THREE-FREQUENCY RESONANCES IN DYNAMICAL SYSTEMS

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We investigate numerically and experimentally dynamical systems having three interacting frequencies: a discrete mapping (a circle map), an exactly solvable model (a system of coupled ordinary differential equations), and an experimental device (an electronic oscillator). We compare the hierarchies of three-frequency resonances we find in each of these systems. All three show similar qualitative behaviour, suggesting the existence of generic features in the parameter-space organization of three-frequency resonances.

1. Introduction

A fundamental goal in the modelling of complex behaviour by means of low-dimensional nonlinear deterministic systems is the understanding of the transition to chaos from quasiperiodic motion on a $\mathbb{T}^j$ torus for small j. This transition is well understood for the case $j = 2$, the paradigmatic model being the periodically forced nonlinear oscillator and its discrete version: the circle map.

In $\mathbb{T}^2$, orbits can be hierarchically arranged by means of rotation numbers [Arrowsmith & Place, 1990]

$$\rho(r,\theta) = \lim_{n\to\infty} \frac{\theta_n - \theta_0}{2\pi n}, \tag{1}$$

periodic orbits — lockings or resonances — being then characterized by rational rotation numbers. If two rational rotation numbers $(p/q, r/s)$ satisfy $|ps - rq| = 1$ they are said to be unimodular or Farey adjacents. Between two periodic solutions characterized by unimodular rotation numbers there exists a periodic solution with a minimal period. This solution is given by a rotation number termed the Farey mediant of the two previous ones: $m/n = p/q \oplus r/s = (p+r)/(q+s)$. Moreover this periodic solution is the most prominent — the largest — in the open interval $(p/q, r/s)$ between the two parents. Continued application of the mediant gives the Farey tree which underlies the organization of the locking intervals of

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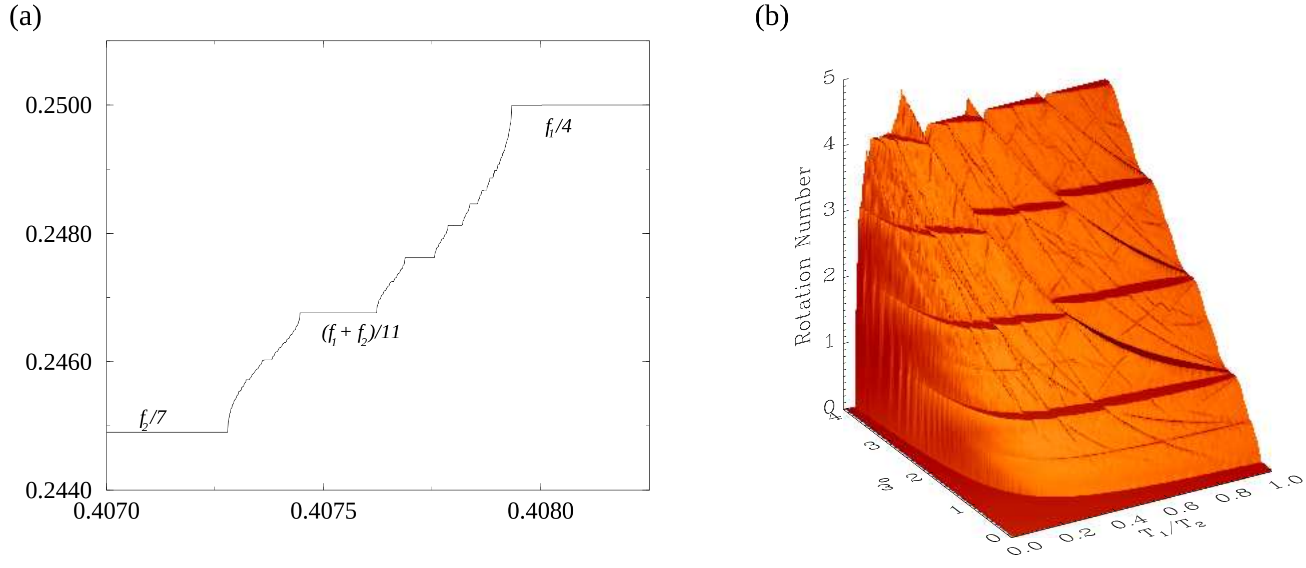

Figure 1: (a) Three-frequency devil's staircase in the quasiperiodically forced circle map. The external frequencies are $f_1 = 1$ and $f_2 = 12/7$. The steps at left and right correspond to the plateaux $(12/7)/7 = 0.2448979\ldots$ and $1/4 = 0.25$; the largest plateau between the two is $(1 + 12/7)/11 = (19/7)/11 = 0.2467532\ldots$, in accordance with the prediction of the generalized Farey mediant. (b) Devil's ramp system: the rotation number as a function of the external frequency ratio and the intrinsic frequency for a quasiperiodically forced circle map representing a doubly kicked nonlinear oscillator. The devil's ramps illustrate the overall ordering of three-frequency resonances.

rational rotation numbers in the space of parameters of two-frequency systems; the so-called devil's staircase [González & Piro, 1983; Aronson *et al.*, 1983; González & Piro, 1985; Cvitanović *et al.*, 1985; Hao, 1989; Arrowsmith *et al.*, 1993]. These properties of rotation numbers are closely related to the continued fraction expansion which gives the best rational approximants to an arbitrary real number.

In order to study $\mathbb{T}^3$ torus breakdown we have generalized this approach to the case of the simultaneous approximation of a pair of real numbers [Cartwright *et al.*, 1999b]. Three-frequency systems also possess a structure of resonances, but this time more complex, for in addition to the rational relations found in two-frequency systems there is now a new type of locking: a three-frequency resonance, given by the nontrivial solutions of the equation $nf_1 + mf_2 + qf_3 = 0$ with n, m, and q nonzero integers. Three-frequency resonances form a web in the parameter space of the frequencies [Baesens *et al.*, 1991].

We have found that [Cartwright *et al.*, 1999b]:

- We can define a subharmonic real interval $(f_1/p, f_2/q)$ inside which the hierarchical organization of three-frequency resonances is well described by a generalized Farey sum.

- In order to implement the generalized Farey sum we must modify the adjacency condition: if p/q is a convergent of the forcing frequency ratio we define two fractions of real numbers f_x/m, f_y/n as adjacents if $|f_x n - f_y m| = |f_1 q - f_2 p|$ where f_1 and f_2 are the external frequencies.

- We can now define the generalized Farey sum between fractions of real numbers which satisfy this adjacency condition: $f_r/k = f_x/m \oplus f_y/n = (f_x + f_y)/(m + n)$.

Here we present numerical and experimental studies of three dynamical systems having three interacting frequencies, in which we can observe three-frequency resonances, and compare our observations with the above predictions.

2. Numerical Simulations

We have investigated a discrete mapping: a quasiperiodically forced circle map given by the equation

$$\phi' = \phi + \Omega_n - \frac{k}{2\pi} \sin 2\pi\phi \bmod 1. \qquad (2)$$

The quasiperiodic sequence Ω_n is the time interval between successive pulses of a sequence composed of the superposition of two periodic subsequences, one of period $T_1 = 1$ and the other of period T_2 ($T_2 < 1$ with no loss of generality), multiplied by the value of the intrinsic frequency of the oscillator.

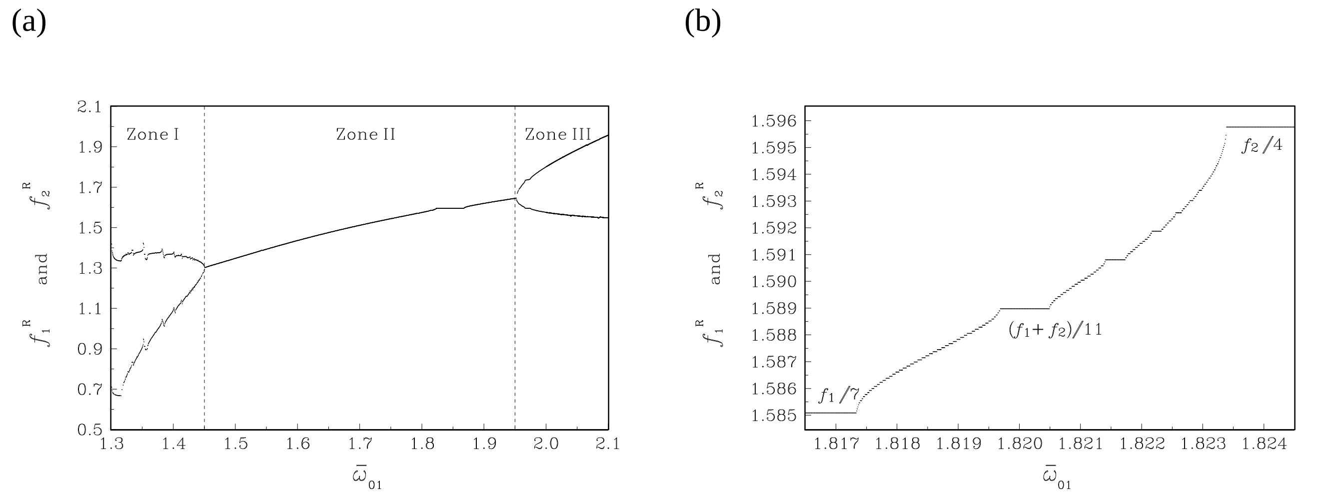

Figure 2: (a) The frequency response of both oscillators of Eq.(3) as a function of the intrinsic frequency of the first oscillator. Zone I is chaotic, with intermittent synchronization, in zone II there is 1/1 synchronization, and in zone III, synchronization other than 1/1. The input frequencies have been set to be equivalent to those of Fig. 1. (b) A magnification of the central zone of (a), where the oscillators are mutually synchronized at 1/1.

In Fig. 1a we can observe that there are intervals of constant frequency response, namely the horizontal segments of the figure. These intervals represent stable three-frequency resonances, that is, responses that satisfy $nf_1 + mf_2 + qf_3 = 0$ with n, m, and q nonzero integers. We can also see in Fig. 1a that the stability widths of the resonances form a hierarchical structure very similar to that of the well known devil's staircase in periodically forced nonlinear oscillators. This is the generalized devil's staircase for three-frequency systems.

The other image, Fig. 1b, shows the devil's ramps. These represent the global hierarchy of three-frequency resonances shown as a function of the external frequency ratio and the intrinsic frequency for a critical ($k = 1$, the map instantaneously has critical behaviour) quasiperiodically forced circle map.

We have also performed numerical simulations with two parametrically coupled nonlinear oscillators, each with an exact analytical solution, forced by means of two impulsive periodic forces. The differential equation for each oscillator [González & Piro, 1983; González & Piro, 1985] is:

$$\ddot{u}_i+(4bu_i^2-2a)\dot{u}_i+b^2u_i^5-2abu_i^3+(\omega_{0i}(t)^2+a^2)u_i = f_i(t) \tag{3}$$

for $i = 1, 2$. The coupling and forcing terms $\omega_{0i}(t) = \tilde{\omega}_{0i} + sgn\, u_i(t)\, sgn\, u_j(t)\Delta_i$ and $f_i(t) = V_{Ei}\sum_n \delta(t - nT_{Ei})$ preserve the piecewise integrability of the system.

We have made a power spectrum analysis of the output of both oscillators. In Fig. 2a we display the most prominent peak in each spectrum versus the intrinsic frequency of oscillator 1 for a parameter region equivalent to that of Fig. 1a. In Fig. 2b we show a magnification of the zone of Fig. 2a in which the two oscillators are synchronized at 1/1. The three principal peaks in the Fourier spectrum satisfy $nf_1+mf_2+qf_3 = 0$ with n, m, and q nonzero integers. All the other peaks in the spectrum can be expressed as linear combinations of this fundamental set.

3. Experimental Results

We have constructed an electronic oscillator — Fig. 3 — a higher-dimensional version of a phase-locked loop (PLL), forced with two independent periodic forces [Calvo *et al.*, 1999]. Our circuit consists of two coupled voltage-controlled oscillators forced with two external forces of frequencies f_1 and f_2. As a basic circuit for both oscillators we use a digital phase-locked loop integrated circuit, the CD 4046A. The outputs of the two phase-locked loops are sent to a type 1 phase comparator. The error signal is fed back to both voltage-controlled oscillators, and passes through an overall adjustable amplifier to provide control over the coupling strength; we are interested in the weak coupling regime in this work. Inverted and direct versions of the error signal are sent to oscillators one and two respectively; this inversion of the error signal in one of the paths is necessary for the stability of the circuit. Feedback signals enter the voltage-controlled-oscillator control pins through appropriate adder circuits. The adders also allow independent coupling with the external forces and tuning of the internal frequencies through application of adjustable DC levels.

In Fig. 4a we plot the fundamental frequency f_3 versus the DC offset of one of the forcing signals. As in the previous numerical simulations, the general-

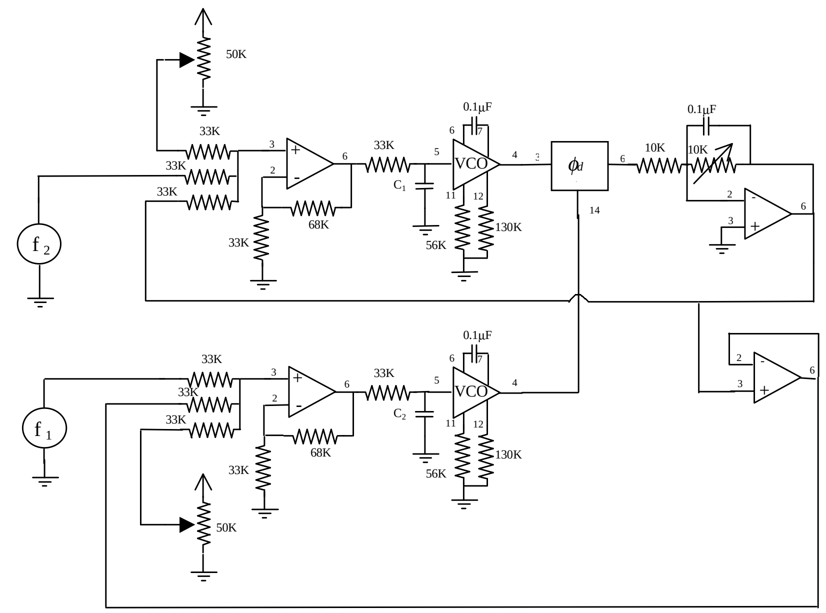

Figure 3: The circuit. The output of the voltage controlled oscillator (VCO) sections of both digital phase-locked loop devices are sent to an exclusive OR port for phase comparison. The phase comparator output is returned, after low-pass filtering, to both VCO inputs. An amplifier in the feedback path controls the coupling strength between the oscillators. Appropriate operational adders on both VCO inputs allow the external forcing of the oscillators. Stable phase-locked responses between oscillators also require an additional unit gain inverter prior to one of the device inputs.

ized mediant is the largest plateau between the two adjacents given by the subharmonics of the forcing frequencies. The circuit of Fig. 3 represents qualitatively a large class of real dynamical systems characterized by some kind of feedback loop with a transference function of the low-pass type, and can be useful in clarifying the physical mechanisms that underlie three-frequency resonances. In Fig. 5a we show the Fourier spectrum of the oscillator response at a point inside the stability region for the three-frequency resonance. In Fig. 5b, a magnification around this frequency value, we can see that harmonics of the main resonance at $\tilde{f}_s = 518.2$ Hz lie at equal distances $\delta = 27.3$ Hz from both external frequencies. The peak at $\tilde{f}_s$ dominates the spectrum, being greater than those corresponding to the forcing frequencies. The distance $\delta = \varepsilon \tilde{f}_s$ appears as a modulation of the spectrum, which shows, on this fine scale, a series of peaks separated precisely by this distance.

The origin of these frequencies can be interpreted in terms of the general structure of the system. Passive nonlinearities are able to generate appropriate frequency harmonics. These, in turn, through nonlinear mixing with the driving frequencies, can generate the frequency δ. Only for the case of the main three-frequency resonance are the two δs of the same value, i.e., $\delta = \delta_1 = \delta_2$. Thus, further nonlinear mixing in this case gives only terms of zero frequency and harmonics of δ, that is, no other frequencies are added to the system. Otherwise, for an arbitrary frequency, the two δs are in general different and, if irrationally related, generate an infinity of low-frequency components that pass through the low-pass filter and are fed back to the system, destabilizing the response for this particular frequency value. This physical mechanism can also explain the ordering of the three-frequency resonances. Only the main resonance is characterized by $\delta_1 = \delta_2$, but successive application of the generalized Farey operation gives distances that are rationally related, that is $\delta_1/\delta_2 = r/s$. Consequently, for all resonances, successive nonlinear mixing through the feedback loop can generate only a finite number of new frequencies, preserving the stability of the corresponding response. Also the greater the integers r, s, the smaller the stability interval, because more destabilizing frequencies are added through the feedback mechanism.

4. Discussion

In Fig. 4b we plot the generalized Farey tree structure formed by the predicted three-frequency resonances [Cartwright *et al.*, 1999b]. The frequencies of the resonances are obtained by recursive application of the generalized Farey sum starting with f_1/q and f_2/p. This structure accurately describes all the three-frequency resonances found in the experiment with phase-locked loops [Calvo *et al.*, 1999] and in the numerical simulations with differential equations and maps. Successive levels in the tree describe the ordering of stability widths in each case. The generalized Farey tree structure is thus found to gov-

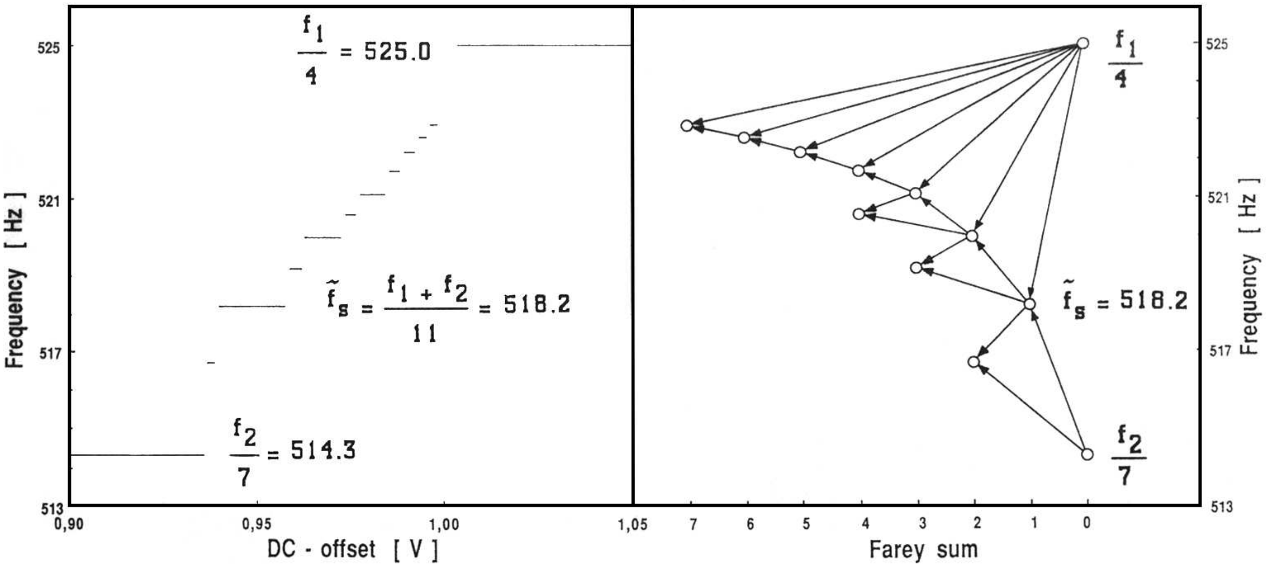

Figure 4: (a) Experimental results from an electronic circuit of quasiperiodically forced phase-locked loops — a three-frequency devil's staircase. The external frequencies f_1 and f_2 are here fixed at 2100 Hz and 3600 Hz, equivalent to those in Fig. 1. We have plotted the third frequency of a three-frequency resonance against a control parameter (the DC offset of one of the external forces) for all resonances with plateaux larger than a certain size. (b) Shows the hierarchy of three-frequency resonances predicted by the generalized mediant starting from the parents $f_1/4$ and $f_2/7$. At each level in the hierarchy, the daughter resonance formed by the mediant between two adjacent parents is seen to be the largest in its interval.

ern the hierarchy of three-frequency resonances in representative dynamical systems with three interacting frequencies. We conjecture from this that the ordering may be universal in three-frequency systems.

From the theorems of Newhouse, Ruelle, and Takens [Ruelle & Takens, 1971; Newhouse *et al.*, 1978] we expect that this hierarchical structure of three-frequency resonances should be relevant to the study of torus breakdown and the transition to chaos in complex arrays of coupled nonlinear oscillators. Such oscillator networks occur in many biological systems, from fireflies and circadian rhythms to physiological and neurological systems such as the heart and brain. We have investigated the application of these ideas to one such problem in biology: that of the mechanism of pitch perception in the auditory system. We find good agreement between dynamical systems theory and perceptual experiments [Cartwright *et al.*, 1999a].

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References

Aronson, D. G., Chory, M. A., Hall, G. R. & McGehee, R. P. [1983] "Bifurcations from an invariant circle for two-parameter families of maps of the plane: A computer-assisted study," *Commun. Math. Phys.* **83**, 303–354.

Arrowsmith, D. K., Cartwright, J. H. E., Lansbury, A. N. & Place, C. M. [1993] "The Bogdanov map: Bifurcations, mode locking, and chaos in a dissipative system," *Int. J. Bifurcation and Chaos* **3**, 803–842.

Arrowsmith, D. K. & Place, C. M. [1990] *An Introduction to Dynamical Systems* (Cambridge University Press).

Baesens, C., Guckenheimer, J., Kim, S. & MacKay, R. S. [1991] "Three coupled oscillators: Mode-locking, global bifurcations and toroidal chaos," *Physica D* **49**, 387–475.

Calvo, O., Cartwright, J. H. E., González, D. L., Piro, O. & Sportolari, F. [1999] "Three-frequency resonances in coupled phase-locked loops," *IEEE Trans. Circuits and Systems, to appear.*

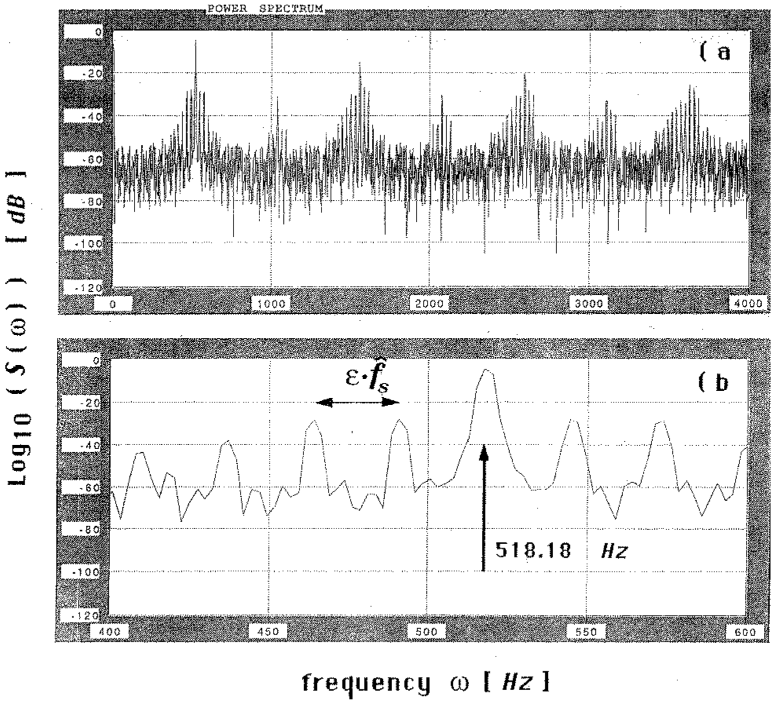

Figure 5: The power spectrum of the output of the circuit for a DC offset of 0.95 V. Other parameters are as in fig. 4. (a) From 0 – 4000 Hz. (b) Detail from 400 – 600 Hz, showing peak at $\tilde{f}_s \approx 518.18$ Hz and minor peaks separated a distance $\delta = \varepsilon \tilde{f}_s$.

Cartwright, J. H. E., González, D. L. & Piro, O. [1999a] "Nonlinear dynamics of the perceived pitch of complex sounds," *Phys. Rev. Lett.* **82**, 5389-5392.

Cartwright, J. H. E., González, D. L. & Piro, O. [1999b] "Universality in three-frequency resonances," *Phys. Rev. E* **59**, 2902–2906.

Cvitanović, P., Shraiman, B. & Söderberg, B. [1985] "Scaling laws for mode lockings in circle maps," *Physica Scripta* **32**, 263–270.

González, D. L. & Piro, O. [1983] "Chaos in a nonlinear driven oscillator with exact solution," *Phys. Rev. Lett.* **50**, 870–872.

González, D. L. & Piro, O. [1985] "Symmetric kicked self oscillators: Iterated maps, strange attractors and the symmetry of the phase locking Farey hierarchy," *Phys. Rev. Lett.* **55**, 17–20.

Hao, B.-L. [1989] *Elementary Symbolic Dynamics and Chaos in Dissipative Systems* (World Scientific).

Newhouse, S. E., Ruelle, D. & Takens, F. [1978] "Occurrence of strange axiom A attractors near quasiperiodic flows on T^m, $m \geq 3$," *Commun. Math. Phys.* **64**, 35–40.

Ruelle, D. & Takens, F. [1971] "On the nature of turbulence," *Commun. Math. Phys.* **20**, 167–192. **23**, 343–344.