Since by Lemma 2.1 the system states $n(\cdot)$ and $N_g(\cdot)$ are nonnegative, we can write (37) as

$$\dot{W}(X) \leq -\min\{\beta b N_{a_0} + c,\, d,\, 2e\} \cdot \left[\frac{N_{a_0}}{2(1-\beta)}(2n + N_g) + \frac{1}{2}(N_a - N_{a_0})^2\right]. \quad (38)$$

Thus, we have the linear differential inequality

$$\dot{W}(t) \leq -\gamma W(t) \quad (39)$$

where

$$\gamma := \min\{\beta b N_{a_0} + c,\, d,\, 2e\} > 0. \quad (40)$$

By a comparison theorem given in [15, p. 2] or [16, p. 3] we conclude that W in (39) satisfies

$$W(X) \leq W(X_0)\exp(-\gamma t) \quad (41)$$

for all $t \geq 0$. Therefore, $W(X)$ tends to zero exponentially, and consequently X tends to X_e. □

IV. Conclusions

In this paper, we considered the rate equations of passively Q-switched lasers. We showed that for nonnegative and bounded inputs, the laser output is bounded. Furthermore, we showed that when the input is switched off, the laser output converges to zero asymptotically.

The dynamics of passively Q-switched lasers are very well represented by the system (1), to which step inputs are applied. In Section II-B, on the one hand, we showed that for some values of the laser parameters and the input amplitudes, the equilibrium points of the system (1) can be unstable. On the other hand, we showed that the system states are bounded, regardless of the values of the laser parameters and the input amplitudes. The instability of the equilibrium points and the boundedness of the system states suggest that the system states can possibly have periodic behavior, which is known as self-pulsation. We have simulated the system (1) for longer periods of time and have observed that the system states behave periodically for certain values of the laser parameters and the input amplitudes. However, we have not been able to prove the existence of periodic behavior rigorously, which is a difficult task. The difficulty stems from the fact that almost all existing mathematical techniques by which the existence of limit cycles is established, such as Poincare–Bendixon theorem, are for two-dimensional (planar) systems. From the practical point of view, it is important to establish the BIBO stability of the laser, which was achieved in this paper.

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An Experimental Setup for Studying the Effect of Noise on Chua's Circuit

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***Abstract*— The analog simulation of nonlinear dynamical systems is advantageous in some cases, i.e., when compared with the study using digital computers and, in particular, when one wishes to investigate the role of noise in these systems. In the present work we introduce two different methods of introducing a noise component in the most widely used chaotic circuit, namely, Chua's circuit, and apply these methods to study the effect of noise on identically driven chaotic circuits.**

***Index Terms*—Chaos, Chua circuit, noise, nonlinear circuits, stochastic systems.**

I. Introduction

The study of the behavior of nonlinear dynamical systems under the influence of noise is usually a difficult task [1]. Normally, one performs this type of study through digital simulation. However, noise usually interacts with numerical methods in a complicated manner,

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often posing a difficult challenge in the study of these stochastic differential equations [2], while the generation of high-quality random numbers in digital devices may prove difficult [3]. One conclusion is that the results of these studies with digital devices are not always clear cut. Thus, in many cases, analog simulations have practical advantages [4] as noise comes from a physical (random) process and has the appropriate correlation properties, and no ambiguities in the formulation of the system exist as one is studying a real physical process. This fact also implies that there are no doubts in the reality of the phenomenon under consideration, its robustness against small parameter mismatches, etc. Moreover, one can easily study the effect of parameter variation by simply tuning a knob.

A topic that has received some attention in the last few years is that of the effect of noise in (uncoupled) chaotic systems. When trying to study this effect in the case of Chua's circuit, we found that introducing an external source of noise, e.g., in the voltage across one of the capacitors of the circuit, was less trivial than expected, although we were able to develop various techniques for this purpose. Our research on this topic started after Maritan and Banavar [5] suggested that identical chaotic systems that are subject to the same noise do synchronize. In this paper, we shall show our results obtained from experiments with analog circuits (Chua's circuits) subjected to external noise [6].

II. Experimental Setup

The aim of this work is to present some suggestions regarding the implementation of two electronic setups used to introduce noise in analogue circuits. In this case we want to introduce the same noise in two identical chaotic circuits to observe if there is synchronization between them. Our experiments are based on Chua's circuit, a well-known paradigm of nonlinear analog circuits exhibiting chaotic behavior [7]. It is defined by the following evolution equations:

$$
\begin{aligned}
C_1 \frac{dV_1}{dt} &= \frac{1}{R}(V_2 - V_1) - h(V_1) \\
C_2 \frac{dV_2}{dt} &= \frac{1}{R}(V_1 - V_2) + i_L \\
L \frac{di_L}{dt} &= -V_2 - r_0 i_L
\end{aligned} \tag{1}
$$

where V_1 and V_2, the voltages across C_1 and C_2, respectively, and i_L, the current through L, are the three variables that describe the dynamical system. The three-segment piecewise linear characteristic of the nonlinear resistor is defined by

$$h(V_1) = G_b V_1 + \tfrac{1}{2}(G_a - G_b)[|V_1 + B_p| - |V_1 - B_p|] \tag{2}$$

where G_a and G_b are the slopes of the inner and outer regions of $h(V_1)$, respectively, and $B_p = 1\ V$ defines the location of the break points of the three-slope nonlinear characteristic $h(V_1)$. The slopes of the nonlinear characteristic $h(V_1)$ in (2) are defined by $G_a = -8/7000$ s and $G_b = -5/7000$ s.

An experimental setup of two identically driven Chua's circuits whose components are defined by $(C_1, C_2, L, r_0, R) = (10$ nF, 100 nF, 10 mH, 20 Ω, 1100 $\Omega)$ has been built. The tolerances of the components employed here are 10% for inductors, 5% for capacitors, and 1% for resistors. The circuits were sampled with a digital oscilloscope (Hewlett-Packard 54 601B) with a maximum sample rate of 20 millions samples per second, with 8-bit A/D resolution and a record length of 4000 points.

The external noise has been generated by using a function generator (Hewlett-Packard 33 120A) and its desired statistical characteristics, namely, a Gaussian distribution and a zero mean in the absence of an offset, have been adequately checked. The external noise has been introduced in two different ways: additively and multiplicatively. In the first case [see Fig. 1(a)] the stochastic voltage produced by the noise generator, $\xi(t)$, has been converted into a current through a voltage controlled current source (VCCS). This contribution is then added to that of the nonlinear element, yielding the following evolution equation for the voltage across capacitor C_1:

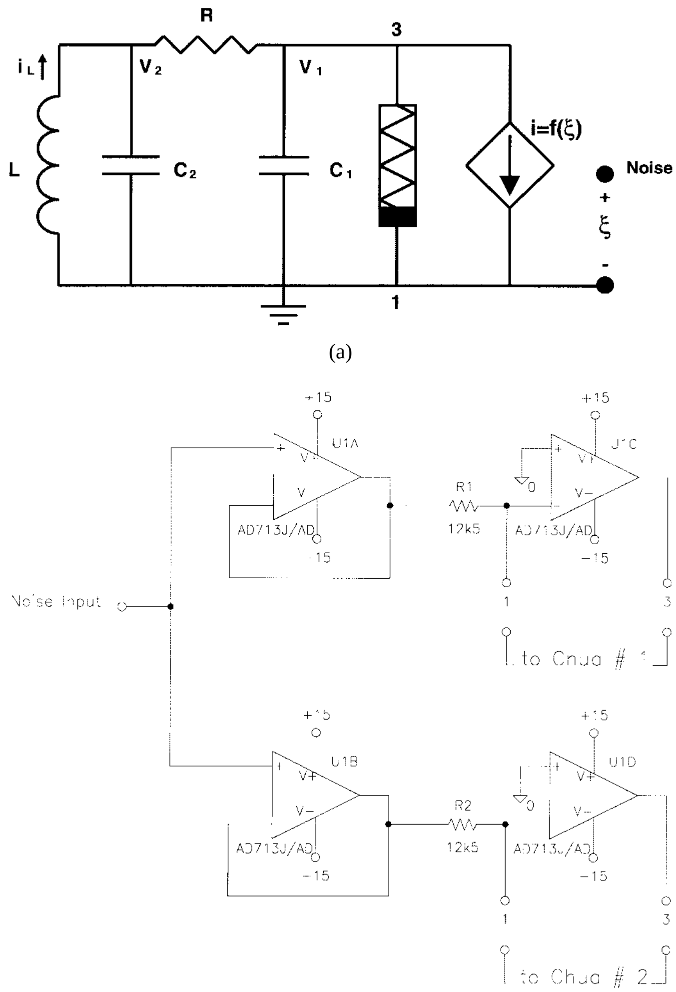


Fig. 1. Schematic diagrams of the experimental setting used for adding noise. (a) Setup corresponding to a single Chua's circuit. Noise enters additively in the form of a stochastic voltage that is converted into a current through a VCCS and added to the current produced by the nonlinear element of the circuit [see (3)]. (b) Detailed schematic of the setup used for adding the same noise to two Chua's circuits. The noise is buffered from both VCCS's ensuring thereby no interaction between the circuits.

$$C_1 \dot{V}_1 = \frac{V_2 - V_1}{R} - h(V_1) - f(\xi) \tag{3}$$

where $f(\xi) = \xi/R'$ with $R' = 12.5$ kΩ.

The details of the electronic configuration are shown in Fig. 1(b). The signal from the noise generator is buffered by op amps U_{1A} and U_{1B} so that no interaction between Chua's circuit one and two is guaranteed. The buffered signals drive two identical VCCS's formed by a resistor R_1 (R_2) and an op amp U_{1C} (U_{1D}). The voltage across resistors R_1 and R_2 is the difference between the outputs of each buffer and the virtual grounds (inverter terminal of each op amp in the VCCS). The resistance value $(12.5\ \text{k}\Omega)$ is selected to match the output voltage range of the noise generator with a proper range in the current applied to the circuit. In this configuration, the current flow across resistors R_1 and R_2 is applied between terminals one and three of the respective Chua's circuit in view of the high input impedance of the op amps (Jfet input).

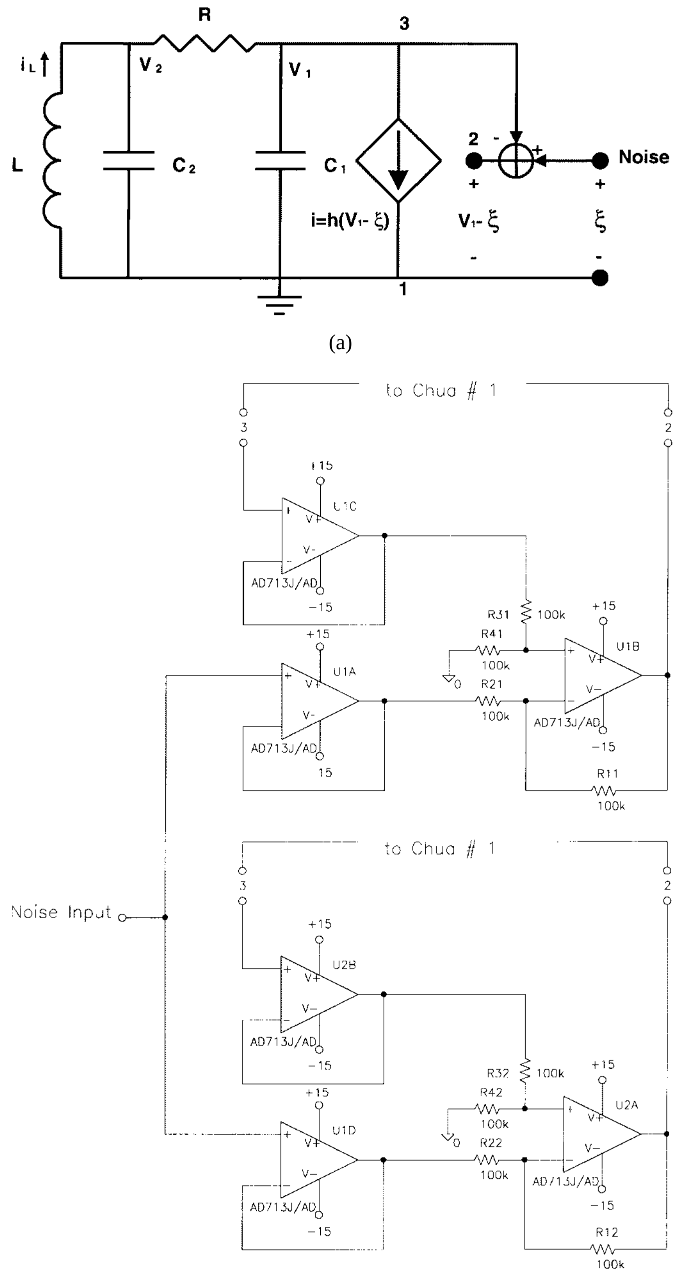
In the second case we want to introduce the noise in a multiplicative way. To achieve this goal we drive the nonlinear element with the voltages from capacitor C_1 and the noise source. Specifically, we have subtracted the stochastic signal to voltage V_1 by using an analog subtractor and the result has been used to drive the nonlinear element. This yields the following evolution equation for the voltage across capacitor C_1:

$$C_1 \dot{V}_1 = \frac{V_2 - V_1}{R} - h(V_1 - \xi) \tag{4}$$

where it is easy to see that the noise term now yields a multiplicative contribution.

As can be seen in (4) the argument of the nonlinear function is not simply the voltage from the capacitor C_1 (V_1). Instead, we have the capacitor voltage V_1 minus the noise voltage ξ. The usual implementation of a nonlinear element in Chua's circuit consists of two negative resistors with different slopes connected in parallel so that a current flows across the two terminals of the element. In this implementation, current flow is controlled by the voltage across these terminals and so its performance is restricted to this voltage (V_1). For this reason, we will need a new implementation of the nonlinear element that will allow driving it from any voltage, not only V_1. The new implementation of the nonlinear element [8] is based on a VCCS with a characteristic defined by (2). With this element, we produce a current [$h(V)$ in (1)] that is controlled with any voltage. In our case, we subtract the same noise to voltage from capacitor C_1 of each Chua's circuit and drive each nonlinear element with the respective signal (4) to try to synchronize the circuits without any connection between them.

The electronic configuration is shown in detail in Fig. 2(b). As in the previous case, the noise from the Generator is buffered through op amps U_{1A} and U_{1D}. The voltage from capacitor C_1 in Chua's circuits is buffered also by op amps U_{1C} and U_{2B}. With this arrangement no interaction between the two circuits and application of the same noise to both circuits is guaranteed. Each analog subtractor consists of one op amp U_{1B} (U_{2A}) and four resistors $R_{11}, R_{21}, R_{31}, R_{41}$, $(R_{12}, R_{22}$, and $R_{32}, R_{42})$, respectively. The difference of the signals is given by application of Kirchhoff's laws at the subtractor. The signal resulting from this operation is used to drive the nonlinear element.



(a)

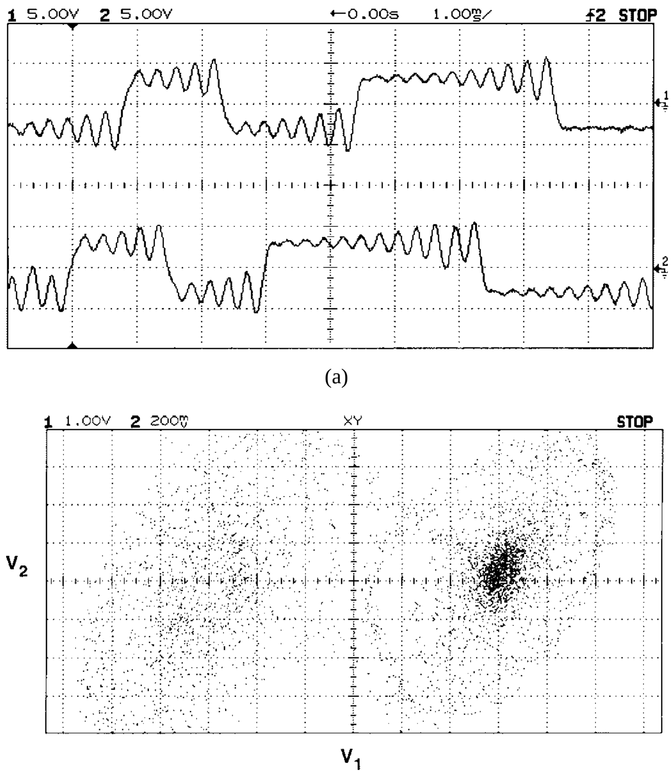
(b)

Fig. 2. Schematic diagrams of the experimental setting used for introducing noise in multiplicative way. (a) Setup corresponding to a single Chua's circuit. Noise is added to voltage from capacitor C_1 and is used to drive the nonlinear element of the circuit [see (4)]. (b) Detailed schematic of the setup used for introducing the same noise into two Chua's circuits. The noise is buffered from both VCCS's, thereby ensuring no interaction between the circuits. For each circuit the buffered noise is added to the signal from capacitor C_1 (also buffered).

III. Results

To illustrate the kind of studies that can be performed with these methods of introducing noise on Chua's circuits, we shall present results corresponding to two identical chaotic systems subject to external noise (a more complete account can be found in [6]). First, by studying theoretically the statistics of trajectory separation in an ensemble of identical systems subject to the effect of external white noise we find that synchronization is attained whenever the largest Lyapunov exponent of each system is negative [9], [10]. If the largest Lyapunov exponent is positive (which occurs in a chaotic system) the identically synchronized systems will not synchronize. Pikovsky [11] had studied the case of the logistic equation under the presence of noise, discussed in [5], concluding that they should not synchronize in such a case. Thus, contradicting the results of [5]. An important feature to be noticed in this case, in comparison to the usual setting of chaotic synchronization (see, e.g., [12]) is that here the systems remain uncoupled. This means that here one no longer has conditioned Lyapunov exponents, transverse or otherwise. Thus, it is not possible for the noise-driven systems to remain chaotic and exhibit synchronization (this means that one is not speaking about chaotic synchronization). Instead, the systems must become periodic in order to exhibit synchronization behavior.

In agreement with previously reported theoretical analyses, the basic result of the present work is that the important aspect in the effect of noise on an ensemble of identically driven chaotic systems depends on whether its mean is zero (symmetrically distributed noise) or nonzero (asymmetrical noise). If the noise has zero mean the identically driven systems do not synchronize, regardless of the amplitude and variance of the noise. Our observations are in line with [13]. Noise produces a loss of the fine fractal details of the



(a)

(b)

Fig. 3. Effect of symmetrically distributed (zero mean amplitude 1.25 V) noise on two Chua's circuits identically driven according to (3) (see also Fig. 1). (a) V_1 taken in both circuits versus time (top and bottom, respectively). (b) Phase portrait in one of the two circuits (V_2 versus V_1).

strange attractor that becomes smeared out, while no evidence of synchronization behavior is observed. This can be seen in Fig. 3 for the case of additive noise (3). The results are completely analogous in the case of multiplicative noise (4). In contrast, when the noise that is introduced in the system has a nonzero mean, the stochastic signal induces qualitative changes in the system [6] in both the additive and the multiplicative case. This confirms the validity in our case of the theory of statistics for trajectory separation in random dynamical systems [9], [10]. Synchronization occurs, in a generalized sense, when the largest Lyapunov exponent becomes negative, although this implies that the system is no longer chaotic.

IV. Conclusions

In the present contribution we have discussed two different methods of introducing an external noise component in Chua's circuits where we have used a recently introduced design of the nonlinear element [8]. The first method adds a noisy current signal to another current signal in a chaotic circuit. This method is particularly suitable to be used with Miller integrators, allowing us to add the noise to a current variable in the system. The second method introduces the noise in the form of a multiplicative contribution to a signal. This method is suitable for introducing noise into a particular place of the evolution equations. These designs have been tested by trying to elucidate the effect of noise on identically driven chaotic systems, showing that synchronization is only expected to appear if the applied noise has a nonzero mean which changes the behavior of the system to periodic.

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