function. The synchronized solution is given by $y_n = x_{n-M+N}$. By studying the evolution of a small perturbation of the solution it was shown that the synchronization is stable for $\eta = 1$. The existence of a threshold value of the coupling, $\eta_c < 1$, above which the synchronized solution is stable, was also shown.

When the chaotic behavior of the master map is not induced by a delay, to the best of our knowledge no analytical conditions for anticipation have been reported. In this letter we study a system composed by a chaotic map (master) unidirectionally coupled to a second chaotic map (slave) which has its own signal reinjected after a certain delay. We consider a master map, x_n, and a slave map, y_n of the form

$$x_{n+1} = f(x_n), \tag{3}$$

$$y_{n+1} = f(y_n) + \gamma(x_{n-N} - y_{n-M}), \tag{4}$$

where f is a nonlinear function. We show that the delayed auto-injection in the slave map leads to anticipation in the synchronization, and present analytic conditions for the stability of anticipated synchronization. An analytical treatment is possible because we consider simple chaotic maps. We exemplify the results with numerical simulations of coupled 1D Bernoulli-like maps and 2D Baker maps.

The synchronization manifold of (3), (4) is given by

$$y_n = x_{n-N+M}, \tag{5}$$

and thus the slave variable is lagged by $N - M$ steps behind the value of the master variable (if $N - M < 0$, the slave map anticipates the dynamics of the master map). To study the stability of the synchronized solution we consider a perturbation of the form

$$y_n = x_{n-N+M} + \delta_n. \tag{6}$$

In the linear regime, the perturbation obeys the following map

$$\delta_{n+1} = f'(x_{n-N+M})\delta_n - \gamma\delta_{n-M}, \tag{7}$$

where $f' = df/dx$. Making the change of variables $\delta_n = (z_n)^n$ gives

$$(z_{n+1})^{n+1} = f'(x_{n-N+M})(z_n)^n - \gamma(z_{n-M})^{n-M}. \tag{8}$$

Notice that the value of N is irrelevant in the long-term behavior, thus the stability of the synchronized solution is the same in the anticipated and in the retarded regimes (similar results were found in [16]). Therefore, and without loss of generality, in the following we consider $N = 0$.

A sufficient condition for the stability of the synchronized solution will be that, for $n > n_0$, where n_0 is some number of transient steps, all the solutions z_n^i of (8) (where i labels the different solutions) satisfy $|z_n^i| < 1$. When f' depends on x_n, an analytical treatment of the stability of the synchronized solution is in general not possible. However, there are particular cases in which the study of Eq. (8) gives insight into the parameter region where synchronization is stable. As a first example we consider the 1D Bernoulli-like map

$$f(x_n) = ax_n \bmod 1 . \tag{9}$$

The map is chaotic for $a > 1$. $f' = a$, and thus the solutions of (7) are linear combinations of functions of the form $\delta_n = z^n$, with constant z. Thus (8) reads

$$z^{M+1} = az^M - \gamma. \tag{10}$$

Stability of the synchronized solution is obtained if all the solutions of (9) satisfy $|z| < 1$.

For $M = 1$ the roots of Eq. (10) are the solutions of a simple quadratic equation, so that it is simple to check that they have $|z| < 1$ when $\gamma \in (a-1, 1)$. For arbitrary M a *necessary* (but not sufficient) condition for the stability of the synchronized solution is $\gamma \in [a-1, a+1]$. To show that, first note that for $\gamma = 0$, Eq. (10) has one root at $z = a > 1$, and the other M roots are at $z = 0$. A simple analysis of perturbations shows that a small γ breaks the degeneracy of the roots at the origin, which then move radially outwards in the complex plane, whereas (if $\gamma > 0$) the root that was located at $z = a$ diminishes its value, moving towards the unit circle. Thus, stability of the synchronized solution will be obtained by increasing γ if this last root enters into the unit circle in the complex plane before some of the other M roots leave it. Finally, for γ large enough, all the roots are outside the unit circle. Eq. (10) can be rewritten as

$$z^M = -\frac{\gamma}{z-a}. \tag{11}$$

Thus, the value of $|z|^M$ is bounded between

$$\frac{\gamma}{a+|z|} \leq |z|^M \leq \frac{\gamma}{a-|z|}. \tag{12}$$

At the limits of the stability region, some root z would satisfy $|z| = 1$, so that

$$\frac{\gamma}{a+1} \leq 1 \leq \frac{\gamma}{a-1}. \tag{13}$$

which leads to $a - 1 \leq \gamma \leq a + 1$. Therefore, only within this interval a root might cross the unit circle, leading to stability changes of the synchronized solution. In consequence the range of γ leading to stable synchronization is inside $\gamma \in [a-1, a+1]$. Further insight can be gained from the study of the roots at the boundaries of this interval. First we consider the case $\gamma = a - 1$. For this value of the coupling Eq. (10) becomes

$$z^{n+1} = az^n - a + 1. \tag{14}$$

Clearly, $z = 1$ is a solution for all a and M. A small perturbation of the value of γ, $\gamma = a - 1 + \delta\gamma$ leads to a modification of the value of z, $z = 1 + \delta z$. To first order, δz and $\delta\gamma$ are related by

$$\delta z = -\frac{\delta\gamma}{1 - M(a-1)}. \tag{15}$$

The denominator is positive if $M < 1/(a-1)$ and negative otherwise. Thus, if $M < 1/(a-1)$, by increasing γ, $(\delta\gamma > 0)$, $\delta z < 0$ and a real root enters into the unit circle. Since this value of γ is the smallest one for which crossing the unit circle becomes possible, all the roots satisfy now $|z| < 1$ and thus the synchronized solution becomes stable. On the other hand, if $M > 1/(a-1)$, by increasing γ, $\delta z > 0$, and a real root leaves the unit circle. In this case the synchronized solution becomes more unstable, in the sense that the rate of escape given by the largest $|z|$ increases by increasing γ. Thus, $M < 1/(a-1)$ gives a limit of the number of steps M for which the synchronized solution can become stable. There is a relation between M and the degree of chaos, associated to the Lyapunov exponent $\log a$, of the master map: the largest the value of a, the lower the stable anticipation times. If $a > 2$, stable anticipated synchronization becomes impossible.

Next, we consider the other boundary of the synchronization region, $\gamma = a+1$. In this case Eq. (10) becomes

$$z^{n+1} = az^n - a - 1. \tag{16}$$

If M is even, $z = -1$ is a solution. Considering a perturbation of the form $\gamma = a + 1 + \delta\gamma$, in the same way as before it can be shown that $z = -1 + \delta z$ with $\delta z = -\delta\gamma/[1 + M(a + 1)]$. Since the denominator is always positive, if γ grows, z decreases, so that the root close to $z = -1$ leaves the unit circle. Thus, we simply confirm that a necessary condition for the stability of the synchronized solution is $\gamma < a + 1$.

Next we show results of numerical simulations that confirm these analytic arguments. Figure 1 shows simulations of Bernoulli-like maps, in which the slave anticipates the master in five steps. During the first 5000 steps the maps evolve independently. Then we set the value of the coupling to $\gamma = 0.15$ and after a very short transient the anticipation of the slave to the master is evident. Figure 2 shows, for the same value of the parameters a and M, and increasing coupling γ, how the roots of Eq. (10) move in the complex plane. For low coupling Eq. (10) has one real root $z_1 \sim a > 1$, one real root $z_2 < 1$, and four complex conjugate roots with modulus less than 1 [Fig. 2 (a)]. As the coupling increases, z_1 decreases while the other roots increase their modulus, approaching the unit circle. For $\gamma \gtrsim 0.1 = a - 1$ all roots of Eq. (8) have modulus less than 1 [Fig. 2 (b,c)] and the synchronized solution is stable (Fig. 2(c) corresponds to Fig. 1). For even larger coupling, pairs of complex-conjugate roots cross the unit circle [Fig. 2 (d)], and synchronization is unstable again.

For values of the coupling such that all roots of Eq. (10) are inside the unit circle, the distance between the two trajectories decreases exponentially, $|x_{n+M} - y_n| \sim |x_M - y_0| \exp(-n/\tau)$, with the transient time to synchronization given by the inverse of the logarithm of the modulus of the largest root, $\tau = -1/\ln|z_1|$. Fig. 3 shows the transient evolution of $|x_{n+M} - y_n|$, for the

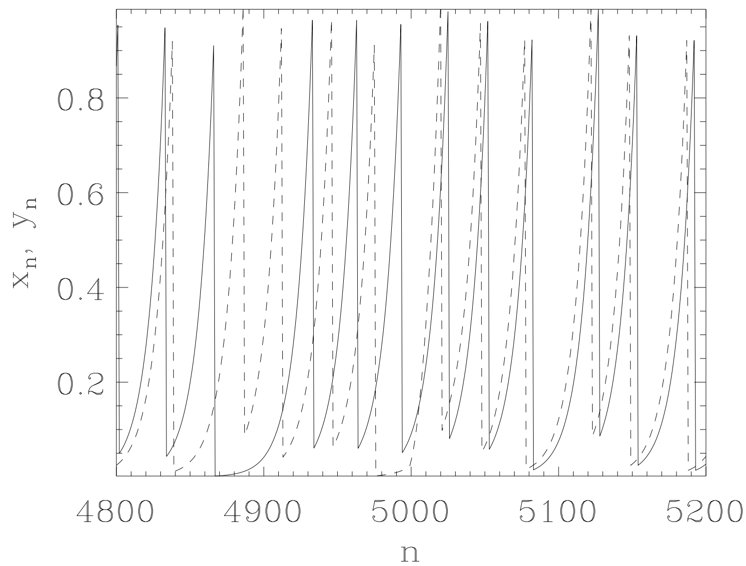

Fig. 1. Time series of Bernoulli-like maps (x_n solid line, y_n dashed line) for $a = 1.1$, $\gamma = 0.15$, $M = 5$. The coupling is set on at $n = 5000$, leading to anticipated synchronization.

parameters of Fig. 1. We observe a damped oscillatory behavior. While the damping time is τ, the frequency is associated to the phase of z_1.

As a second example we consider the 2D Baker's map that transform the unit square into two non overlapping rectangles: The master map, (x_n^m, y_n^m), and the slave map, (x_n^s, y_n^s), are

$$\begin{cases} x_{n+1}^m = f_x(x_n^m, y_n^m) \\ y_{n+1}^m = f_y(x_n^m, y_n^m) \end{cases} \tag{17}$$

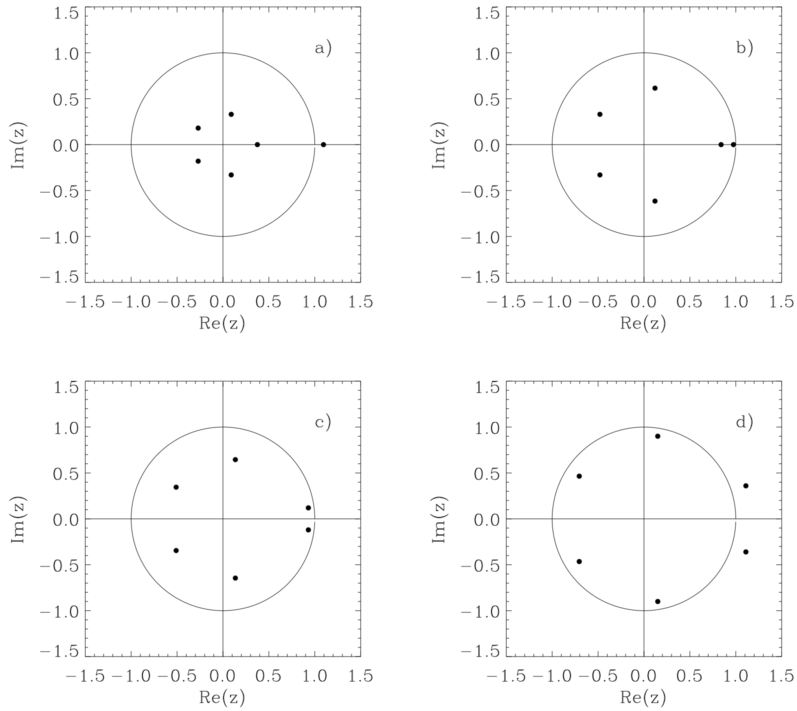

Fig. 2. Roots of Eq. (10) for $a = 1.1$, $M = 5$ and (a) $\gamma = 0.005$, (b) $\gamma = 0.11$, (c) $\gamma = 0.15$, and (d) $\gamma = 0.8$.

$$\begin{cases} x^s_{n+1} = f_x(x^s_n, y^s_n) + \gamma(x^m_n - x^s_{n-M}) \\ y^s_{n+1} = f_y(x^s_n, y^s_n) + \gamma(y^m_n - y^s_{n-M}). \end{cases} \tag{18}$$

where

$$f_x = \begin{cases} ax_n & \text{if } x_n < 1/a, \\ a(x_n - 1/a) & \text{if } x_n \geq 1/a, \end{cases} \tag{19}$$

$$f_y = \begin{cases} by_n & \text{if } x_n < 1/a, \\ by_n + (1-b) & \text{if } x_n \geq 1/a, \end{cases} \tag{20}$$

$a > 1$ and $b < 1$ are the expansion and contraction rates, respectively. It is easy to see that for the stability of the synchronized solution we obtain a pair of equations of the same form as Eq. (10), where now a is equal to the expansion and to the contraction rate, respectively. Both equations must have roots with modulus less than 1, for the synchronized solution to be stable. In Fig. 4 we present results of numerical simulations that show anticipated synchronization by one step. The coupling is set on at $n = 30$, and after a

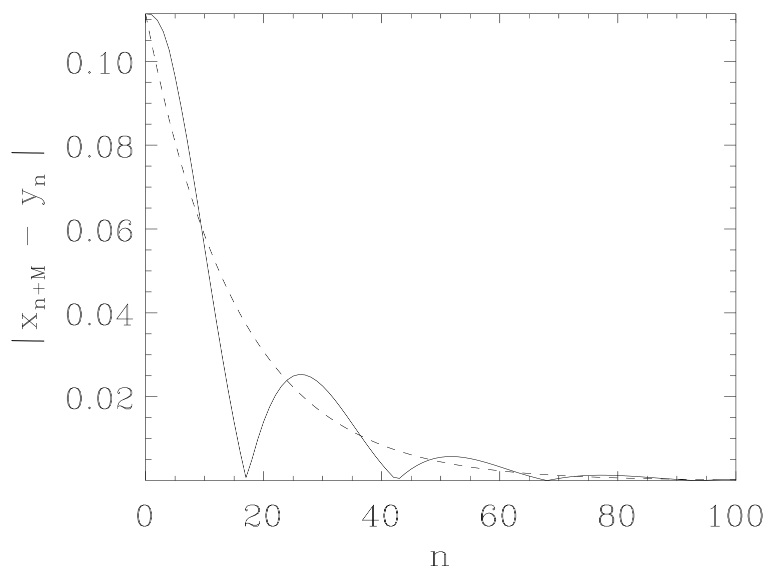

Fig. 3. Transient decay of $|x_{n+M} - y_n|$ for $a = 1.1$, $M = 5$ and $\gamma = 0.15$. The solid line indicates the value of $|x_{n+M} - y_n|$ and the dashed line indicates the value of $|x_M - y_0| \exp(-n/\tau)$ with $\tau = -1/\ln|z_1| \sim 15.55$ as explained in the text.

transient the map solution approaches the anticipated synchronization state. We find synchronized solutions for parameter values such that the equations analogous to (10) have roots with modulus less than 1. The duration of the transient, again, is related to the modulus of the largest root.

In summary, we have studied the regime of anticipated synchronization in unidirectionally coupled chaotic maps. In a general case is not possible to give analytic conditions for the parameter region where anticipated synchronization occurs, but we have presented two class of maps, 1D Bernoulli-like maps

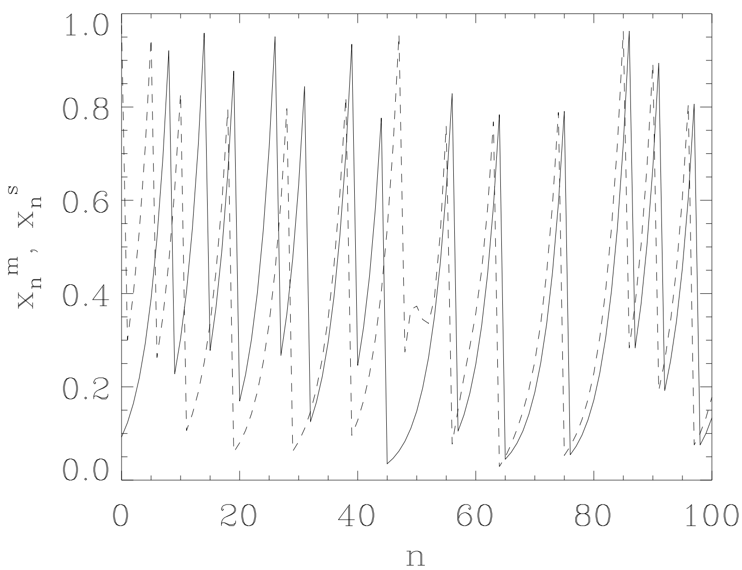

Fig. 4. Time series of the 2D Baker map (x_n^m solid line, x_n^s dashed line) for $a = 1.333$, $b = 0.777$, $\gamma = 0.7$, and $M = 1$. The coupling is set on at $n = 30$, leading to anticipated synchronization.

and 2D Baker maps, in which an analytic treatment of the stability of the synchronized solution is possible. The results of numerical simulations are in good agreement with the analytic predictions.

C. Masoller was supported by Proyecto de Desarrollo de Ciencias Básicas (PEDECIBA), Comisión Sectorial de Investigación Científica (CSIC), Uruguay, and Universitat de les Illes Balears, Spain. E. Hernández-García and C. Mirasso acknowledge support from MCyT (Spain), project CONOCE BMF2000-1108.

References

[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. 69 (1983), 32; A. S. Pikovsky, Z. Phys. B 55(1984), 149; L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. 64 (1990) 821.

[2] M. Rosenblum, A. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76** (1996) 1804.

[3] M. G. Rosenblum et al., Phys. Rev. Lett. 78 (1997) 4193.

[4] N. F. Rulkov et al., Phys. Rev. E 51 (1995) 980.

[5] L. Kocarev and U. Parlitz, Phys. Rev. Lett. 76 (1996) 1816.

[6] S. Rim, D.-U. Hwang, I. Kim, and C.-M. Kim, Phys. Rev. Lett. 85 (2000) 2304.

[7] R. Toral, C. R. Mirasso, E. Hernandez- García, and O. Piro, Chaos 11 (2001) 665; ibid. "*Unsolved Problems of Noise and Fluctuations: UPON'99, Second International Conference*". D. Abbot, L. B. Kish, eds. American Institute of Physics (Melville, NY, 2000).

[8] H. U. Voss, Phys. Rev. E 61 (2000) 5115.

[9] H. U. Voss, Phys. Lett. A 279 (2001) 207.

[10] C. Masoller, Phys. Rev. Lett. 86 (2001) 2782.

[11] H. U. Voss, Phys. Rev. Lett. 87 (2001) 014102.

[12] A. Locquet, F. Rogister, M. Sciamanna, P. Megret, and M. Blondel, Phys. Rev. E 64 (2001) 045203(R).

[13] I.V. Koryukin and P. Mandel, Phys. Rev. E 65 (2002), 026201.

[14] S. Sivaprakasam, E. M. Shahverdiev, P. S. Spencer, and K. A. Shore, Phys. Rev. Lett. 87 (2001) 154101.

[15] H.U. Voss, to appear in the Int. J. of Bifurcation and Chaos (2002).

[16] C. Masoller, and D. Zanette, Physica A 300 (2001) 359.