Three-Frequency Resonances in Coupled Phase-Locked Loops

Oscar Calvo, Julyan H. E. Cartwright, Diego L. González,
Oreste Piro, & Francesco Sportolari

Abstract—**We construct an experimental model of a nonlinear dynamical system with three frequencies. With this analog electronic circuit made up of quasiperiodically forced coupled phase-locked loops, we investigate the structure of the scaling of three-frequency resonances or lockings in the dynamics. We hypothesize and confirm experimentally that for weak coupling the three-frequency resonance with the largest frequency-locked plateau in the space of parameters in the interval between two adjacent resonances p/q and r/s is given by the mediant $(p+r)/(q+s)$. We expect this to be universal behavior in systems of three coupled oscillators.**

I. Introduction

THE last decades of this millennium have seen a revolution in our understanding of complex and nonlinear systems driven by the application of the mathematics of dynamical systems theory to diverse fields of science, ranging from physics, through engineering, to chemistry and biology.

Many of these breakthroughs have been possible thanks to insight into the behavior of discrete and continuous systems gained through intensive numerical calculations. The bulk of these calculations have been performed using digital circuits. However, there is a growing interest in analog electronic circuits as systems where theoretical predictions can be tested in a more realistic environment that that of numerical model simulation, as experimental systems where new interesting dynamical properties can be discovered, and as powerful emulators of real complex systems in practical applications [1], [2], [3], [4], [5], [6], [7].

A particularly important aspect of the new viewpoint that we have arrived at in the study of dynamical systems is that of universality. There are many facets of the behavior of dynamical systems that are universal across classes of systems in very different fields of science. Within a class, we can predict the qualitative and in some cases quantitative behavior of the system without having investigated the details of each case.

The dynamics of nonlinear systems with two interacting frequencies has been thoroughly investigated in many theoretical and experimental studies, including many studies of electronic circuits consisting of forced and coupled nonlinear oscillators. It is now well understood how the resonances or lockings found in these systems arise, and how they are distributed in the parameter space [8], [9], [10], [11], [12]. Only recently, however, has attention been brought to bear on the more complex problem of three interacting frequencies. It is known that three-frequency systems also have a structure of resonances, but here there are three-frequency resonances as well as the type found in systems with only two frequencies. Previous investigation of three-frequency systems has shown that there is a web of resonances throughout the parameter space [13]. Experiments with electronic circuits [14], [15], [16] have confirmed theoretical results, and provided much data for theorists. The broad picture of three-frequency systems is clear from these studies, but the question of scaling laws for the relative sizes of resonances in small intervals of parameter space has not been addressed.

In this paper we investigate universal aspects of the fine structure of the nonlinear dynamics of a three-frequency system with weak coupling. We develop a theoretical prediction about the distribution and hierarchy of three-frequency resonances in the parameter space. We then construct an analog circuit of quasiperiodically forced coupled phase-locked loops. We focus our attention on the three-frequency resonances in the output of the circuit, and use it to test our scaling hypothesis.

The paper is organized as follows: in Section II we give a succinct review of the theoretical basis for the scaling of resonances in two-frequency systems, and we build on that to propose a hypothesis for the scaling of three-frequency resonances in three-frequency systems. In Section III we describe the analog circuit of coupled phase-locked loops we use, in Section IV we analyse the behavior of the circuit and test the validity of our scaling hypothesis; we end in Section V with conclusions.

II. Three-Frequency Resonances

Nonlinear systems with competing frequencies show the phenomenon of resonance, also termed phase-, mode- or frequency locking, in which the system locks into a resonant periodic response which has a rational frequency ratio. Resonance was first observed and explained in 1665 by Huygens [17], [18], [19] in two pendulum clocks coupled by a common mounting. The amount of resonance increases with coupling strength, from none in the uncoupled linear regime, to a critical situation where the system is locked into resonance at all values of the frequency ratio. The subcritical system has quasiperiodic responses between different lockings, while at supercritical values of the coupling strength, chaotic as well as periodic and quasiperiodic responses may occur. Resonance phenomena were first noted in the context of electronics by van der Pol [20], [21]. In the intervening seventy years, resonances have been investigated theoretically and experimentally in many nonlinear systems, and their distribution in parameter space is now well understood, from the number theoretical concept of Farey trees [8], [9], [10], [11],

Oscar Calvo is with L.E.I.C.I. at the Universidad Nacional de La Plata, Argentina. Email: calvo@athos.fisica.unlp.edu.ar.

Julyan Cartwright is with the Instituto Andaluz de Ciencias de la Tierra, Granada, Spain. Email: julyan@galiota.uib.es, Web http://formentor.uib.es/∼julyan.

Diego González and Francesco Sportolari are with the Istituto Lamel, CNR, Bologna, Italy. Email: gonzalez@lamel.bo.cnr.it and sportolari@lamel.bo.cnr.it.

Oreste Piro is with IMEDEA on the campus of the Universitat de les Illes Balears, Spain. Email: piro@imedea.uib.es, Web http://www.imedea.uib.es/∼piro.

[12]. However, all this applies to resonances generated by the interaction of two frequencies. Far less is known, by comparison, when there are three or more interacting frequencies.

Adding another frequency allows new phenomena to take place. Now as well as the possibility of all three frequencies having a rational relation, to form a (two-frequency) resonance as before, there is a further possibility: that of a three-frequency resonance, also known as a weak resonance or partial mode locking. Three-frequency resonances are given by the nontrivial solutions of the equation $af_0 + bf_1 + cf_2 = 0$, where a, b, and c are integers, f_1 and f_2 are the forcing frequencies, and f_0 is the resonant response. Three-frequency resonances form a web in the parameter space of the frequencies [13], [14], [15], [16]. In this section we review the novel number-theoretical results [22], [23] leading to the present investigations. We should like to discover the local scaling laws governing the relative sizes of neighboring three-frequency resonances. To do this we must have recourse to some number theory of rational approximations. Firstly, we revise the situation for the case of two frequencies, then we extend this to systems in which there are three interacting frequencies.

A. Continued fraction approximations for two frequencies

Consider a two-frequency system with autonomous frequency f_0 and external frequency f_1. Let $\tilde{f} = f_1/f_0$. Our aim is to define a sequence of rationals that converges to $\tilde{f}$. Strong convergence is measured with the metric (Kinchin's metric of the second kind [24])

$$\left\| \tilde{f} - \frac{p_i}{q_i} \right\| = |q_i \tilde{f} - p_i|. \tag{1}$$

p_n/q_n is a best rational approximation if

$$\left\| \tilde{f} - \frac{p_n}{q_n} \right\| < \left\| \tilde{f} - \frac{p_i}{q_i} \right\| \tag{2}$$

for all (p_i, q_i) for any $q_i \leq q_n$. Given $\tilde{f}$, p_n and q_n are produced by expanding $\tilde{f}$ in continued fractions $\tilde{f} = (a_1, a_2, a_3, \ldots)$, and truncating the expansion as $p_n/q_n = (a_1, a_2, a_3, \ldots a_n)$ [25]. The p_n/q_n are then the strong convergents of $\tilde{f}$. They give the sequence of fractions with lowest monotonically increasing denominators that converges to $\tilde{f}$.

B. The Farey tree for two frequencies

The physically reasonable hypothesis invoked to explain the local ordering of the hierarchy of (two-frequency) resonances is that the smaller the denominator, the larger the width of the resonance in parameter space. The fraction with smallest denominator between p/q and r/s, if they are sufficiently close so that $|qr - ps| = 1$, when they are called adjacents, is $(p+r)/(q+s)$. This fraction, known as the mediant, then gives the most important resonance in the interval between the resonances p/q and r/s. Repeatedly performing the mediant operation

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s} \tag{3}$$

on a pair of adjacent rationals, we obtain a Farey tree. For weak coupling, the Farey tree provides a qualitative local ordering of two-frequency resonances [8], [9], [10], [11], [12].

C. Continued fraction approximations for three frequencies

Now consider the case of three frequencies, one internal f_0, and two external f_1 and f_2. We may divide through by the autonomous frequency f_0, to give $f_1^\dagger = f_1/f_0$, and $f_2^\dagger = f_2/f_0$. We now aim to come up with two convergent sequences of rationals with the same denominator, p_n/k_n and q_n/k_n, which are strong convergents to $f_1^\dagger$ and $f_2^\dagger$ respectively.

As before, strong convergence is measured through the metric

$$\left\| (f_1^\dagger, f_2^\dagger) - (\frac{p_i}{k_i}, \frac{q_i}{k_i}) \right\| = |k_i(f_1^\dagger, f_2^\dagger) - (p_i, q_i)|. \tag{4}$$

Thus $(p_n/k_n, q_n/k_n)$ are best rational approximants if

$$\left\| (f_1^\dagger, f_2^\dagger) - (\frac{p_n}{k_n}, \frac{q_n}{k_n}) \right\| < \left\| (f_1^\dagger, f_2^\dagger) - (\frac{p_i}{k_i}, \frac{q_i}{k_i}) \right\| \tag{5}$$

for all triplets of integers (p_i, q_i, k_i) for any $k_i \leq k_n$.

So we may write

$$\varepsilon_1 = \left\| \frac{p_n}{k_n} - f_1^\dagger \right\| = |k_n f_1^\dagger - p_n|, \tag{6}$$

$$\varepsilon_2 = \left\| \frac{q_n}{k_n} - f_2^\dagger \right\| = |k_n f_2^\dagger - q_n|, \tag{7}$$

where we wish to obtain the integers p_n, q_n and k_n. This general problem has not been solved [26], [27], however, we may set $\varepsilon_1 = \varepsilon_2$, so that both approximations should be equally good or bad. Taking $\varepsilon_1 = \varepsilon_2$ is an ansatz to simplify the problem that may, or may not, prove correct; as we shall see later, it leads to results that are confirmed both in numerical simulations and experimentally. Note that the work cited above of Kim & Ostlund [26], [27], is dedicated to the organization of two-frequency resonances in three-frequency systems, whereas here we are concerned with the more general three-frequency resonances.

If we then set $\varepsilon_1 = \varepsilon_2$, we can equate

$$|k_n f_1^\dagger - p_n| = |k_n f_2^\dagger - q_n|, \tag{8}$$

and ask what is k_n. There are two solutions

$$k_n = \frac{q_n \pm p_n}{f_2^\dagger \pm f_1^\dagger}. \tag{9}$$

At this point we must remember that k_n is an integer, so these solutions require that the frequencies be rescaled by $f_2^\dagger \pm f_1^\dagger$. For which we define for the first solution

$$\tilde{f}_1 = \frac{f_1^\dagger}{f_1^\dagger + f_2^\dagger}, \qquad \tilde{f}_2 = \frac{f_2^\dagger}{f_1^\dagger + f_2^\dagger}, \tag{10}$$

and similarly for the other solution

$$\tilde{f}_1^* = \frac{f_1^\dagger}{f_2^\dagger - f_1^\dagger}, \qquad \tilde{f}_2^* = \frac{f_2^\dagger}{f_2^\dagger - f_1^\dagger}. \tag{11}$$

The two solutions give rise to different ε's

$$\varepsilon = |(p_n + q_n)\tilde{f}_1 - p_n| = |(p_n + q_n)\tilde{f}_2 - q_n|, \tag{12}$$

$$\varepsilon^* = |(q_n - p_n)\tilde{f}_1^* - p_n| = |(q_n - p_n)\tilde{f}_2^* - q_n|, \tag{13}$$

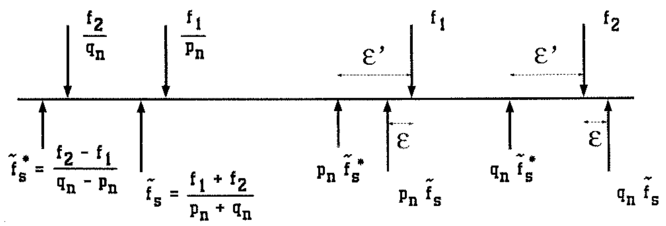

Fig. 1. A schematic diagram of the frequency line shows the external forces f_1 and f_2, their subharmonics f_1/p_n and f_2/q_n, and the most important three-frequency resonances associated with the mediant $\tilde{f}_s$ and the generalized adjacent $\tilde{f}_s^*$ of these subharmonics.

from which one can obtain

$$\frac{\varepsilon}{\varepsilon^*} = \left| \frac{f_2 - f_1}{f_1 + f_2} \right| < 1. \tag{14}$$

So in this sense the $(\tilde{f}_1, \tilde{f}_2)$ solution is better than the $(\tilde{f}_1^*, \tilde{f}_2^*)$ solution.

Now sticking with the $(\tilde{f}_1, \tilde{f}_2)$ solution, p_n, $(p_n + q_n)$, and q_n are obtained from the continued fraction expansions of $\tilde{f}_1$ and $\tilde{f}_2$. Since

$$\tilde{f}_1 = \frac{f_1^\dagger}{f_1^\dagger + f_2^\dagger} = \frac{f_1}{f_1 + f_2} = \frac{1}{1 + \frac{1}{\frac{f_1}{f_2}}}, \tag{15}$$

$$\tilde{f}_2 = \frac{f_2^\dagger}{f_1^\dagger + f_2^\dagger} = \frac{f_2}{f_1 + f_2} = \frac{1}{1 + \frac{f_1}{f_2}}, \tag{16}$$

if we have the continued fraction expansion of $f_1/f_2 = (a_1, a_2, a_3, \ldots)$, that of $\tilde{f}_1 = (a_1 + 1, a_2, a_3, \ldots)$, and $\tilde{f}_2 = (1, a_1, a_2, a_3, \ldots)$. Hence if p_n/q_n is the nth strong convergent of f_1/f_2, or equivalently of $\tilde{f}_1/\tilde{f}_2$, given by this continued fraction expansion, $p_n/(p_n + q_n)$ and $q_n/(p_n + q_n)$ are the strong convergents of $\tilde{f}_1$ and $\tilde{f}_2$ respectively.

D. The generalized Farey tree for three frequencies

Suppose that p_n/q_n is a convergent of f_1/f_2. We may define as generalized adjacents any pair of f_i/r_i, f_j/r_j, with f real and r integer, that satisfy

$$|f_i r_j - f_j r_i| = |f_1 q_n - f_2 p_n| = |\Delta|. \tag{17}$$

The subharmonics f_1/p_n and f_2/q_n are obviously generalized adjacents, and the mediant between them is

$$\tilde{f}_s = \frac{f_1 + f_2}{p_n + q_n}, \tag{18}$$

which by extension from the two-frequency case we hypothesize to be the largest resonance in parameter space between f_1/p_n and f_2/q_n (see fig. 1).

Let us take as an example a three-frequency system with the two external frequencies set to $f_1 = 2100$ Hz and $f_2 = 3600$ Hz. The frequency ratio f_1/f_2 is then 7/12. The continued fraction expansion for f_1/f_2 is $(1, 1, 2, 1, 1)$, and the different truncations of this produce the convergents of 7/12, which are 1/1, 1/2, 3/5, 4/7, and 7/12. Now the rescaled external frequencies are $\tilde{f}_1 = f_1/(f_1 + f_2) = 7/19$, with convergents 1/2, 1/3, 3/8, 4/11, and 7/19, and $\tilde{f}_2 = f_2/(f_1 + f_2) = 12/19$, with convergents 1/2, 2/3, 5/8, 7/11, and 12/19. The mediant frequencies $\tilde{f}_s = (f_1 + f_2)/(p_i + q_i)$ are $5700/2 = 2850$ Hz, $5700/3 = 1900$ Hz, $5700/8 = 712.5$ Hz, $5700/11 = 518.18$ Hz, and $5700/19 = 300$ Hz.

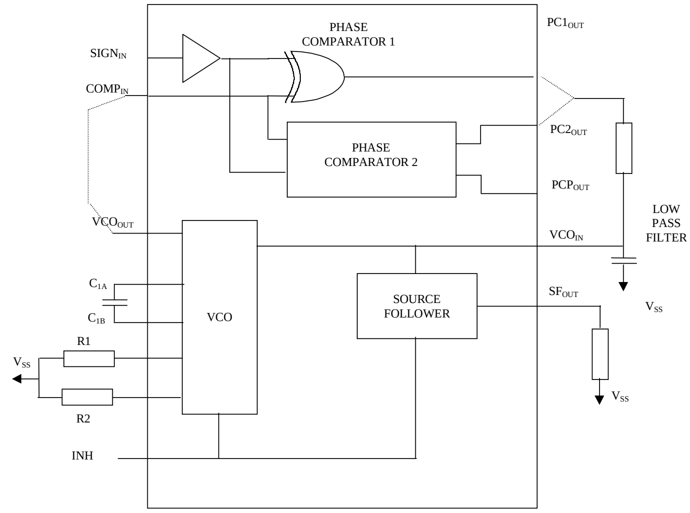

Fig. 2. Block diagram of the CD 4046A phase-locked loop we use, made up of a low-power voltage-controlled oscillator, two different phase-comparator sections with a common-input amplifier, a voltage-regulator zener, and a source-follower section.

We are interested in this paper in the relative importance or sizes of the different resonances. The hypothesis we have introduced here is that for weak coupling the daughter resonance formed by the mediant

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p + r}{q + s} \tag{19}$$

is the largest three-frequency resonance in the interval between its two parents p/q and r/s when they are generalized adjacents. We shall use the electronic circuit to investigate our hypothesis experimentally in a real three-frequency nonlinear dynamical system.

III. The Circuit

We investigate a system consisting of two externally forced coupled nonlinear oscillators. As a basic circuit for both oscillators we use a digital phase-locked loop integrated circuit, the CD 4046A. This is made up of a low-power voltage-controlled oscillator, two different phase-comparator sections with a common-input amplifier, a voltage-regulator zener, and a source-follower section useful in demodulation applications. In fig. 2 we show a block diagram of the CD 4046A phase-locked loop.

Phase-locked loop circuits are found in many industrial applications [28]: for example, in communication systems they are used for AM and FM demodulation and digital signal transmission over telephone lines. They are also interesting when considered as dynamical systems: FM demodulation phase-locked loops can show periodic, quasiperiodic, and chaotic responses in some parameter intervals [1], [3], [4], [5], [29], [6].

Coupled phase-locked loop circuits are also of interest to industry. For example, two coupled phase-locked loops are used to synchronize geographically separated timing clocks [30]. These kinds of systems have also been studied as dynamical systems in order to determine the boundaries of regions of safe operation, that is, regions free of irregular or chaotic behavior [2], [7].

Here our circuit consists of two coupled voltage-controlled oscillators forced with two independent external forces of frequencies f_1 and f_2. In fig 3 we show the electrical scheme of the circuit implementation. The outputs of the two phase-locked loops are sent to a type 1 phase comparator. The type 1 phase comparator of CD 4046A is an exclusive OR network that exhibits a triangular shaped response after low-pass filtering. The error signal is fed back to both voltage-controlled oscillators, and passes through an overall adjustable amplifier to provide control over the coupling strength; we are interested in the weak coupling regime in this work. Inverted and direct versions of the error signal are sent to oscillators one and two respectively; this inversion of the error signal in one of the paths is necessary for the stability of the circuit. Feedback signals enter the voltage-controlled-oscillator control pins through appropriate adder circuits. The adders also allow independent coupling with the external forces and tuning of the internal frequencies through application of adjustable DC levels.

In a phase-locked loop one normally uses the voltage-controlled oscillator and one of the two phase comparators from the integrated circuit. In our experiment we use a voltage-controlled oscillator for each phase-locked loop and a type 1 phase comparator of one of the two CD 4046As. The rest of the circuit consists of external linear adders and amplifiers. The low-pass filter is also external as in normal phase-locked-loop circuits.

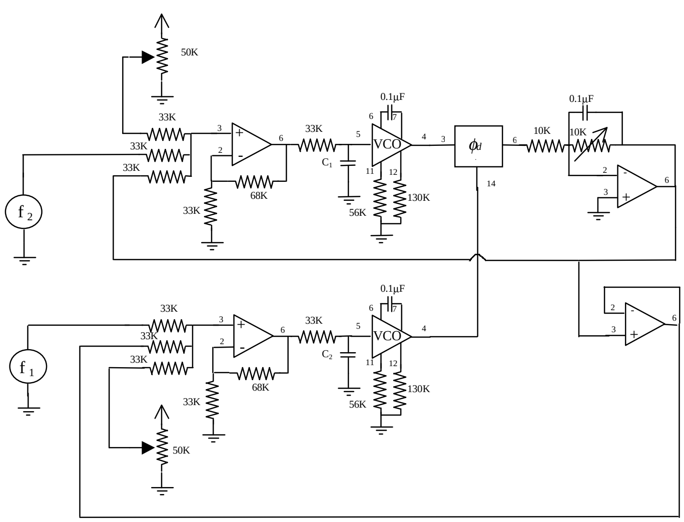

Fig. 3. The circuit. The output of the voltage controlled oscillator (VCO) sections of both digital phase-locked loop devices are sent to an exclusive OR port for phase comparison. The phase comparator output is returned, after low-pass filtering, to both VCO inputs. An amplifier in the feedback path controls the coupling strength between the oscillators. Appropriate operational adders on both VCO inputs allow the external forcing of the oscillators. Stable phase-locked responses between oscillators also require an additional unit gain inverter prior to one of the device inputs.

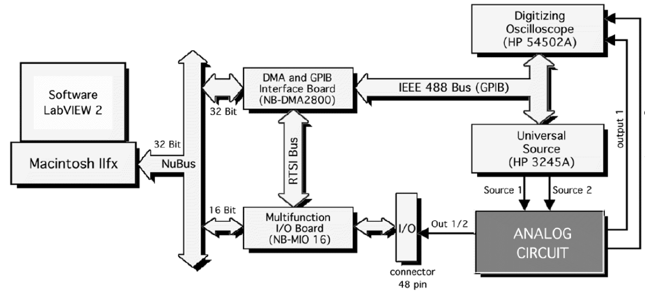

Fig. 4. The experimental setup. Apart from the circuit itself, we use an HP 3425A function generator, a digitizer, and a Macintosh running the graphical control software LabView 2 for overall control of the experiment.

IV. Experimental Setup and Results

Figure 4 shows a block diagram of the experimental setup. The external forces are generated by an HP 3245A two-channel digital function generator. As a control parameter we use the DC offset of one of the external forces; a change in the mean value of an external force gives a linear change in the natural frequency of the associated oscillator: we vary the DC offset of f_1 in steps of 1 mV. The outputs of the oscillators are digitized at a rate of 40 kHz, and saved for subsequent analysis. Graphical control software (a Macintosh running LabView 2) is used to control the overall experiment. A controllable time from the start of any set of data can be chosen in order to allow transients to die out. After this time the fundamental frequency is calculated by a virtual instrument frequency meter.

The circuit shows 1/1 synchronization between the two oscillators for a wide range of excitation and system parameters. Typically, the fundamental frequency of both outputs remains constant in a finite interval of DC offset. The frequency value of the fundamental determines, together with the frequency values of the two external forces, a three-frequency resonance. Experimental data from the circuit are shown in fig. 5 together with the predicted hierarchy of resonances described by our mediant hypothesis of Section II. The two external frequencies are here fixed at f_1 = 2100 Hz and f_2 = 3600 Hz respectively. These values correspond to the example we gave in Section II*D*.

In fig. 5 the first level is defined by the two adjacents $f_2/7 \approx 514.3$ Hz and $f_1/4 = 525$ Hz, which are subharmonics of the external frequencies. The mediant $\tilde{f}_s$ between these two solutions corresponds to $(f_1 + f_2)/(4 + 7) \approx 518.2$ Hz. Clearly, this is the largest stability region in this parameter interval. Subsequent levels in the hierarchy confirm the mediant hypothesis up to the limit of resolution of the graphic. In fig. 6 we show a power spectrum for one of the outputs in the parameter region of $\tilde{f}_s$. The spectrum is dominated by the $\tilde{f}_s$ peak, which is at least 10 dB greater than the components at the external frequencies. This shows that the subharmonic frequency mainly determines the dynamics of the system. Moreover, we can see that the spectrum is very complicated in the low frequency region, owing principally to the presence of several minor peaks. The principal spacing between peaks is $\varepsilon\tilde{f}_s \approx 27.29$ Hz, which corresponds to beats between the external forces and appropriate harmonics of $\tilde{f}_s$ (see fig. 1).

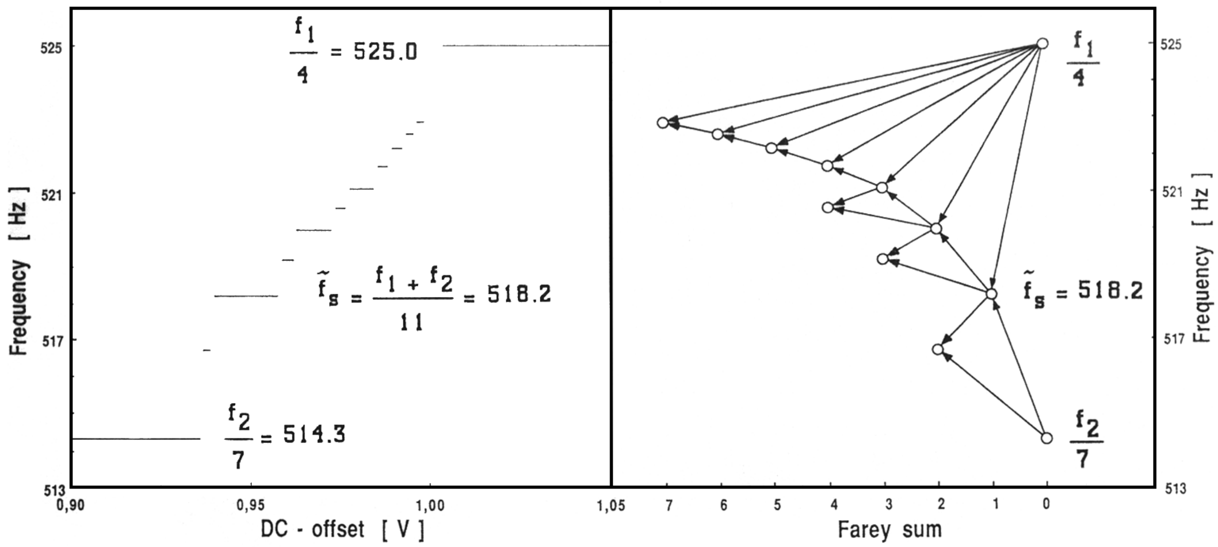

Fig. 5. Experimental results — a three-frequency Devil's staircase. The external frequencies f_1 and f_2 are here fixed at 2100 Hz and 3600 Hz. On the left is plotted the third frequency of a three-frequency resonance against a control parameter (the DC offset of one of the external forces) for all resonances with plateaux larger than a certain size. On the right we see the hierarchy of three-frequency resonances predicted by the mediant hypothesis starting from the parents $f_1/4$ and $f_2/7$. At each level in the hierarchy, the daughter resonance formed by the mediant between two adjacent parents is seen to be the largest in its interval.

As $4/7$ is a convergent of f_1/f_2, the first level in fig. 5 is given by the two primary adjacents, $f_2/7 \approx 514.3$ Hz and $f_2/4 = 525$ Hz. These are subharmonics of the external frequencies and define the value of Δ in (17), i.e., $\Delta = (7f_1 - 4f_2) = 300$ Hz. The mediant between these two solutions corresponds to $\tilde{f}_s = (f_1 + f_2)/(4+7) \approx 518.2$ Hz. Clearly, this is the largest stability region in this parameter interval. Observe that $\tilde{f}_s$ is a generalized adjacent of both $f_1/4$ and $f_2/7$, (equal Δ in (17)) and thus the mediant operation (18) can be implemented to obtain the second level of the generalized Farey tree i.e., $\tilde{f}_s + f_1/4 = (2f_1+f_2)/(2\cdot 4+7)$ and $\tilde{f}_s + f_2/7 = (f_1+2f_2)/(4+2\cdot 7)$. It is clear that the procedure can be iteratively applied to give successive levels of the tree. Up to the limit of the resolution of fig. 5, all resonances found are represented in some level of the generalized tree and the mediant hypothesis of hierarchical ordering is confirmed.

We have repeated the experiment with different frequency ratios; the qualitative features of the responses of the device are insensitive to this change, and plots such as fig. 5 are qualitatively the same. Moreover, they are also analogous for different input waveforms, whether of sine, square, or sawtooth waves, and do not show a qualitative dependence on the details of the circuit. These facts lead us to suggest that the behavior described is robust and universal for a whole class of systems.

V. Conclusions

We have investigated universal aspects of the nonlinear dynamics of three-frequency systems. We have concentrated on the three-frequency resonances that we find in such systems, and we have formulated a hypothesis about the local scaling laws for the relative sizes of different resonances: that the daughter resonance formed by the mediant is the largest in the interval between its two parents when they are generalized adjacents.

We have constructed an experimental model of a nonlinear dynamical system with three frequencies. With this circuit, made up of quasiperiodically forced coupled phase-locked loops, we have investigated the fine structure of three-frequency resonances in the dynamics. We have confirmed experimentally that for weak coupling the most important three-frequency resonance — being that with the largest plateau in the space of parameters — in the interval between two adjacent parent resonances is given by the mediant. We emphasize that since we have been interested in universal behavior, we expect that the results we have obtained are not dependent on details of the construction of the circuit, but rather represent the behavior of any dynamical system of this class.

What importance might these findings have for phase-locked-loop circuits? The advantage of finding the widest plateaux is evident: these represent the responses that are most stable to arbitrary perturbations, and thus are those that are most likely to be found in a real system. Also, it is important to understand what happens when another frequency is added in parasitic fashion to a periodically-forced system; that is one considers one of the two forces as a perturbation. This can produce undesired effects in the normal phase locking of the system, and as such from an engineering viewpoint it is important to know the regions of parameter space where the behaviour is dependable. It may be that a technique based on the simultaneous synchronization of two reference signals offers some advantages in certain technological applications. For example, it may be less prone to the influence of perturbations, or may be able to maintain the system in a certain frequency range when one of the two frequencies is

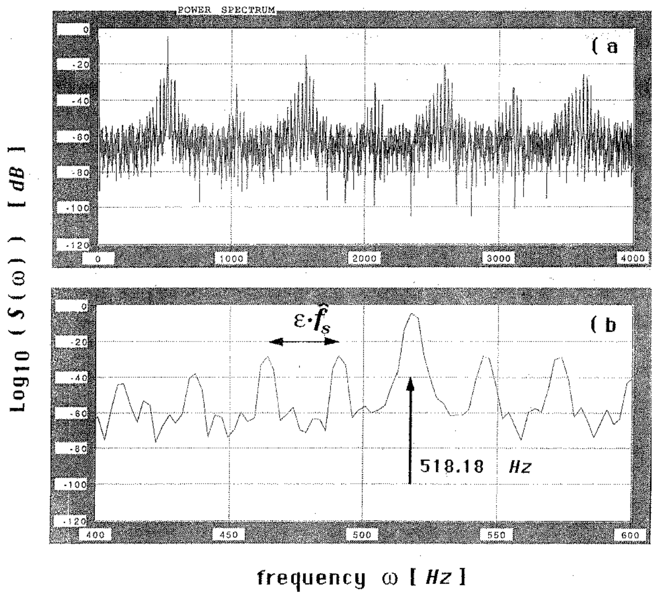

Fig. 6. The power spectrum of the output of the cicuit for a DC offset of 0.95 V. Other parameters are as in fig. 5. (a) From 0 – 4000 Hz. (b) Detail from 400 – 600 Hz, showing peak at $\tilde{f}_s \approx 518.18$ Hz and minor peaks separated a distance $\varepsilon \tilde{f}_s$.

not present, perhaps through system failure, or owing to external conditions (for example, weather conditions in the case of atmospheric propagation of radio signals). Clearly, when there is only one reference frequency, the absence of that frequency will leave the system in an unpredictable state. The presence of two reference frequencies in synchronization applications can then lead to systems that are more robust and reliable in the face of perturbations and failures.

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Oscar Calvo received his BSEE in electrical engineering from the National University of La Plata (Argentina) in 1979, and his MSEE in computer and electrical engineering from the Illinois Institute of Technology, (Chicago, IL, USA) in 1988. He has been a guest scientist at the Fermi National Accelerator Laboratory (Chicago, IL, USA), sponsored research staff at the Massachusetts Institute of Technology (Boston, MA, USA), and invited professor at the University of the Balearic Islands (Palma de Mallorca, Spain). He holds a position as adjunct professor and researcher at the National University of La Plata. His research is in the areas of control systems and VLSI devices.

Julyan Cartwright received his BSc in theoretical physics from the University of Newcastle upon Tyne (UK) in 1988, and his PhD in applied mathematics from Queen Mary and Westfield College of the University of London (UK) in 1992. Since then he has been working in Spain, until recently with the University of the Balearic Islands (Palma de Mallorca, Spain). He is now with the Andalusian Institute for Earth Sciences (IACT, CSIC–UGR) in Granada, Spain. His research encompasses a broad range of interdisciplinary themes within the field of nonlinear science.

Diego González received his BSc in physics from the National University of La Plata (Argentina) in 1981 and his PhD in theoretical physics from the same university in 1987. Since then he has been with the Lamel Institute (Bologna, Italy) of the Italian National Research Council (CNR). His research is mainly in the field of low-dimensional dynamical systems and its application to the modeling of complex biological systems.

Oreste Piro obtained his PhD in physics from the National University of La Plata (Argentina) in 1984. At the postdoctoral level he worked at the James Franck Institute of the University of Chicago (IL, USA), as a visiting scholar, at the Brookhaven National Laboratory (Upton, NY, USA), as a guest scientist, and at Queen Mary and Westfield College of the University of London (UK) as a senior researcher. He has been a faculty member of the National University of La Plata and fellow of the the National Council for Scientific and Technical Investigations (CONICET) in Argentina, fellow of the National Center for Scientific Research (CNRS) at the Nonlinear Institute (INLN) in Nice, France, and is currently professor at the University of the Balearic Islands and member of the Mediterranean Institute for Advanced Studies (IMEDEA, CSIC–UIB) in Mallorca, Spain. His research interests range over many areas of the theory of nonlinear dynamical systems and its applications to engineering, biology, and physics.

Francesco Sportolari received his BSEE in electronic engineering from Bologna University (Italy) in 1986. He has worked in different private companies in the area of software development. Since 1992 he has had a research contract from the Lamel Institute of the CNR (Bologna, Italy) for the modeling and visualization of complex systems. His research is mainly in the field of nonlinear modeling of complex systems with particular emphasis on simulation by means of analog electronic circuits.