DOMAIN GROWTH AND COARSENING INHIBITION IN A NON POTENTIAL SYSTEM

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Abstract

We present a study of interface dynamics in two spatial dimensions for a non-relaxational system that describes the temporal evolution of three competing real fields. This and similar models have been used to get insight into problems like Rayleigh-Bénard convection in a rotating cell or population competition dynamics in predator-key systems. A notable feature is that the non-potential dynamics stops the coarsening process as long as the system size is large enough. For certain values of the parameters, the system switches to a chaotic dynamical state known as the Küppers-Lortz (KL) instability. When isotropic spatial derivatives are used, the intrinsic period of the KL instability diverges with time. On the contrary, anisotropic derivatives stabilize the KL period.

1 Introduction and model

A topic that has deserved recently much attention is the influence of nonpotential dynamics (those that cannot be obtained from the minimization of a potential function) on coarsening processes. In this paper we show how, for nonpotential systems, different types of spatial derivatives can modify completely the nature of the coarsening process. For this purpose, we use a theoretical model proposed in the context of convection in a rotating cell by Busse and Heikes [1] to which spatial dependent terms have been added:

$$\begin{aligned}
\partial_t A_1 &= \mathcal{L}_1 A_1 + A_1 \left[1 - A_1^2 - (\eta+\delta)\, A_2^2 - (\eta-\delta)\, A_3^2\right], \\
\partial_t A_2 &= \mathcal{L}_2 A_2 + A_2 \left[1 - A_2^2 - (\eta+\delta)\, A_3^2 - (\eta-\delta)\, A_1^2\right], \\
\partial_t A_3 &= \mathcal{L}_3 A_3 + A_3 \left[1 - A_3^2 - (\eta+\delta)\, A_1^2 - (\eta-\delta)\, A_2^2\right].
\end{aligned} \tag{1}$$

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Here $\mathcal{L}_i$ $(i = 1, 2, 3)$ are linear differential operators. The fields A_i $(i = 1, 2, 3)$ are the (real) amplitudes of three set of convection rolls oriented 60 degrees at each other; the parameter δ bears on the rotation angular velocity of the cell and η is related to other physical properties of the fluid. Similar models have been used to study competition population dynamics in predator-prey systems [2,4]. A one-dimensional version of (1) is studied in reference [5].

It is convenient to split (1) into potential and non-potential contributions: $\partial_t A_i = \delta\mathcal{F}/\delta A_i + \delta \cdot f_i$ $(i = 1, 2, 3)$, where $\mathcal{F}$ is a real functional. When $\delta = 0$, $\mathcal{F}$ is a Lyapunov potential, so that the dynamics relaxes towards the minima of $\mathcal{F}$. On the other hand, $\delta \neq 0$ implies a non-relaxational dynamics governing the system. In the fluid analogy this means a nonzero rotation angular velocity.

Eqs. (1) have three stationary homogeneous "roll" solutions $A_i = 1$, $A_j = 0$, $j \neq i = 1, 2, 3$ that are stable for $|\delta| < \eta - 1$. For every $\eta > 1$ there exists a critical value $\delta_c(\eta)$ such that, when $|\delta| > \delta_c$, the roll solutions are no longer stable. Then the system breaks up into a chaotic dynamical state. In the reference system that rotates with the cell, the convective rolls alternatively change between three preferred directions. This phenomenon is known as the Küppers-Lortz (KL) instability. The 0-dimensional model ($\mathcal{L}_i = 0,\ i = 1, 2, 3$) shows the unwanted feature that the period between successive alternations of the dominating modes A_1, A_2 and A_3 diverges with time. In order to avoid this behavior, two solutions have been proposed. Originally, Busse and Heikes suggested that noise (present at all times) could lead to a constant period (fluctuating around a mean value). This is checked in numerical simulations [6]. Another solution to circumvent this problem is to add spatial dependent terms to the equations. Starting from a small perturbation of the unstable state $A_1 = A_2 = A_3 = 0$ in the case $\eta > 1$, the system develops a spatial structure consisting of domains of rolls separated by rather abrupt interfaces. In the bulk of each domain, one amplitude is close to one and the other two close to zero. The KL instability now occurs in the bulk of the domains and leads to an alternation of the modes at every point of space. Below the KL point the system may exhibit domain growth until it reaches a final stationary state.

2 The role of the spatial derivatives

We use two choices for the linear differential operators $\mathcal{L}_i$ of model (1):

First we consider terms of the simplest diffusive form, i.e., $\mathcal{L}_i = \nabla^2$, $i = 1, 2, 3$. This kind of spatial dependent terms have also been used in biological models [2].

The isotropic nature of the spatial dependent terms make the fronts move in the normal direction at each point. It is possible to show [7] that the (normal) front velocity is given by $v_n(\mathbf{r}, t; \eta, \delta) = -\kappa(\mathbf{r}, t) + \delta \cdot f(\eta)$, where κ is the local curvature of the front line. The term $v(\delta) \equiv \delta \cdot f(\eta)$ is the planar front velocity which is proportional to δ at lowest order. For η, δ fixed, there exists a critical value of the curvature, $\kappa_c \equiv \delta \cdot f(\eta)$ such that an interface does not propagate. In spherical symmetry this fact entails the existence of a droplet of critical radius R_c; any drop with radius $R > R_c$ grows and if $R < R_c$ it shrinks.

In the potential limit, $\delta = 0$, the system tends to reduce the total interfacial area by decreasing the curvature of interfaces. As a consequence, domains grow in time and the system shows coarsening. The growth law is found to be $R(t) \sim t^{1/2}$ as corresponds to a nonconserved scalar order parameter. However when $\delta \neq 0$, the nonpotential dynamics can stop the coarsening process even if $\delta < \delta_c$. The role of the non-potential terms is mainly to make the front lines rotate around points (vertices) where the three amplitudes coexist and this prevents the system from coarsening for system sizes sufficiently large. It is important to realize that the alternation between the modes in a fixed point above the KL instability is the result of two combined phenomena: the KL instability itself in the bulk of the domains and the interface rotation around the vertices. For relative small systems, the vertices are close to each other and those with opposite sense of rotation may annihilate each other leading to coarsening. In a noncoarsening situation, for a large enough system, the number of vertices is found to be constant for long times. The vertices interact each other and move through the system with a time scale much larger than that associated with interface motion. The two images of the upper row in figure 1 show two configurations in two dimensions at long times below and beyond the instability point. Apart from the typical size of the domains, it appears that there is no qualitative difference between them. The reason is that, just like in the zero dimensional case, the KL period T increases with time. Therefore, at long times, T is so large that we only see rotating interfaces around vertices as below the instability point.

2.2 *Anisotropic derivatives*

The second choice for the spatial derivatives is a simplified version of the Newell-Whitehead-Segel type terms retaining dominant contributions as made in [3]. Now $\mathcal{L}_i = \partial^2_{\hat{x}_i}$, $i = 1, 2, 3$, where $\hat{x}_1, \hat{x}_2, \hat{x}_3$ represent three directions with a relative orientation of $60°$. The anisotropic nature of these spatial terms changes the dynamics significantly with respect to the isotropic derivatives. In particular the propagation of the interfaces no longer follows the normal direction at each point. A consequence is that closed domains adopt an elliptic shape rather than a spherical one. Moreover the number of vertices is not constant for long times but they annihilate each other and also originate from the collision of interfaces (two interfaces rotating in the same direction may collide and generate new vertices).

The returning time of the KL instability now saturates to a finite value so directional derivatives seem to be more indicated to study the KL instability than isotropic ones. However it is more difficult to make analytical studies of interface dynamics.

The two images on the lower row of figure 1 represent two typical configurations below and beyond the KL instability with directional spatial derivatives terms. Below the KL instability we observe rotating interfaces that inhibit coarsening as in the case with isotropic derivatives. In the KL regime we observe in addition domains of one phase emerging in the bulk of other domain.

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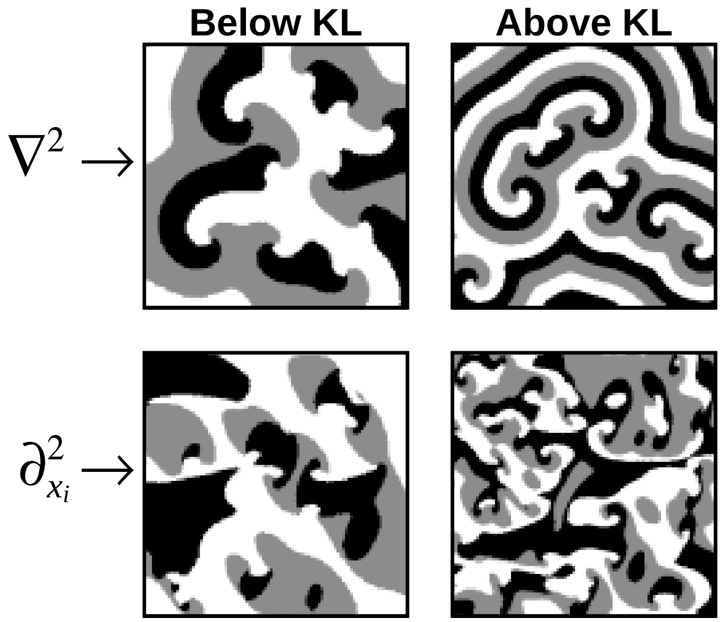

Fig. 1. Snapshots corresponding to a numerical simulation in two dimensions of the system (1) for both isotropic and directional derivatives below and beyond the KL instability. The white, grey and black regions are the regions occupied by the modes A_1, A_2 and A_3 respectively. To integrate the system of equations (1) we have used a finite differential scheme in a rectangular mesh with null boundary conditions.