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TRANSITION TO CHAOTIC ROTATING WAVES IN ARRAYS OF COUPLED LORENZ OSCILLATORS

E. SÁNCHEZ*
Escuela Técnica Superior de Ingeniería Industrial, Universidad de Salamanca, E-37700 Béjar (Salamanca), Spain

M. A. MATÍAS†
Física Teórica, Facultad de Ciencias, Universidad de Salamanca, E–37008 Salamanca, Spain

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In the present paper we report experimental observations of discrete spatiotemporal structures in rings of unidirectionally coupled analog Lorenz systems. This is an extension of the results previously reported in [Sánchez & Matías, 1998], studying now other parameter regimes. These studies are based on the use of an analog circuit corresponding to Lorenz equations. The more interesting new result is that a route has been found connecting periodic and chaotic rotating waves in a new kind of transition involving two symmetry related quasiperiodic attractors that merge at the transition point.

1. Introduction

Pattern formation in far-from-equilibrium systems is a topic of much recent interest [Cross & Hohenberg, 1993]. Much attention has been devoted to the study of continuous systems, but the goal of this issue is to precisely study pattern formation in discretely coupled dynamical systems. In the present work, we shall consider in which coupled dynamical systems there are chaotic oscillators that undergo pattern formation from a spatially homogeneous initial state (that is temporally chaotic). Pattern formation in coupled oscillators was studied long ago by Turing [1952] in a seminal contribution.

In particular, we shall consider the case of oscillators coupled in a ring geometry, i.e. the case of a one-dimensional array with periodic boundary conditions. This kind of systems may have interesting applications in a number of biological systems [Collins & Stewart, 1994], specially in the case of CPGs (Central Pattern Generators), that have been shown to play an important role in peripheral neural systems, locomotion, etc. [Collins & Stewart, 1994; Abarbanel *et al.*, 1996]. Some studies have considered a double ring geometry [Golubitsky *et al.*, 1998; Mariño *et al.*, 1999]. These CPGs perform *computations* through transitions between different patterns, whose structure and relationship has to be understood in this context.

One could wonder why the local dynamics of the cells is chaotic if the CPGs are to have a neural significance. The answer is that we are taking into account recent results that support the role of deterministic chaos in the behavior of single neurons [Hayashi & Ishizuka, 1992; Abarbanel *et al.*, 1996; Makarenko & Llinás, 1998], in addition to the studies that support the view that the activity of the brain as a whole is chaotic [Skarda & Freeman, 1987].

*E-mail: esteban@sonia.usal.es
†Present address: IMEDEA (CSIC-UIB), E-07071 Palma de Mallorca, Spain.
E-mail: manuel@imedea.uib.es

Regarding the behavior of coupled chaotic oscillators, some recent studies have been devoted to this problem [Heagy *et al.*, 1994a, 1994b; Matías *et al.*, 1997a, 1997b; Pecora *et al.*, 1997; Pecora, 1998; Pecora & Carroll, 1998; Hu *et al.*, 1998; Sánchez *et al.*, 1999]. The main conclusion of these studies is that global (or uniform) chaotic synchronization is not stable in the case of systems with periodic boundary conditions (rings). More interesting is the recent finding that in some cases, past the onset of instability, the system engages in a new type of highly coherent behavior, corresponding to (discrete) spatiotemporal structures. Two different such structures have been found according to the type of oscillator: Rings of coupled Lorenz systems yield a periodic rotating wave (through a type of Hopf bifurcation) [Matías *et al.*, 1997b; Sánchez & Matías, 1998], while rings of coupled Chua circuits yield a so-called chaotic rotating wave [Matías *et al.*, 1997a; Sánchez *et al.*, 1999].

In our studies we have characterized pattern formation in these systems by taking as reference the spatially uniform (chaotic) solution, that corresponds to synchronized chaotic behavior, and by studying instabilities at this level. Thus, this topic is also related to chaotic synchronization, that has received much attention recently [Fujisaka & Yamada, 1983; Pecora & Carroll, 1990; Pecora *et al.*, 1997].

The aim of the present paper is to present experimental results corresponding to rings of analog Lorenz systems in the chaotic regime for other parameter values. A more complete analysis of this system will be published elsewhere [Sánchez & Matías, 1999]. The main result of the present contribution is the unveiling of a route by which periodic rotating waves transform into chaotic rotating waves. Thus, one may understand that their behavior as rotating waves arises from a symmetric Hopf bifurcation, e.g. one that arises due to the symmetry of the ring. The transition from periodic to chaotic rotating waves involves a symmetry breaking bifurcation, in which two symmetry related periodic attractors are born, followed by an ordinary Hopf bifurcation that leads to two symmetry related quasiperiodic attractors. It is precisely the interaction of these attractors that leads to the chaotic rotating waves, in a kind of merging crisis. Some of these transitions are quite new and, apparently, have not been reported so far. They probably arise from the new degrees of freedom introduced in the system by the higher dimensionality of phase space.

2. Results

In the present paper we are reporting results obtained from an experimental setup [Sánchez & Matías, 1998] consisting of three analog Lorenz oscillators coupled accordingly to the method introduced in [Güémez & Matías, 1995]. In particular, the array has built according to the evolution equations,

$$\left.\begin{aligned} \dot{u}_j &= \sigma(v_j - u_j) \\ \dot{v}_j &= R\overline{u}_j - v_j - u_j w_j \\ \dot{w}_j &= u_j v_j - b w_j \end{aligned}\right\} \quad j = 1, \ldots, N\,, \tag{1}$$

i.e. the coupling enters in the Ru term, and where $\overline{u}_j = u_{j-1}$ ($\overline{u}_1 = u_N$ as we are considering periodic boundary conditions). The corresponding experimental setup, including the analog circuit representing Lorenz system [Lorenz, 1963], including details such as the necessary rescaling of variables and the coupling among the circuits is discussed in detail in [Sánchez & Matías, 1998]. The schematic of such a circuit is shown in Fig. 1.

All the results presented here correspond to the case of three oscillators ($N = 3$). While in our previous study [Sánchez & Matías, 1998] we considered the behavior of the system corresponding to fixed values of the parameters $R = 28$, $\sigma = 10$, and $b = 8/3$, in the present study we have studied other regions of parameter space. Logically, this has been achieved by the appropriate use of potentiometers to set the value of the relevant resistances such that the desired parameters are obtained (the relationship between the model parameters and the electronic components is explained in detail in [Sánchez & Matías, 1998]).

One of the main conclusions of the present study is that the above-described setup of Lorenz oscillators is capable of yielding a much richer range of behaviors than thought of before. Thus, in [Sánchez & Matías, 1998] it was shown that the system exhibited a transition to a rotating periodic wave, with a characteristic time scale that is one order of magnitude shorter than that of the original chaotic oscillator. On the other hand, in recent studies we described a so-called *chaotic rotating wave* [Matías *et al.*, 1997a; Sánchez *et al.*, 1999], characterized by a chaotic waveform that is

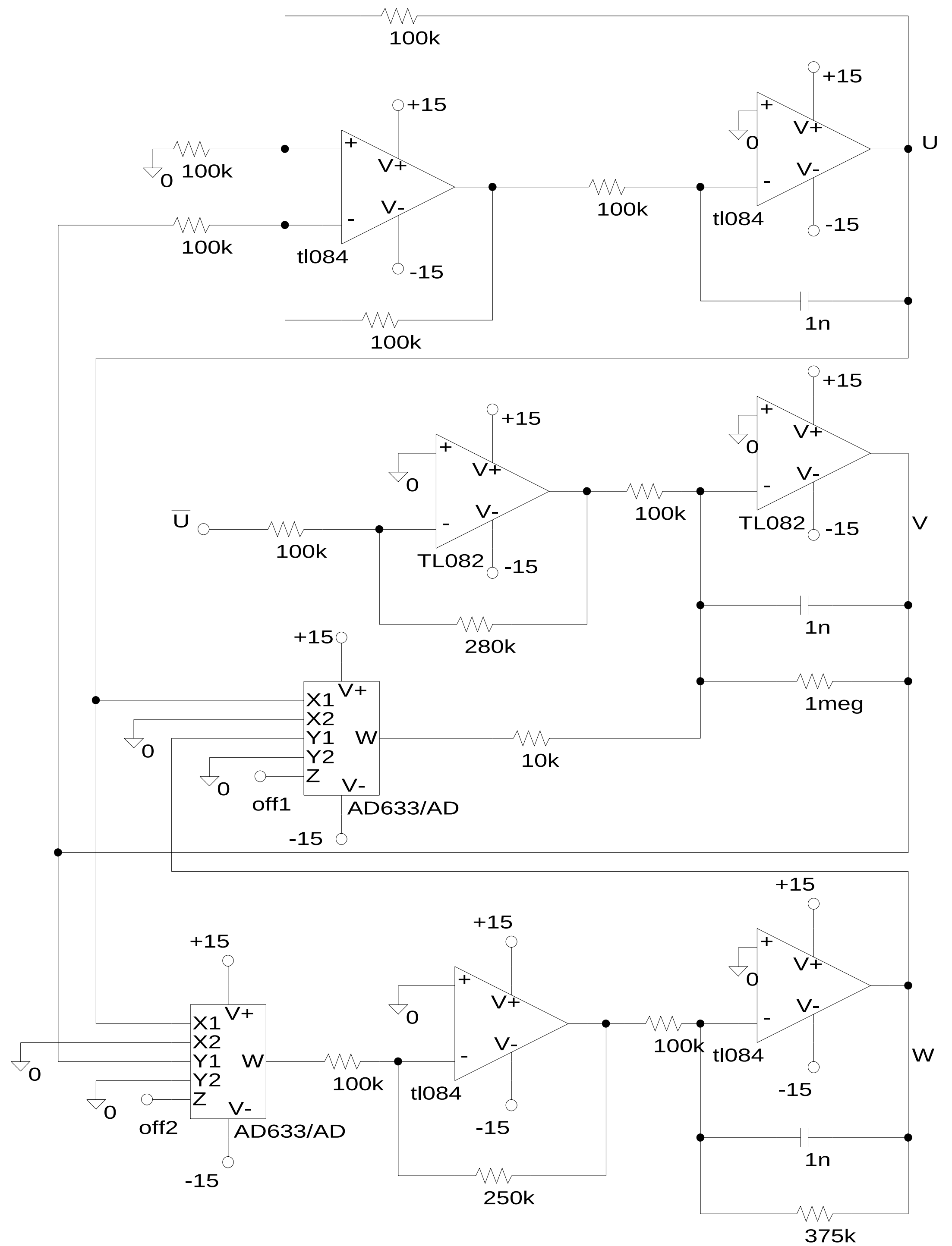
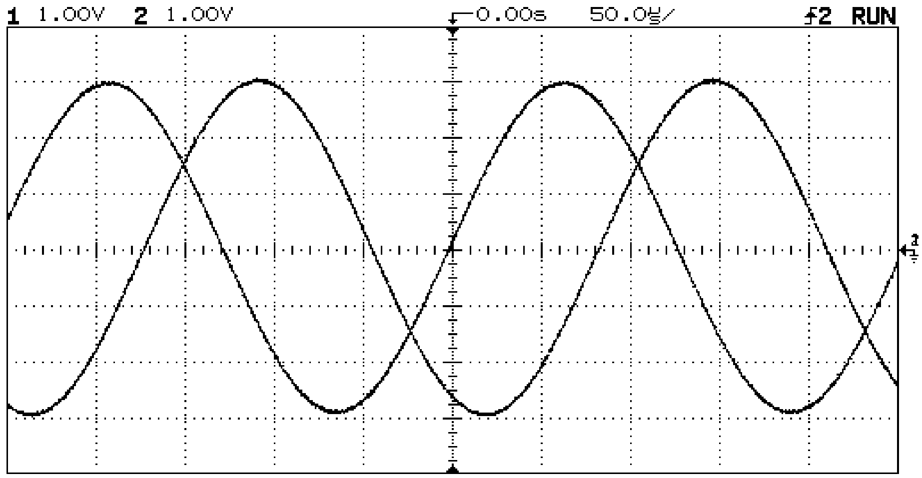


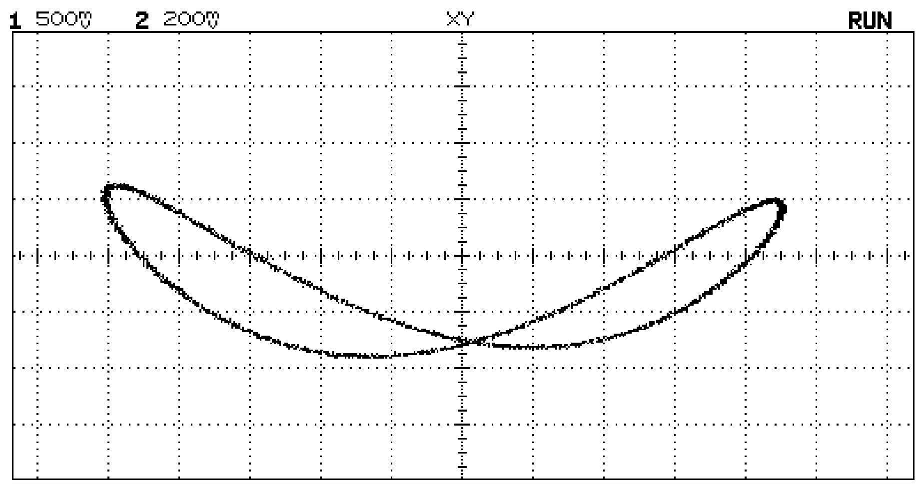
Fig. 1. Schematic of the Lorenz-based chaotic circuit, including the coupling with neighbors. The coupling is such that the Ru term is generated from the u voltage coming from the corresponding driving circuit.

modulated in an approximately periodic wave. This structure was found in studies of rings of coupled Chua circuits, and in [Matías *et al.*, 1997a] it was argued that this kind of structure appeared due to the special structure of the nonlinear term of Chua's circuit.

Instead, one of the main findings of the present work is that this kind of structure is not specific of this system, and that appears also in rings of Lorenz oscillators, if one works in the appropriate range of parameters. In particular, we have been able to report a route that links chaotic synchronized



(a)

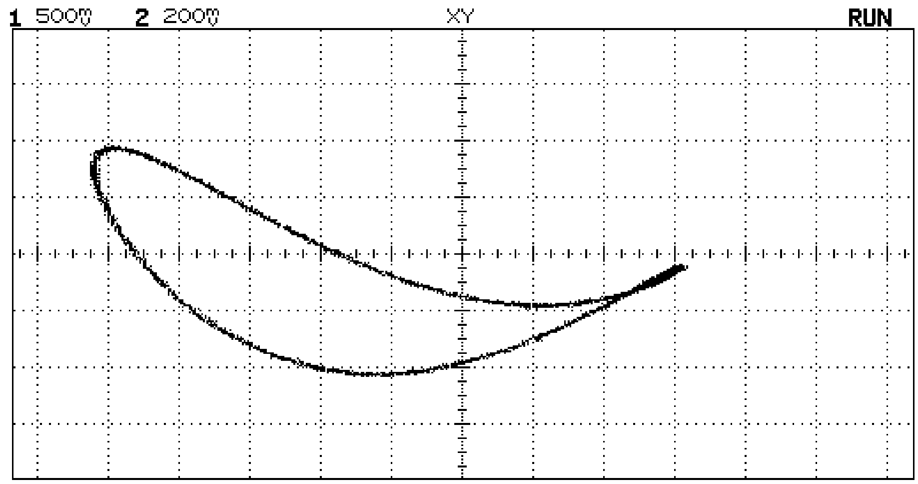


(b)

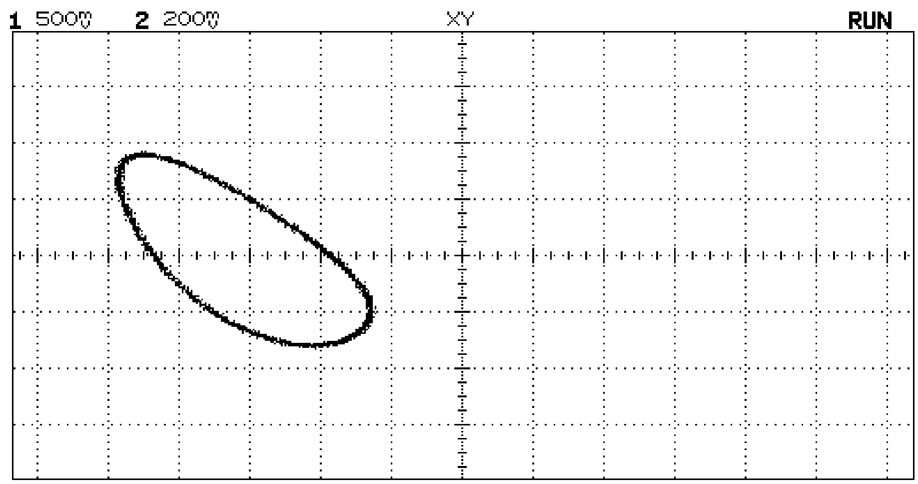
Fig. 2. Experimental results corresponding to $N = 3$ coupled Lorenz systems according to Eq. (1) for $\sigma = 20$, $b = 2.5$ and $R = 41$: (a) representation of u versus time for two contiguous oscillators; (b) phase portrait w versus u. The behavior of the array corresponds to a symmetric periodic rotating wave solution.

behavior with periodic rotating waves passing through chaotic rotating waves and quasiperiodic behavior. In the remainder of this section we are going to describe this route, that can be obtained by fixing two of the parameters to take the values: $\sigma = 20$ and $b = 2.5$, while R is varied.

Thus, Fig. 2 presents the results obtained for $R = 41$, that are completely analogous to those presented in [Sánchez & Matías, 1998], i.e. the behavior of the system corresponds to a periodic rotating wave. This attractor is quite robust in parameter space (the system exhibits this behavior at least in the range $R \in [40, 60]$. Another feature of the behavior of the system for this parameter is the fact that the attractor has the same symmetry as the evolution equations (1), namely regarding discrete symmetries, like the reflections $R : (x, y) \to (-x, -y)$. If one lowers R the system exhibits a symmetry breaking that leads to two symmetry related attractors with respect to the symmetry operation R, that still exhibit limit cycle behavior. In Fig. 3 the process of symmetry breaking is illustrated for two different parameter values, namely $R = 39$ in panel (a), and $R = 37$ in panel (b). The following step is that these symmetry related attractors exhibit a Hopf bifurcation



(a)

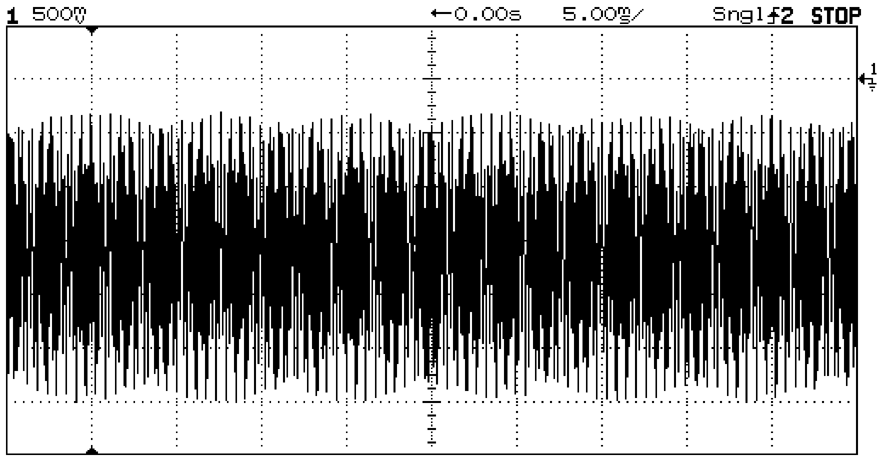


(b)

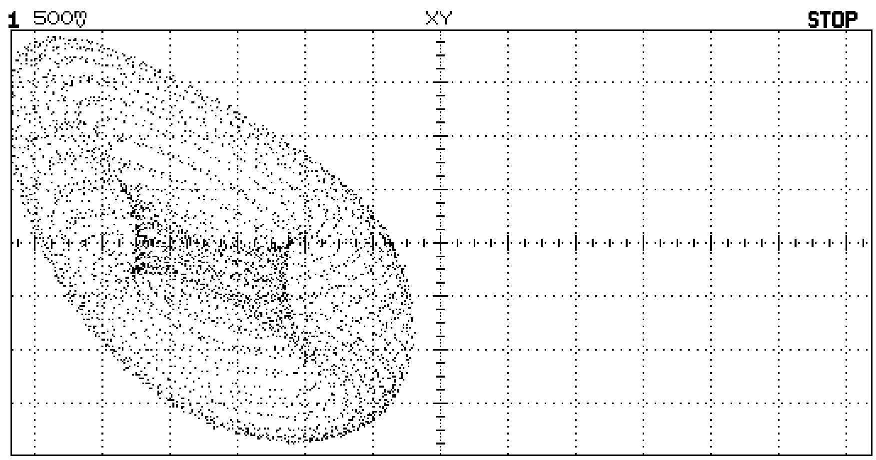
Fig. 3. Experimental results corresponding to $N = 3$ coupled Lorenz systems according to Eq. (1) for $\sigma = 20$ and $b = 2.5$: (a) phase portrait w versus u for $R = 39$; (b) phase portrait w versus u for $R = 37$. The behavior of the array corresponds in both cases to one of the two symmetry broken periodic rotating wave solutions that exist for these values of the parameters.

from about $R \sim 36.5$ that leads to quasiperiodic behavior, as can be seen from Fig. 4. There are two symmetry broken quasiperiodic solutions (the second one is obviously obtained by reflecting the one presented in this figure). Another feature of the quasiperiodic behavior is that the system retains the $2\pi/N$ between neighbor oscillators.

A new type of behavior happens from $R = 36.3$. An attractor merging bifurcation, crisis-like although the attractors are not chaotic, occurs, and the two tori merge leading to a symmetric, chaotic attractor. For values of R that are close to the transition, the system exhibits a type of behavior that is analogous to crisis-induced intermittency, implying that the system spends lapses of time alternatively in each of the formerly symmetry related attractors. The behavior that characterizes the system is analogous to the one found for rings of Chua's circuits in the previous works [Matías *et al.*, 1997a; Sánchez *et al.*, 1999], i.e. is the one we called a chaotic rotating wave, as can be seen from Fig. 5. In particular, it means that this behavior is characterized by an instantaneous phase difference of $2\pi/N$ between neighbor oscillators, stemming from a symmetric Hopf bifurcation, i.e. one in which two modes with a symmetry related by the ring



(a)



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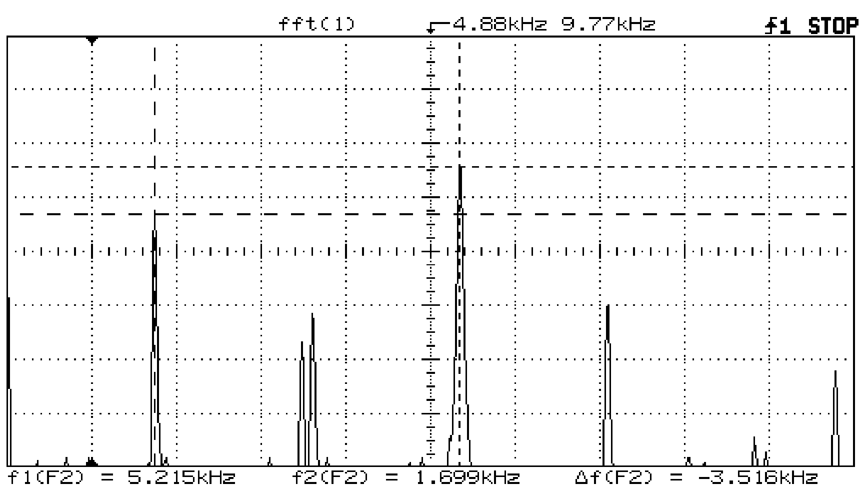
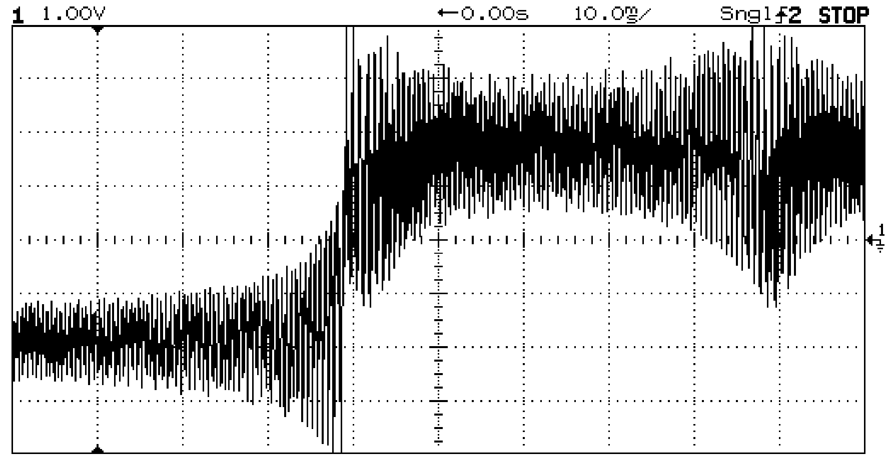
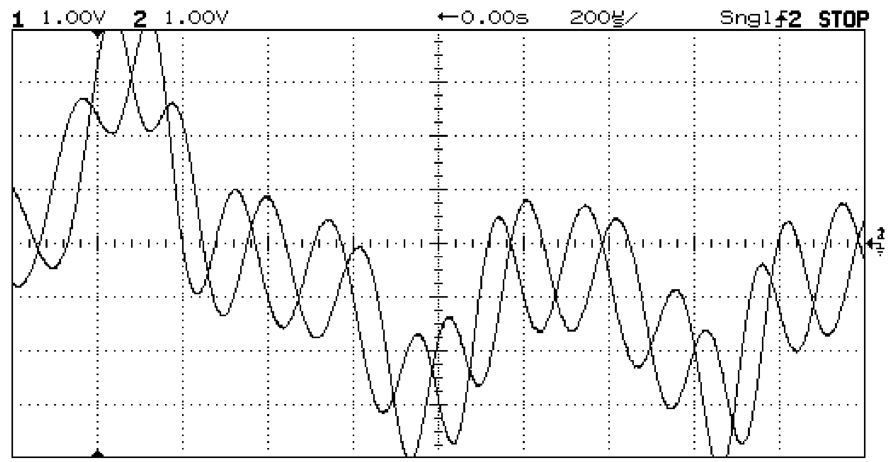


(c)

Fig. 4. Experimental results corresponding to $N = 3$ coupled Lorenz systems according to Eq. (1) for $\sigma = 20$ and $b = 2.5$ and $R = 36.5$: (a) representation of u versus time for one oscillator; (b) phase portrait w versus u; (c) power spectrum as a function of the frequency. As can be seen from the figure, the behavior of the array corresponds to a quasiperiodic attractor. In fact, there are two of such symmetry related attractors.



(a)



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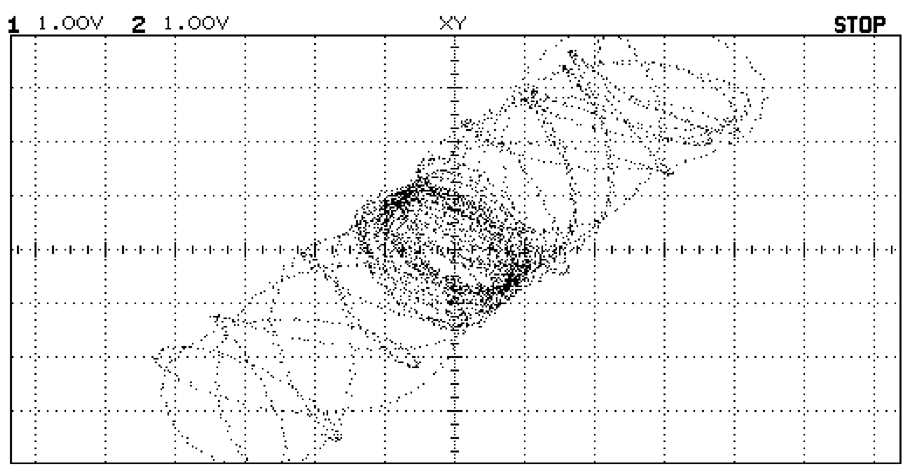
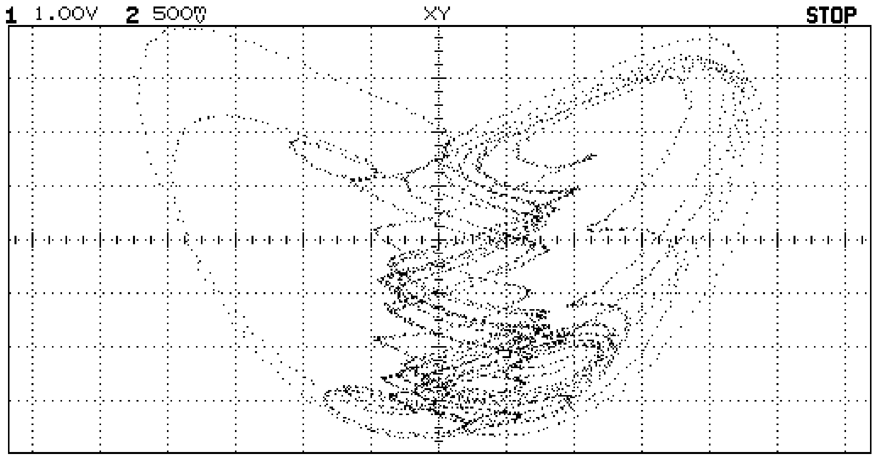


(c)

Fig. 5. Experimental results corresponding to $N = 3$ coupled Lorenz systems according to Eq. (1) for $\sigma = 20$ and $b = 2.5$ and $R = 36$: (a) representation of u versus time; (b) representation of u versus time for two contiguous oscillators to show the phase relationship; (c) representation of u_2 versus u_1; (d) representation of w versus u for a single oscillator. The behavior of the system for these values of the parameters corresponds to a chaotic rotating wave, see text.



(d)

Fig. 5. (*Continued*)

interact. The new light shed by our present study is that the chaotic rotating behavior can be understood as arising from the merging of two symmetry related quasiperiodic attractors. Their merging originates a chaotic attractor that stills retains some features of the two tori, and among them the phase relationship between neighbors, that is now instantaneous due to the underlying chaotic dynamics. Finally, by lowering R one gets synchronized chaotic behavior. The transition is continuous in this case (not abrupt, as happens for the transition at $R = 36.3$ from the quasiperiodic attractor to the chaotic rotating wave behavior.

3. Conclusions

To summarize the main features of the present preliminary communication, we have reported experimental results corresponding to the behavior of small arrays of identical coupled chaotic Lorenz oscillators. These results correspond to regions of parameter space that had not been covered before. The main conclusion is that these results, that correspond to a single type of oscillator, exhibit a quite rich behavior. The behaviors presented here are due to instabilities in the chaotic uniform state of the array. In particular, we have shown that an array of Lorenz systems exhibits a kind of behavior, chaotic rotating waves, that was previously reported for rings of Chua's oscillators. In the route presented in this contribution, this behavior appears from a crisis-type bifurcation in two symmetry related quasiperiodic attractors. These attractors appear when the symmetry broken periodic attractors, from the previously reported periodic rotating wave, exhibit a Hopf bifurcation.

The present work continues our investigation of emergent behaviors, i.e. behaviors that are not contained in those of the corresponding uncoupled systems. Thus, in addition to the interest in the knowledge of the different instabilities that may arise in this kind of coupled systems, they might be useful in the design of Central Pattern Generators, i.e. rings of neurons, able to perform a rich variety of dynamical behaviors. As a design caveat one could arrive at the conclusion that incorporating the known dynamical behaviors in the design of the isolated neurons may be not so relevant.

Acknowledgments

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