Chaotic Synchronization in Small Assemblies of Driven Chua's Circuits

Esteban Sánchez, Manuel A. Matías, and Vicente Pérez-Muñuzuri

***Abstract*—Chaotic synchronization is studied in experiments performed on dynamic arrays of Chua's circuits that are connected by using a recently introduced driving method especially suited for the design of such arrays. Namely, the driven circuit has the same number of energy storage elements as the driving circuit. The experimental results, which are supported by theoretical analysis, are different depending on the geometric arrangement of the array. In the case of linear arrays, the first circuit always imposes its behavior to the rest of the chain at a finite velocity. Instead, in the case of ring geometries, the chaotic synchronized state is only stable up to a certain size of the ring. Beyond this critical size a desynchronizing bifurcation occurs, leading to a chaotic rotating wave that travels through the array. This instability is explained by performing an analysis in terms of modes.**

***Index Terms*—Chaotic synchronization, Chua's circuit, dynamic arrays, pattern forming instabilities.**

I. Introduction

THE phenomenon of synchronization among coupled nonlinear oscillators, first studied by Huygens in the 17th century, has proven to be rather fruitful in a variety of scientific fields. One may mention the use of this paradigm in the study of biological rhythms [1], [2], chemical oscillators [1], [3], arrays of Josephson junctions [4], and arrays of lasers [5].

Less intuitive is probably the finding that chaotic systems can be also made to synchronize in spite of the sensitive dependence of these systems on the initial conditions. Thus, chaos has been described as a situation in which a system gets out of synchronization with itself [6]. However, chaotic synchronization was demonstrated by Pecora and Carroll on analog circuits [7]–[9] by splitting a (response) system in two subsystems, one of which is held in common with the drive. This work based on previous theoretical work by Fujisaka and Yamada [10] and Rabinovich *et al.* [11] and experimental work by Volkovskii and Rulkov [12] that considered the case of linear (resistive) coupling implemented with analog circuits in [13]–[15]. One of the most important promises of chaotic synchronization, beyond the interest of the phenomenon in itself, is that it could be useful in the field of secure communications [16]–[21] and in spread spectrum communications [21]. Note, however, that some recent studies have shown that a simple masking scheme may be easily deciphered [22], [23].

Synchronization is also relevant in biology. Its role has already been explored in the case of periodic (limit cycle) systems, e.g., [1]. In addition, recent studies suggest that the brain may use deterministic chaos in perceptive processes [24]–[26], while some evidence points to the role of chaos in the behavior of single neurons [27], [28]. In this representation, the brain would be chaotic at rest, while perceptive processes would be associated to transitions to synchronous oscillatory behavior (that could be related to the 40-Hz oscillations found in the brain of mammals by physiologists [29]).

We shall concentrate in the behavior of rings of coupled oscillators that may be also relevant in a biological context, as in the case of morphogenesis [30] or in the context of neural systems. An example of the latter case are central pattern generators (CPG's), i.e., assemblies of a small number of neurons that are capable of providing the necessary rhythm of muscular activity even in the absence of external stimuli. These CPG's, that may operate independently from the brain, have been studied in some degree of detail for a few lower animals, e.g., lobsters [31]. In the study of CPG's the important points to be considered [28] are the dynamics of the isolated neurons, e.g., periodic or chaotic, the interaction between the oscillators, and the way in which information is processed. An important aspect is that the resulting spatio-temporal patterns can be analyzed through symmetry arguments [32] that allow one to study the different possible behaviors, stemming from symmetry breaking bifurcations, and the transition between different types of gaits has been explained in this way by considering a model formed by a ring of coupled oscillators [33], [34]. A different point of view is that these arrays can be also discussed in the framework of cellular neural networks (CNN's) [35], [36]. Thus, it is possible to foresee applications of these networks as information processing units.

One of the most quickly acknowledged features of neural systems is the unidirectional character of information transmission in these systems. In the field of chaotic synchronization a first approach along these lines is the driving method introduced by Pecora and Carroll (PC). In the present contribution we shall implement experimentally the method introduced in [37] and [38] in the case of arrays of Chua's circuits. A useful feature of this approach [37] is that the dynamical evolution of the variable in the response circuit that corresponds to the driving signal is not suppressed. Thus, the result of a single connection does

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the same function as a cascade within the original PC scheme, while it is guaranteed that the response circuit is homologous to the drive circuit (in the electronic implementation this implies that the circuits are identical). This useful property also has the advantage of allowing us to design different types of networks with different geometries of arrangement, because the dimensionality of the circuit is not reduced.

The aim of the present work is to explore the behavior of arrays of Chua's circuits in two types of arrangements, namely, linear arrays and rings. In the first case, the result is that synchronization propagates through the array through a synchronization wave. More interesting, perhaps, is the case of rings, as in this case the chaotic synchronized behavior cannot be sustained for arbitrary sizes of the system. Thus, one finds that for a given size of the system, there is a critical size beyond which an instability that destroys synchronization appears. The behavior of the system past the instability, which had not been reported before, consists of a rotating wave with chaotic amplitude that travels through the array.

II. Experimental Setup and Mathematical Model

In the context of the present work we have built an experimental setup composed of six Chua's circuits operating in the chaotic double-scroll regime. The dynamical evolution of an isolated circuit (see, e.g., [39]) can be characterized by the following set of differential equations:

$$\begin{aligned} C_1\frac{dV_1}{dt} &= \frac{V_2 - V_1}{R} - g(V_1)\\ C_2\frac{dV_2}{dt} &= \frac{V_1 - V_2}{R} + i_L\\ L\frac{di_L}{dt} &= -V_2 - i_L r_0 \end{aligned} \tag{1}$$

where the variables V_1 and V_2 are the voltages across the two capacitances, C_1 and C_2 and i_L is the current passing through the inductor L that has an internal resistance r_0. R is the resistance that couples C_1 and C_2 and the whole circuit is kept active by a suitable nonlinear element that, in the case of a single circuit, is driven by the voltage V_1 across C_1. This element is characterized by its negative conductance curve (see, e.g., [39]), defined by

$$g(V) = \{G_b V + \tfrac{1}{2}(G_a - G_b)\,[|V + B_p| - |V - B_p|]\} \tag{2}$$

where $-B_p$ and B_p are the voltages corresponding to the two breaking points, G_a is the conductance for $|V| < B_p$, while G_b is the conductance for $|V| > B_p$ ($B_p = 1$ V in our circuit).

The Chua's circuits that we have studied experimentally, in both linear and ring geometries, are characterized by the following values of the electronic components: (C_1, C_2, L, r_0, R) = (10 nF, 100 nF, 10 mH, 20 Ω, 1100 Ω), where one should take into account the tolerances of the available discrete components (10% in the case of the inductances, 5% in the case of the capacitances, and 1% in the case of the resistances). The slopes of the nonlinear element (2) are defined by $G_a = -8/7000$ S and $G_b = -5/7000$ S, respectively. The circuits have been sampled with a digital oscilloscope (Hewlett-Packard 54 601B) with a maximum sample rate of 20 million samples per second, with eight-bit A/D resolution and a record length of 4000 points.

The circuits are connected through unidirectional driving, such that the voltage across capacitor C_1 of a given circuit, say $(j-1)$, is used to drive the nonlinear element of the next circuit, j. This is not the only possible connection resulting from the application of the method of [37] that is stable from the point of view of synchronization, but it can be shown that the other two stable connections reduce to cascades resulting from the application of the PC method.

This coupling implies the following evolution equations for the jth coupled circuit in the array:

$$\begin{aligned} C_1\frac{dV_{1,j}}{dt} &= \frac{V_{2,j} - V_{1,j}}{R} - g(\overline{V_{1,j}})\\ C_2\frac{dV_{2,j}}{dt} &= \frac{V_{1,j} - V_{2,j}}{R} + i_{L,j}\\ L\frac{di_{L,j}}{dt} &= -V_{2,j} - i_{L,j} r_0 \end{aligned} \tag{3}$$

with $j = 1, \cdots, N$ and where coupling enters in the nonlinear element as the corresponding term, $g(\overline{V_{1,j}})$ is not driven, in principle, by the voltage across capacitor C_1 of the jth circuit, but by the voltage across capacitor C_1 of the $(j-1)$th circuit, $\overline{V_{1,j}} = V_{1,j-1}$ for $j \neq 1$. The value of $\overline{V_{1,j}}$ for $j = 1$ depends on the geometry of the arrangement, and it is $\overline{V_{1,1}} = V_{1,1}$ for linear arrays and $\overline{V_{1,1}} = V_{1,N}$ for rings.

This way of coupling implies a generalization in the design of the nonlinear element in Chua's circuit. The reason is that in the usual implementation of Chua's circuit (see, e.g., [40]) the nonlinear element is simply a negative-sloped nonlinear resistor $g(V_1)$, that, as can be seen from (1), has as argument the voltage V_1 of capacitor C_1 that is physically attached in parallel with the nonlinear resistor. Instead, looking at (3) it is clear that now the nonlinear element has as an argument a voltage that is different to the voltage across its extremes. Thus, this function must be realized by using a voltage controlled current source (VCCS) that produces the required current stemming from the nonlinear characteristic, as imposed by Kirchhoff's laws. The voltage taken at capacitor C_1 of a given circuit is used to drive the nonlinear element in the next circuit. The implementation of the nonlinear element that has been used in the present work is explained in more detail in the Appendix (see also Fig. 1).

In the work reported here, simulations have been carried out by numerically integrating the evolution equations (3) while, later, more realistic simulations have been performed by using SPICE [41] with accurate models of the electronic devices. In all cases there is a perfect agreement between the different levels of simulations, thus indicating that the suggested circuit represents the electronic implementation of (3). The behavior of the experimental setting is also in agreement with the simulations.

III. Results

A. Linear Arrays of Chua's Circuits

This type of geometry is characterized by the dominance of the first circuit in the array that will ultimately impose its behavior to the whole system. This implies that the asymptotic behavior of the system will be synchronized chaos. In [42] it was

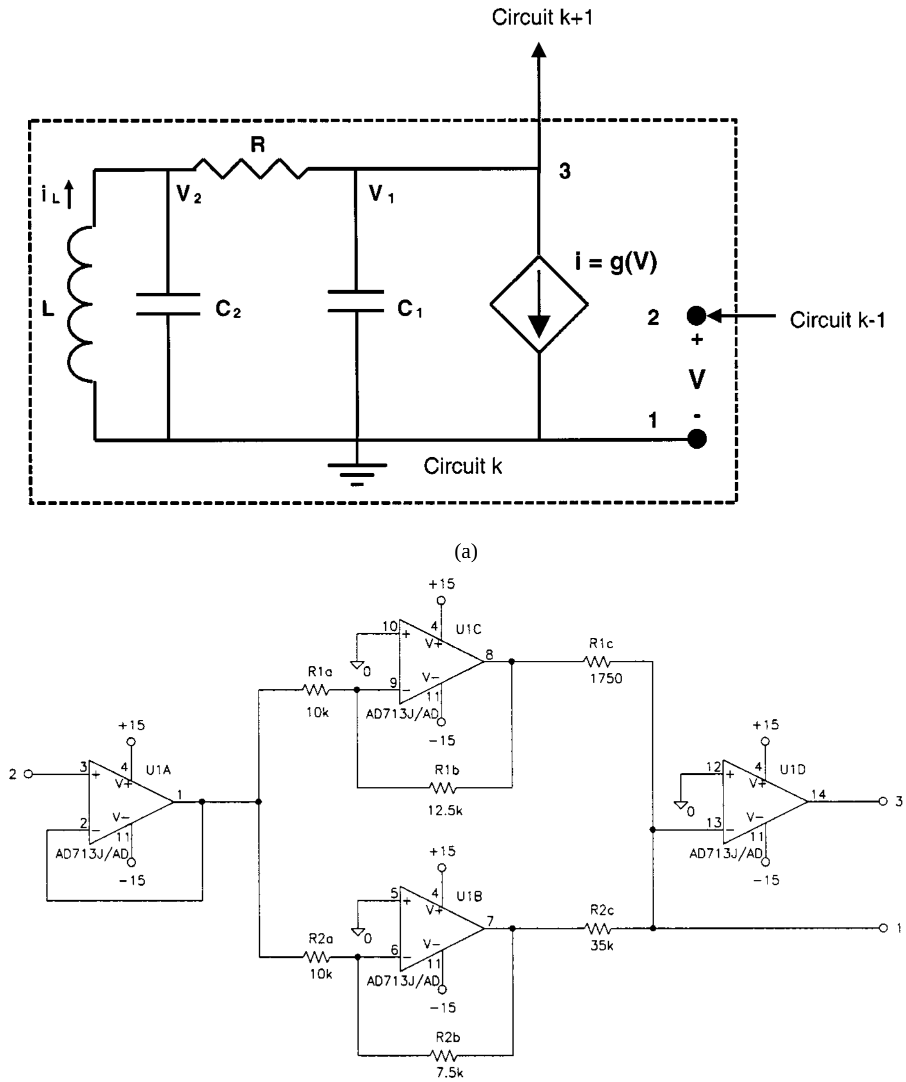


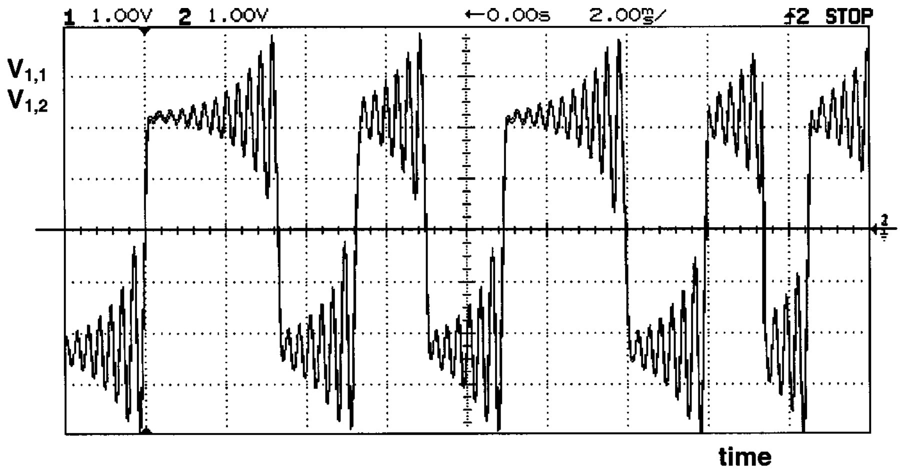
Fig. 1. (a) Schematic representation of a Chua's circuit, say k in the array, coupled to its neighbors according to (3). (b) Schematic representation of the nonlinear element, introducing the third terminal needed to implement the coupling.

shown that chaotic synchronization occurs as a synchronization wave spreads through the medium. The velocity of propagation of this wave was shown to depend linearly on the largest transverse Lyapunov exponent for a drive-response couple. This implies that chaotic synchronization happens consecutively as two contiguous circuits become synchronized (see, however, [43]).

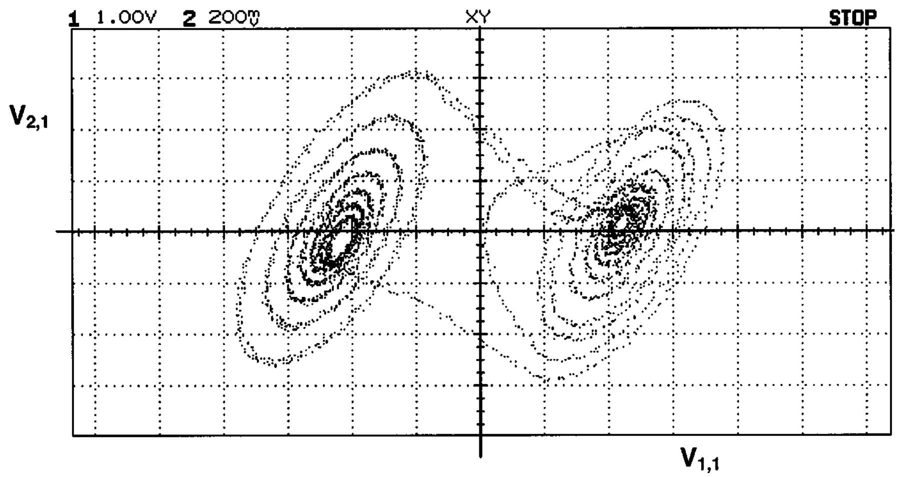
The time needed to achieve synchronization depends on the initial conditions, but the asymptotic state (attained only in the limit when time tends to infinity) will always be synchronized behavior, i.e., with no phase lag between contiguous circuits. This is at variance with what happens with resistive coupling [15], [44] where a phase difference between contiguous circuits that depends on the initial conditions is observed. At first sight, the fact that the whole array ultimately exhibits synchronized chaotic behavior would appear to contradict the predictions stemming from the work in [45] and [46]. By using very simple considerations, these authors showed the impossibility of finding stable uniform chaotic behavior for systems above a certain critical size. However, if the coupling is unidirectional, as in our case, it can be shown that the chaotic uniform synchronized state will be stable (see, e.g., [47] and [48]).

Now, we shall show that, indeed, what is observed is that all the circuits in the array become synchronized with the first one, that exhibits the well-known Chua's double-scroll chaotic attractor (see Fig. 2). A phase portrait of this attractor can be found in Fig. 2(b) (the voltage V_2 across capacitor C_2, is plotted versus the voltage V_1 across capacitor C_1). In turn, in Fig. 2(c) V_1 for the sixth circuit, i.e., $V_{1,6}$, is represented versus the same quantity corresponding to the first circuit, $V_{1,1}$. Synchronization is expressed by the straight line relationship $V_{1,6} = V_{1,1}$, although the line has some thickness due to the fact that the circuits are not completely identical.

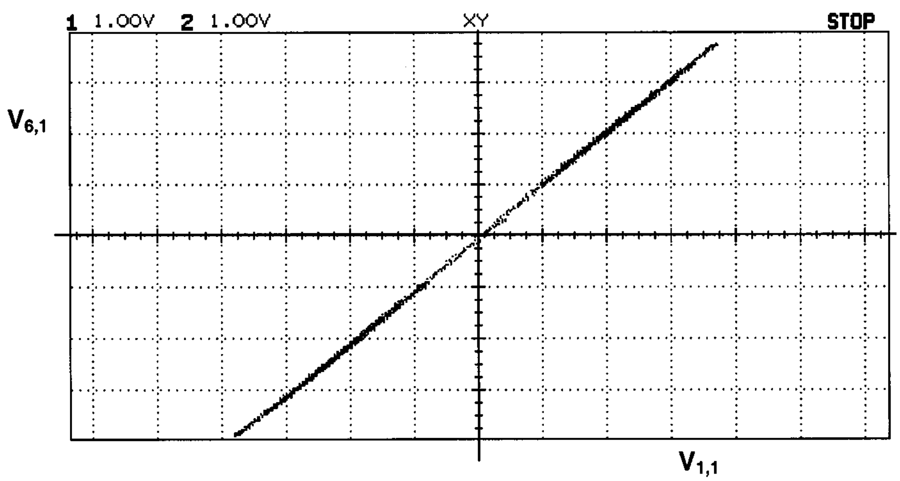
Synchronization implies a collapse of the dynamics of the array from the compound phase space of all the interacting systems to a low-dimensional invariant manifold, that, in principle, is characterized by the equality of all the variables of the system while the behavior of the system is chaotic. A convenient way of analyzing the stability of this situation is



(a)



(b)



(c)

Fig. 2. Experimental results for a linear array of six Chua's circuits coupled according to (3). (a) Time series of V_1 in two contiguous circuits, i.e., $V_{1,1}$ and $V_{1,2}$ versus time. (b) Phase plane of the two voltages for the first circuit, i.e., $V_{2,1}$ versus $V_{1,1}$. (c) Representation of the voltage V_1 for the last circuit versus the same quantity for the first one, i.e., $V_{6,1}$ versus $V_{1,1}$. The parameters of the circuit (1–2) are: $(C_1, C_2, L, r_0, R, G_a, G_b) = (10$ nF; 100 nF; 10 mH; 20 Ω; 1100 Ω; $-8/7000$ mS; and $-5/7000$ mS).

by considering separately the behavior of small perturbations within this invariant manifold from the behavior of small perturbations transverse to this manifold. The linearized evolution analysis of these perturbations, that amount to differences between the synchronized systems, yields the concept of transverse Lyapunov spectrum.

In [37] it was shown that for a pair of connected systems this spectrum can be obtained by setting to zero the entries corresponding to the connections, i.e., to the term in which the driving signal enters. Thus, if one considers the linear stability matrix of an isolated Chua's circuit

$$\mathbf{Z} = \begin{pmatrix} -\alpha[1+g'(x)] & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix} \tag{4}$$

then, for the connection corresponding to driving the nonlinear element with the voltage V_1 across the capacitor C_1 of the previous circuit, i.e., (3), the transverse Lyapunov spectrum is obtained by considering the matrix in which the term corresponding to $g'(x)$ is set to zero. In this case, this yields a completely linear problem and it is not necessary to take the average when time goes to infinity, but simply the real part of the eigenvalues. The transverse Lyapunov spectrum corresponding to a drive-response pair of Chua's circuit coupled in the form given by (3) is $[-0.1570, -0.1570, -10.9059]$ where the fact that the three exponents are negative shows the stability of the connection.

B. Rings of Chua's Circuits

This way of arranging the chaotic circuits leads to a different behavior of the array. The basic reason is that in this case the kind of arguments advanced in [45] and [46] apply. These authors discussed the stability of the mean field approximation to the system in the limit of infinitely many interacting units, also called the thermodynamic limit, concluding that if the uniform state is chaotic it should be unstable to small perturbations having some characteristic length. In the case that we are considering here, rings of chaotic electronic oscillators, another form of stating these results is to say that the uniform synchronized state of these rings will be unstable when that the number of circuits in the ring exceeds a certain threshold.

This instability recently has been analyzed in [49] for the case of rings of Chua's circuits through theoretical analysis and confirmed by numerical simulations. In essence, the idea is that the circulant structure of the ring allows to cast the linear problem, that results when one studies the evolution of small perturbations to the synchronized state, in a particularly simple form through the use of the discrete Fourier transform. Thus, the study of a ring composed out of N Chua's circuits would yield a quite cumbersome $(3N) \times (3N)$-dimensional problem, but the circulant symmetry of the arrangement allows us to write the problem as the superposition of N (3×3) Fourier modes that takes the form [49], [50]

$$\dot{\boldsymbol{\eta}}^{(k)} = \mathbf{C}^{(k)}\boldsymbol{\eta}^{(k)} \tag{5}$$

where $\mathbf{C}^{(k)}$ is the Fourier transform of the Jacobian of the flow, taking the form

$$\mathbf{C}^{(k)} = \begin{pmatrix} -\alpha[1+g'(x)e_k] & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix} \tag{6}$$

where $\boldsymbol{\eta}$ is the Fourier transform of $\delta\mathbf{x}$, $e_k = \exp(i2\pi k/N)$ and $k = 0, \cdots, (N-1)$ are the Fourier modes.

In this framework, the uniform synchronized state of the ring corresponds to the Fourier mode with the longest wavelength, i.e., $k = 0$. The stability condition of this mode to arbitrary perturbations is that the Lyapunov transverse spectrum corresponding to the modes $k = 1 \cdots (N-1)$, be negative. On the other hand, the condition to be fulfilled such as to have chaotic temporal dynamics is that the Lyapunov exponent of the $k = 0$ mode is positive. This fact allows to argue in a simple way that the spatially homogeneous and temporally chaotic state cannot be stable for arbitrarily large sizes of the array (in agreement to the ideas in [45] an [46]). An alternative way of phrasing these arguments in our case is that if we define the continuous variable $q = k/N$, the function $\lambda(q)$, representing the largest Lyapunov exponent of the system, will be continuous. On the other hand, as we are considering a synchronizing connection, $\lambda(q)$ will be typically negative for some q, normally for $q = 1/2$, which corresponds to $k = N/2$. Then, the continuity of the function implies that $\lambda(q_c) = 0$ for some critical value q_c, and the shape of $\lambda(q)$ implies that those wavenumbers that correspond to $q \in [0, q_c]$ will have a positive transverse Lyapunov exponent, indicating an instability that will destroy the chaotic synchronized state (see also [51] for another exposition of this simple argument). Thus, the difference between nonchaotic and chaotic dynamics is that, although in the first case an instability may occur, it is guaranteed to appear in the case that the uniform dynamics is chaotic (in the nonchaotic case $\lambda(0) \leq 0$).

The implication of this fact for the case of coupled circuits is that the instability must occur for some finite size N. If we consider the $k = 1$ mode, which will be the one in which the instability will appear first, this critical size will be the first integer that obeys $N_c \times q_c \geq 1$. If one is considering a value of $q \in [0, q_c]$ in the context of the linear stability theory, these perturbations should grow as $\exp(\lambda(q)t)$ with $q = k/N$. Notice the resemblance of this instability of the synchronized state with the Turing instability [30] (see also [52] and [53] for a discussion in the context of electronic circuits), although in this case the type of coupling is completely different. For other studies on coupled oscillator systems with periodic boundary conditions that exhibit instabilities in their uniform synchronized state, one may mention studies on coupled periodic oscillators such as the studies on coupled Josephson junctions [54], phase locked loops [55], and other systems [56], while there are a few studies of instabilities in coupled chaotic oscillators [50], [57], [58], [51], [42], [43], [49], [59], [60].

A useful way of characterizing the stability of rings of circuits and of determining the critical size at which the uniform chaotic synchronized state becomes unstable is to determine the largest Lyapunov exponent $\lambda(q)$ as a function of $q = k/N$ [49]. In this work we have calculated this quantity by extending Wolf's method [61] to the case of vector spaces over the complex field. The complex character of the vectors comes from the presence of the Nth root of unity in (6). In principle, and due to the stabilizing nature of the connections, one expects that $\lambda(1/2) < 0$, while the value q_c for which $\lambda(q_c) = 0$ signals the onset of the instability. We have determined $\lambda(q)$ as a function of q (see Fig. 3) from which the critical value of q has been determined to be $q_c \sim 0.1725$. This implies that the Fourier mode with the shortest wavelength, $k = 1$, becomes unstable for a size N_c for

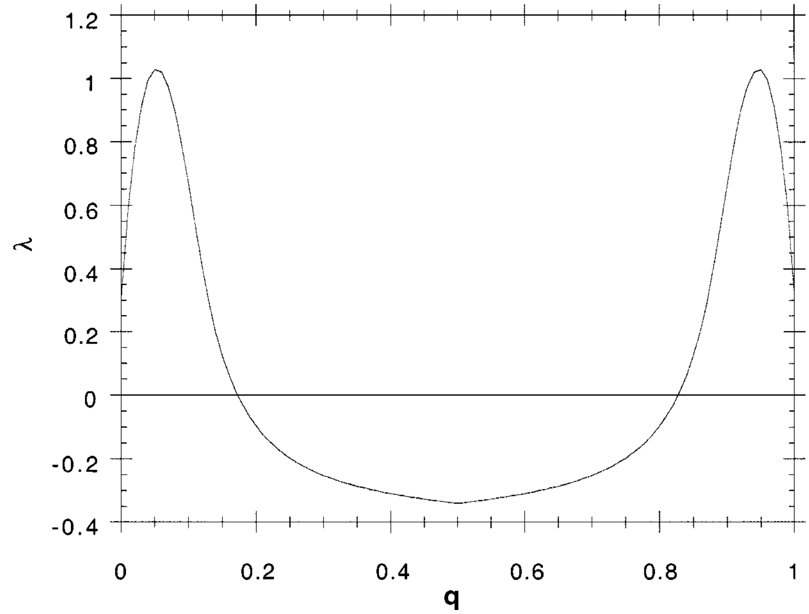


Fig. 3. Representation of the largest Lyapunov exponent $\lambda(q)$ as a function of $q = k/N$, i.e., $\lambda(q)$ versus q (see text).

which $q_c = 1/N_c = 0.1725$ is fulfilled, i.e., for the smallest value N_c for which $N_c q_c \geq 1$ which, in our case, is $N_c = 6$.

To study experimentally the issue of the stability of the chaotic synchronized state we have built rings of Chua's circuits with different sizes, coupled as explained in Section III. It is striking that the experimental results are in very good agreement with the theoretical predictions. Thus, Figs. 4 and 5 contain experimental results corresponding to rings with different numbers of coupled Chua's circuits, namely, five and six. The observed behavior shows clearly that for $N = 5$ the synchronized state is stable, while it becomes unstable for $N = 6$. This critical value is very dependent on the actual values of the electronic components ($N_c = 5$ for the values of the parameters employed in [49]). Before the onset of instability the relationship between voltages across the same capacitor corresponding to contiguous circuits is the $y = x$ straight line corresponding to perfect synchronization.

The behavior of the system for $N = 6$, i.e., immediately past the instability, can be characterized by looking at the simultaneous representation of the voltages across one capacitor, say C_1, versus time for the six circuits [see Figs. 5(a)–6(a)]. One can see that the voltages differ by an almost constant phase, while their amplitudes appear to be modulated by a wave with a smaller, apparently aperiodic, frequency. By analogy with the behavior of beat linear waves one could refer to this behavior with the name of aperiodic beat waves. The underlying aperiodicity in the basic waveform accounts for the plot of two of these voltages [see Figs. 5(b)–6(b)]. One obtains a kind of ergodic Lissajoux figure in a region of state space, indicating a phase relationship between these voltages. An interesting point is the almost perfect coincidence that one finds after comparing the experimental results with realistic SPICE simulations (shown in Figs. 5–6) and also with the results obtained by numerical integration of the evolution differential equations (not shown here: see [49]).

Another interesting property of the behavior of the circuits past the instability is that the shape of the usual double-scroll attractor is changed. This is already apparent from the shape of a time series corresponding to V_1 but can be more clearly seen when one considers the state-space plot of V_2 versus V_1 for the same circuit [see Figs. 5(c)–6(c) and compare it with Fig. 4(b)]. In an isolated Chua's circuit, the trajectories spirals out from a focus in each scroll. Instead trajectories wander around the two foci in a less ordered way, describing ellipses. From these figures we can see that the way trajectories describes around attractors in one circuit is conditioned by the behavior of the previous circuit.

For larger sizes, the behavior is different because the instability causes the unstable mode to grow to infinity. In practice this is not possible as long as the signals are limited by the power supply (-15 V, $+15$ V). Instead of going to infinity, trajectories in state–space describe a big limit cycle similar to the one described in [62]. The interesting feature is that here one has a ring of limit cycle oscillators with a phase shift of $2\pi/N$. In some sense one can consider that the ring exhibits some sort of generalized chaotic synchronization [63] for sizes above the critical number of circuits.

IV. Discussion

In the present work we have considered the synchronization of chaotic Chua's circuits along the lines of the method recently introduced in [37], a generalization of the PC synchronization method [7], in which synchronization is achieved through the injection of a signal coming from a drive system into a precise place of the response system to be synchronized. The feasibility of the method discussed here has been proven through its experimental implementation in terms of Chua's circuits, combined with realistic simulation of the circuits through SPICE, numerical integration of the evolution equations obtained by straight-

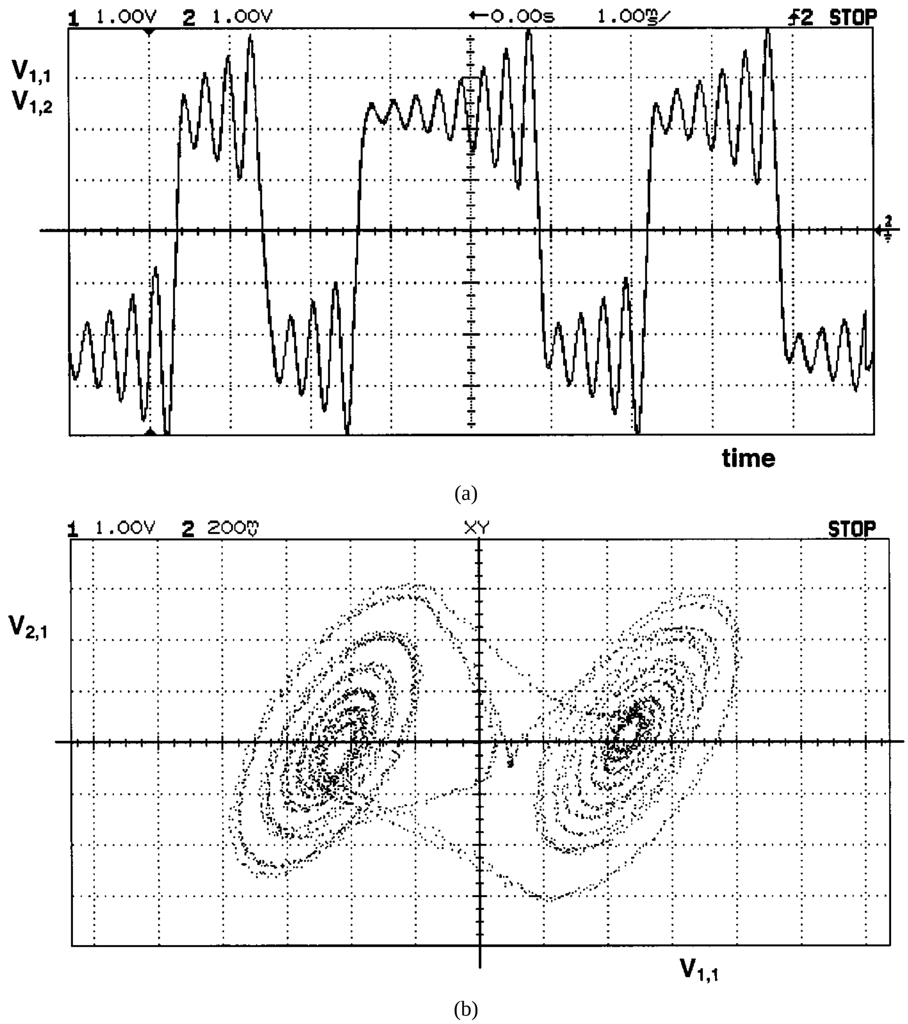


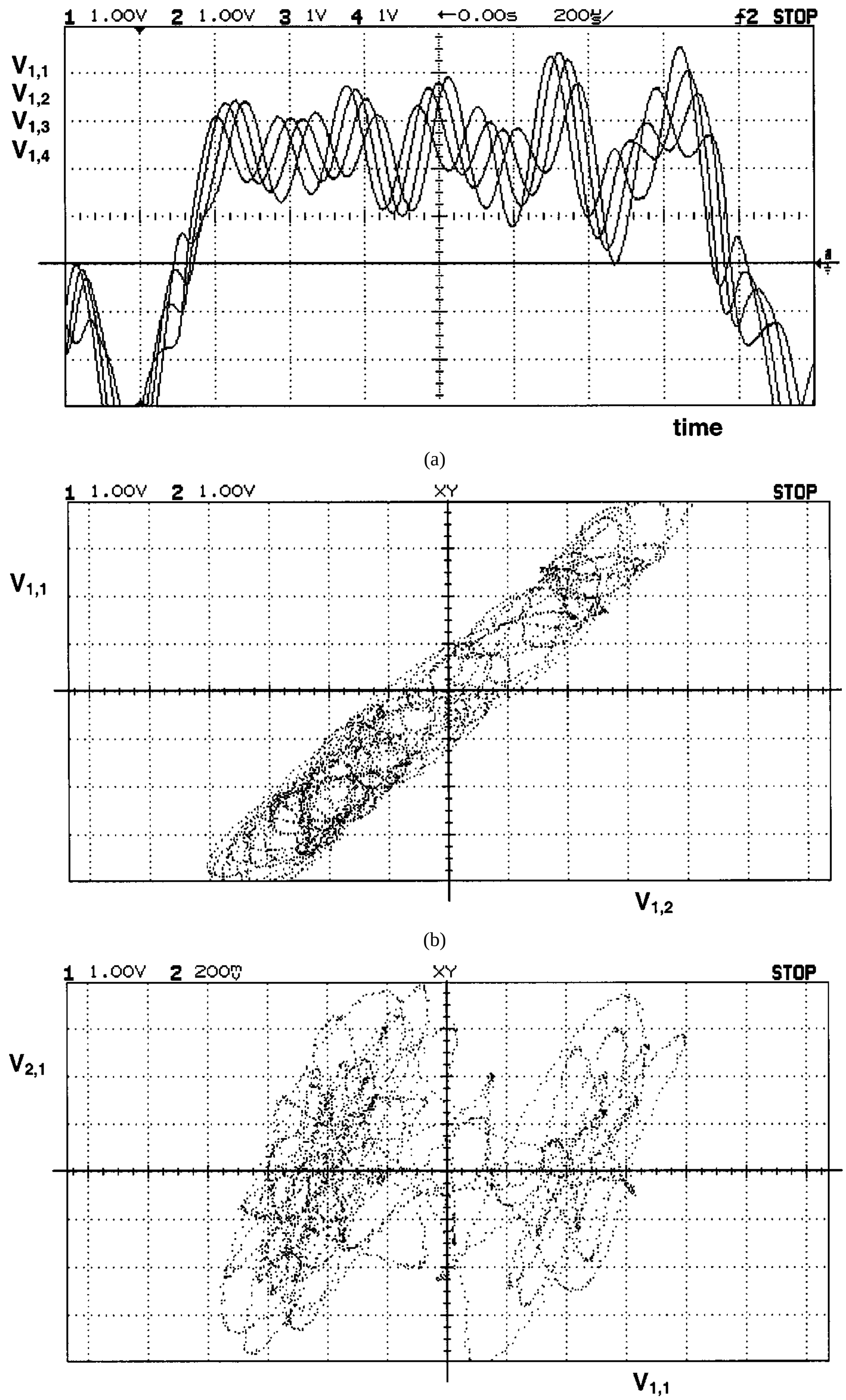
Fig. 4. Experimental esults for a ring of five Chua's circuits coupled according to (3). (a) Time series of the voltage in two contiguous circuits, i.e., $V_{1,1}$ and $V_{1,2}$ versus time. (b) Phase plane of the two voltages of one circuit, i.e., $V_{2,1}$ versus $V_{1,1}$. Note that in this case the ring is below the onset of instability.

forward application of Kirchhoff laws, and some analytical reasoning. In the case of Chua's circuit, the method of [37] suggests a new synchronizing connection that cannot be expressed as a cascade of PC connections. It consists of the use of the voltage across the capacitor in parallel with the nonlinear element of one circuit to drive the response circuit. From the electronic point of view this implies the implementation of the nonlinear element in a different way compared to the usual one [40] (for further details see the Appendix).

We have considered arrays making use of these unidirectional connections in two different arrangements: linear arrays and rings. The different boundary conditions impose quite different behaviors in these two settings. In the case of linear arrays the result is that the first array of the circuit imposes its behavior, i.e., dictated by its initial conditions, on the rest of the array in a finite time. This time depends linearly on the number of connections in the system multiplied by the synchronization time, the time needed by a pair of circuits to synchronize. The kind of synchronized behavior that is obtained is such that all circuits are doing the same after synchronization is achieved, i.e., they do not differ in phase. In this sense, this differs from what one observes in the case of resistive coupled arrays of Chua's circuits [15], [44], in which a phase difference is found that depends on the initial conditions. The process of synchronization can be viewed as a constant velocity wave, called the synchronization wave, that propagates through the array. The velocity of this wave depends linearly on the largest transverse Lyapunov exponent of the connection.

This wave has different properties compared to other types of waves. Thus, in contrast with the case of classical waves in linear systems or autowaves in dissipative media [64], chaotic synchronization waves may carry information in a finite time through an array of cells. This information could be encoded through changes in the initial conditions at unit $k = 1$, which will define different outputs at the opposite side of the array $k = N$ for a time longer than N/V_s where V_s is the velocity of this wave. In particular, the study of the properties of this synchronization wave may shed some light in understanding the transmission of signals in CNN's [35] that can be used for image processing.

In the case of rings, the qualitative behavior of the system is different as the uniform synchronized state of the system becomes unstable for some critical number of circuits in the ring [49]. This instability has been characterized by using a linear stability theory for small perturbations to the uniform chaotic synchronized state of the ring. The circulant symmetry of the



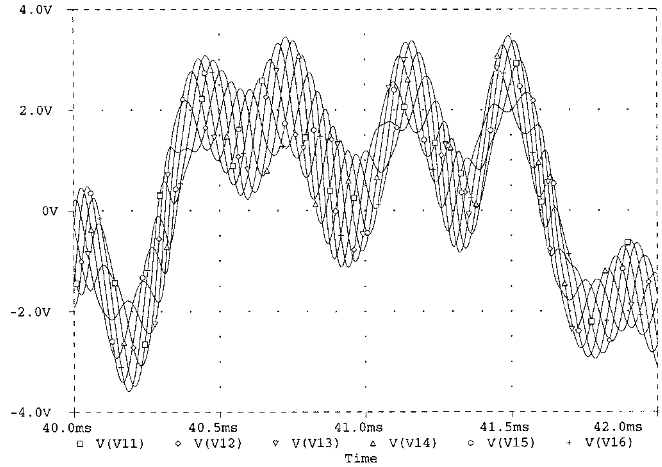
(a)

(b)

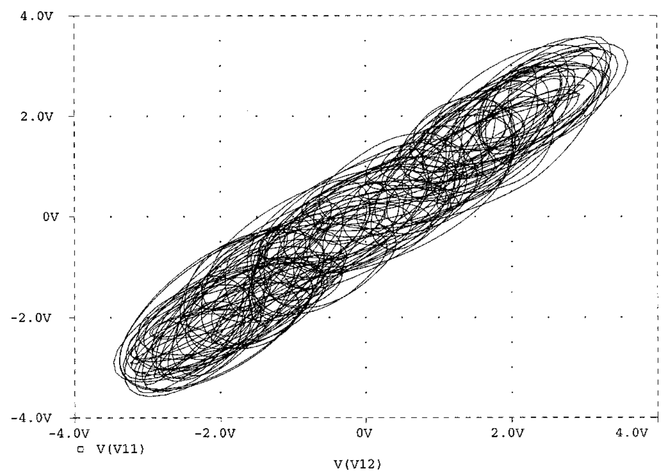
(c)

Fig. 5. Experimental results for a ring of six Chua's circuits coupled according to (3). (a) Time series of the voltage of capacitor in four contiguous circuits, i.e., $V_{1,1}$, $V_{1,2}$, $V_{1,3}$, $V_{1,4}$ versus time. (b) Representation of voltages in two contiguous circuits one versus the other one, i.e., $V_{1,1}$ versus $V_{1,2}$. (c) Phase plane of the two voltages of one circuit, i.e., $V_{2,1}$ versus $V_{1,1}$. Note that in this case the ring is above the onset of instability.

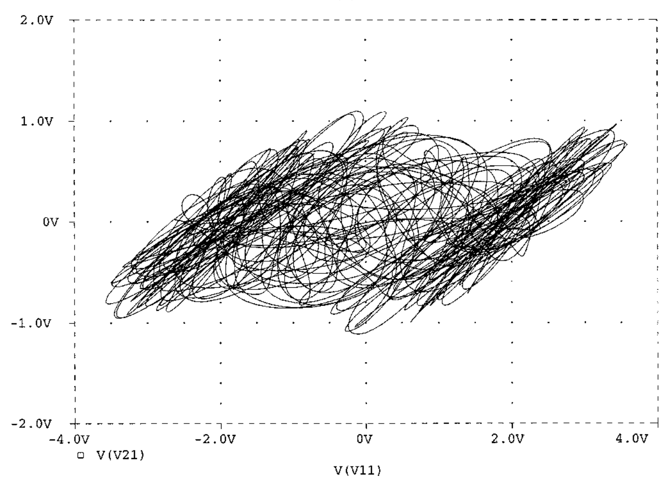
problem allows us to transform the linear stability problem of the whole ring to the superposition of a set of Fourier modes. The first Fourier mode, that implies correlations along the whole ring, becomes unstable when the size of the array grows as Fourier modes with progressively shorter wavelengths become unstable (in the order $k = 1, 2, \cdots$). In this example, the instability occurs through some sort of chaotic rotating wave, such that contiguous circuits differ by a phase equal approximately to the period divided by the number of circuits in the system. Of course, this period is not constant (and thus is not a period in the strong meaning of the word). Here it serves to characterize the average distance between peaks [49]. When



(a)



(b)



(c)

Fig. 6. Results from SPICE simulation for a ring of six Chua's circuits coupled according to (3). (a) Time series of the voltage V_1 in the six circuits. (b) Representation of voltages in two contiguous circuits one versus the other one, i.e., $V_{1,1}$ versus $V_{1,2}$. (c) Phase plane of the two voltages of one circuit, i.e., $V_{2,1}$ versus $V_{1,1}$. Note that in this case the ring is above the onset of instability.

the instability is stronger (bigger sizes) the behavior is not chaotic but periodic and all of the circuits in the ring describe a large limit cycle with a constant phase shift between them.

Regarding future possible extensions of the present work, one may mention the study of two-dimensional (2-D) arrays [62], [65], [66] case in which different connections between the individual units could be considered. These 2-D networks offer a discrete representation of spatially extended systems formed by chaotic units that, in the case of excitable media, have already been shown as very useful in the study of a great number of interesting spatio-temporal phenomena [52]. In addition, as we have already remarked, these arrays can be regarded as a generalization of CNN's [35], [36], usually based on cells with bistable dynamical behaviors. The difference with these CNN's, which have already been shown to be able of sustaining a rich range of spatio-temporal patterns [52], [53], is the type of coupling between the cells. Chaotic CNN's offer some promise in explaining experimental findings in biology in light of Freeman's suggestion of chaotic activity in the brain [24]. It is also interesting to mention the appearance of spatio-temporal structures in three-dimensional 3-D networks in which the type of coupling used in the present work is used [67].

Appendix
Details of the Electronic Implementation

The evolution equations defining the coupled array (3) imply that the nonlinear characteristic of Chua's circuit is driven by voltage $\overline{V_1}$ that is different from the voltage across the extremes of the capacitor C_1 of the circuit, say k, while it is taken from circuit $k-1$. Thus, it is necessary to implement an electronic component that is able to carry out this function, i.e., a voltage-controlled current source. In the usual implementation of Chua's circuit by using op-amps [40] two terminals connected in parallel with capacitor C_1 are used and across these terminals flows a current that is determined by the voltage.

Instead, in the method implemented for the present study it is necessary to have a third (input) terminal, as the voltage that drives the nonlinear element may come from a different Chua's circuit. Having this goal in mind, the nonlinear element shown in Fig. 1 has been built. It implements a voltage-controlled current source in three steps. The first step is formed by operational amplifier U1 (a buffer) that guarantees that one takes just voltage and not the current from capacitor C_1 of the Chua's circuit that is acting as the drive of the present circuit. Thus, the aim of this step is to be sure that the coupling is unidirectional. The second step serves to generate the two slopes, m_0 and m_1 [40] of the nonlinear characteristic $i_R = g(V_R)$ of the circuit. More precisely, op-amp U2 and resistances R_{1a} and R_{1b} generate a constant negative slope in the range between -10 V and 10 V, while op-amp U3, resistances R_{2a} and R_{2b}, and diodes D1 and D2 generate a negative slope in the range between -1 V and 1 V, while it is saturated outside this range. Then, op-amp U4 and resistances R_{4a} and R_{4b} perform the operations of summing these two voltages and yielding a current i proportional to this sum. This current flows between the out terminal of the U4 op-amp and the inverting terminal that acts as a virtual ground. Note that this virtual ground will be different for each Chua's circuit in the array, although every op-amp is referenced to the ground of the whole setting, thereby ensuring that all virtual grounds have the same potential.

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