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BOUNDARY EFFECTS IN THE COMPLEX GINZBURG–LANDAU EQUATION

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The effect of a finite geometry on the two-dimensional complex Ginzburg–Landau equation is addressed. Boundary effects induce the formation of novel states. For example, target-like solutions appear as robust solutions under Dirichlet boundary conditions. Synchronization of plane waves emitted by boundaries, entrainment by corner emission, and anchoring of defects by shock lines are also reported.

1. Introduction

The complex Ginzburg–Landau equation (CGL) is the generic model describing the slow phase and amplitude modulations of a spatially distributed assembly of coupled oscillators near its Hopf bifurcation [van Saarloos, 1994]. It contains much of the typical behavior observed in spatially-extended nonlinear systems whenever oscillations and waves are present. After proper scaling it can be written as:

$$\partial_t A = A - (1 + i\beta)|A|^2 A + (1 + i\alpha)\nabla^2 A \qquad (1)$$

where A is a complex field describing the modulations of the oscillator field, and α and β are two real control parameters. The first two terms in the r.h.s. of Eq. (1) describe the local dynamics of the oscillators: the first one is a linear instability mechanism leading to oscillations, and the second produces nonlinear amplitude saturation and frequency renormalization. The last term is the spatial coupling which accounts both for diffusion and dispersion of the oscillatory motion.

The power of our analytical tools to study nonlinear partial differential equations in general, and the CGL equation in particular, is very limited. Roughly speaking, only relatively simple solutions satisfying simple boundary conditions, usually in infinite domains, are amenable to analysis. Examples of these are plane and spiral waves. Nevertheless, sustained spatiotemporally disordered regimes have been found and thoroughly investigated numerically. Detailed phase diagrams displaying the transitions between different regimes have been charted for the cases of one and two spatial dimensions [Shraiman *et al.*, 1992; Chaté, 1994; Chaté & Manneville, 1996]. However, we want to stress that most of these numerical studies have been performed only under periodic boundary conditions, with the underlying idea that in the limit of very large systems the boundary conditions would not influence the overall dynamics. As a consequence of this belief, and despite its importance for the description of real systems, a systematic study of less trivial boundary conditions has been largely

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postponed. This is the case not only for the CGL equation but also for other nonlinear extended dynamical systems, and only few aspects of this problem have been collaterally addressed so far [Cross *et al.*, 1980, 1983; Sirovich *et al.*, 1990]. The purpose of this paper is to report on the initial steps of a program aiming towards such a systematic study. We will focus here on the behavior of the two-dimensional CGL equation on domains of different shapes and with different types of boundary conditions (Dirichlet or Neumann, for example).

For the purpose of comparison we first summarize the behavior observed numerically on two-dimensional rectangular domains under the commonly used periodic boundary conditions. Let us remind that in the so-called *Benjamin–Feir* (BF) *stable* region of the parameter space defined by $1 + \alpha\beta > 0$, there is always a plane wave solution of arbitrarily large wavelength that is linearly stable. In particular, for parameters in that region, and initializing the system with a homogenous condition (a wave of wavenumber $k = 0$) it will remain oscillating homogeneously. If we now vary the parameters slowly towards crossing the BF line, all the plane wave loss stability and small perturbations bring the system to a spatiotemporally disordered cellular state (the so-called *phase turbulence*). It is known that the behavior close to the BF line can be approximated by the Kuramoto–Sivashinsky equation.

Further change of the parameters to go deeper inside the BF unstable region eventually leads to the generation of defects, i.e. points where $A = 0$, and a kind of turbulent evolution characterized by the presence of these defects sets in. This is the so-called *defect* or *amplitude turbulence*. If we now trace back to the initial parameter values from the state dominated by defects, the system does not recover the initial uniformly oscillatory state. The spontaneous generation of defects ceases at parameter values still inside the BF unstable region. At these parameter values, the system usually reaches a state consisting of a spiral wave whose core is a defect. This spiral occupies most of the domain and it is limited by the shock-lines where the arms of the spiral meet themselves. Defects without spiral arms appear at the crossings of such shock-lines. In this regime, the amplitude of the field is time independent and its phase evolves quite regularly in time. In general, the configurations that share these two properties are called *frozen states*. These states persist while we vary the parameters all the way back to the BF stable region. Starting at values corresponding to a defect-dominated evolution, and suddenly setting the parameters to values in the stable BF regime, the stationary solution will be also a frozen state but in this case several domains, each one containing a spiral wave, may form. The size of these domains vary with the initial conditions, but the typical scale is controlled by the parameters. Shock lines where the arms of different spirals collide now proliferate and nonspiral defects are usually present at the crossings between them.

2. Boundary Effects

Let us consider first parameter values such that with periodic boundary conditions the long-time asymptotic states are *frozen* and look at how the behavior is modified by changing the boundary conditions. We apply null Dirichlet ($A = 0$), and Neumann (vanishing of the normal derivative of A) boundary conditions. For the former, we consider three different boundary shapes: square, circle, and stadium-shaped domains. Comparison between square and circle will allow us to investigate the influence of corners. On the other hand, our interest in the stadium arose from considerations of ray chaos, but it will be presented here as a combination of circle and square geometries.

In the Dirichlet case, the zero amplitude boundaries facilitates the formation of defects near the walls. Starting from random initial conditions, defects are actively created in the early stages of the evolution. After some time, however, all the points on the boundaries synchronize and oscillate in phase so that plane waves are emitted. Defect formation ceases, and the waves emitted by the walls push the remaining defects towards the central region of the domain. There the defects annihilate in pairs of opposite charge and as a result of this process a bound state is formed by the surviving set of equal-charge defects. The orientation of the waves emitted by the boundaries also changes during the evolution. The synchronized emission of the early stages proceeds, obviously, perpendicular to the boundary but later the wavevector tilts to some emission angle of approximately 45°. This angle depends on both the parameter values and the geometry of the boundaries. The fact that this angle is not exactly 45° is made evident by a mismatch of the waves coming from orthogonal walls. Finally the system reaches a frozen state of the type displayed in Fig. 1. The defects are confined to the center

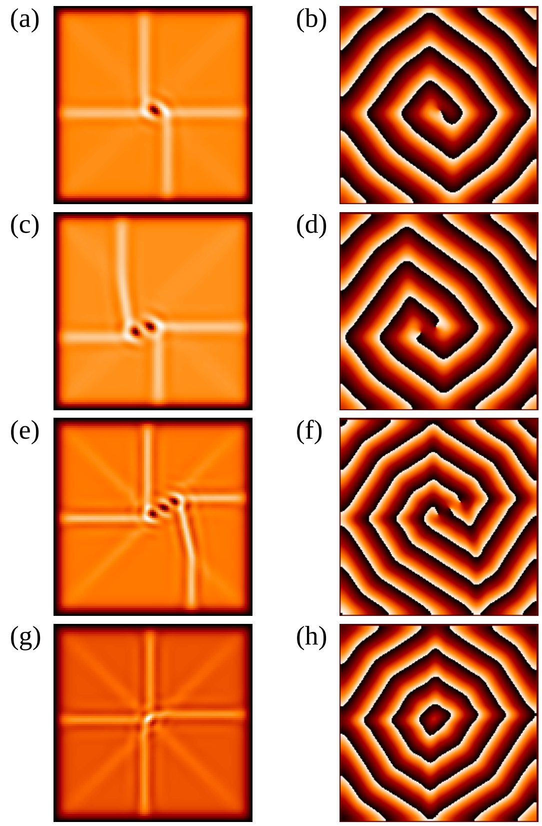

Fig. 1. Frozen structures under null Dirichlet boundary conditions in a square of size 100×100. Parameter values are $\alpha = 2$, $\beta = -0.2$ (a–d), and $\alpha = 2$, $\beta = -0.6$ (e–h). Snapshots of the modulus $|A|$ of the field are shown in the left column and snapshots of the phase in the right column. Color scale runs from black (minimum) to white (maximum).

of the domain forming a rigid static chain. The constant phase lines travel from the boundaries towards the center of the domain. Shock lines appear where waves from different sides of the contour collide. The strongest shocks are attached perpendicularly to the walls. If for a particular initial condition all defects annihilate the asymptotic state is a defect-free *target* solution. This kind of solutions is not seen in simulations with periodic boundary conditions.

It is known [Hagan, 1982] that the phase velocity of the usual spiral waves in infinite systems could point either inwards or outwards from the defect core depending on the parameter values. In our simulations in the square geometry with Dirichlet conditions, however, the direction of the phase velocity is always from the boundary to the core. We can understand this better by applying null Dirichlet conditions to only one of the walls. The synchronized emission that we observe is a straightforward generalization to two-dimensions of the one-dimensional Nozaki–Bekki emitting hole solution [Nozaki & Bekki, 1985]. We have verified [Equíluz *et al.*, 1998], for instance, that the direction of the emitted waves (inwards or outwards) can be changed with parameters as predicted by the analytic computations [Hagan, 1982]. However, when several of the walls are lines of zeros (the four sides of the square, for example) the direction of the phase velocity becomes determined by the angle between these lines. In other words, corners effectively entrain the whole system.

In a circular domain (Fig. 2), the frozen structures are either targets (no defects) or a single central defect. Groups of defects of the same charge can also form bound states, but instead of freezing they rotate together. This contrasts with the behavior of the square domains and is correlated with the absence of shock lines linking the boundaries to the center in the case of the circular domains. These links are probably responsible for providing rigidity to the stationary configuration in the square

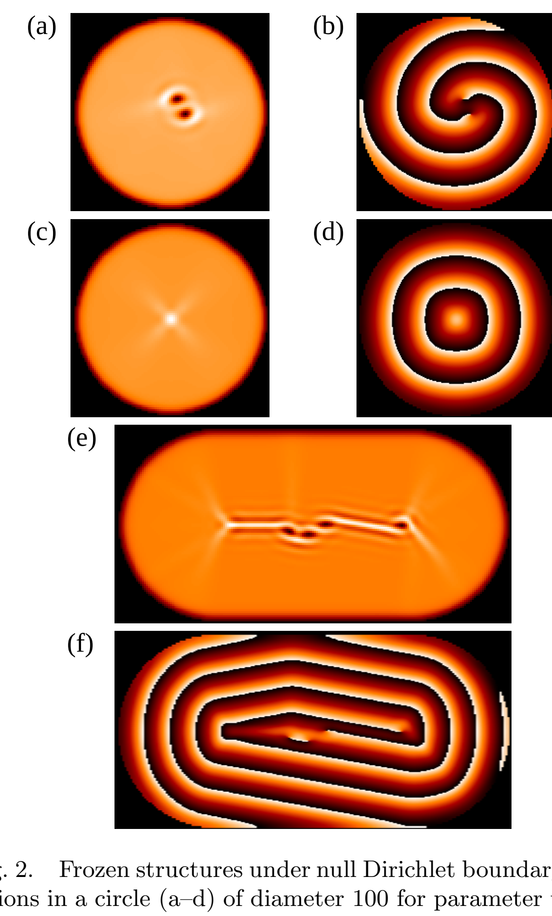

Fig. 2. Frozen structures under null Dirichlet boundary conditions in a circle (a–d) of diameter 100 for parameter values $\alpha = 2$, $\beta = -0.2$, and in a stadium (e–f) of size 200×100, for parameter values $\alpha = 2$, $\beta = -0.6$. Snapshots of the modulus $|A|$ are shown in the left column and (e) the phase is shown in the right column and (f). Color scale as in Fig. 1.

case. Tiny shock lines associated to small departures from circularity in the lines of constant phase can be observed also in the circle but these lines end in the bulk of the region before reaching the boundaries. On the other hand, the constant phase lines reach the boundaries nearly tangentially in contrast to what we observe in the square. In addition, we observe that for circular domains the phase velocity direction can be changed by controlling the parameters. This is probably a consequence of the absence of the corners that synchronize the emission from the boundaries in the square case.

The stadium shape (Fig. 2) mixes features of the two geometries previously studied: It has both straight and circular borders. In this case, the curves of constant phase arrange themselves to combine the two behaviors described above. On one hand the lines meet the straight portions of the border of the stadium with some characteristic angle, as it happens in square domains. However, these lines bend to become nearly tangent to the semicircles in the places where they meet with these portions of the boundaries. A typical frozen solution displays a shock line connecting the centers of the circular portions of the domain. This shock line usually contains defects. It is also possible to find defect-free target solutions as in the case of the circle, and the behavior of the phase velocity is also similar in the sense that its direction can be changed by modifying the parameters.

The behavior under Neumann boundary conditions is rather similar to the case of periodic boundary conditions. However, the Neumann conditions induce several subtle features to the dynamics. For example, shock lines are now forced to reach orthogonally the boundaries. In addition, defects can be irreversibly absorbed by the boundaries, a process that is obviously impossible with periodic boundary conditions. During the evolution a spiral defect behaves as if it were interacting with a mirror image of itself with opposite charge located outside the domain [Aranson *et al.*, 1993]. This reflects in few characteristic phenomena. On one hand, an isolated defect tends to move parallel to a nearby Neumann wall. On the other hand, mutual annihilation of a defect and its image is also possible accounting for the absorption of this defect by the boundary. Finally, when a defect closely approaches a corner, its evolution gains in complexity possibly as a result of the mutual interaction with two different images. Figure 3 displays a typical evolution of the pattern. Initially starting at random, a number of dynamically active spiral defects is created. These move around eventually annihilating mutually or sometimes being absorbed by the walls while the dynamics progressively slows down. Normally one large spiral wave grows until it fills the whole domain at the expense of the smaller ones that are pushed out of the boundaries.

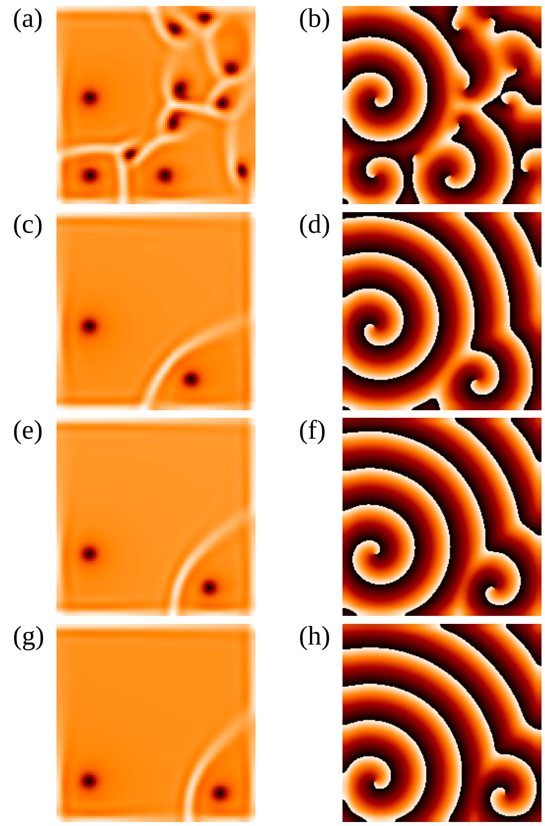

Fig. 3. Snapshots of the field $|A|$ (left column) and phase (right column) in color scale as in Fig. 1 at times $t = 2.5 \times 10^4$ (a–b), $t = 5.0 \times 10^4$ (c–d), $t = 7.5 \times 10^4$ (e–f), and $t = 10.0 \times 10^4$ (g–h) under Neumann boundary conditions in a square domain of size 100×100. Parameter values are $\alpha = 2$, $\beta = -0.2$.

Finally, we have studied the changes induced by the boundaries for parameter values such that active spatiotemporal chaos (i.e. nonfrozen states) is found for periodic boundary conditions. Far from the boundaries, spatiotemporally chaotic solutions behave similarly to those satisfying periodic boundary conditions. However, a boundary layer with different behavior shows up near the borders. In Fig. 4 we can see plane waves emitted by the boundaries and rapidly fading inside the domain where spatiotemporal chaos evolves. In small domains the boundaries could synchronize the whole system. However, as the system size increases, full synchronization ceases.

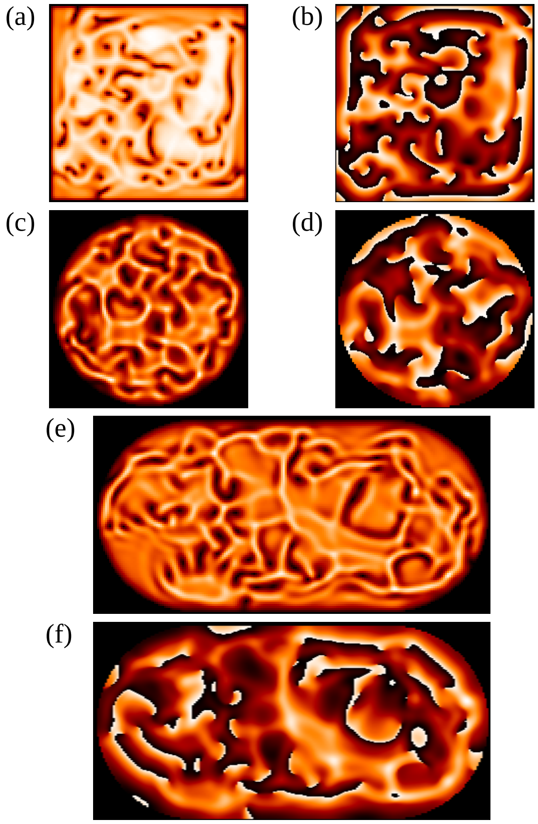

Fig. 4. Dynamical solutions under Dirichlet boundary conditions. Snapshots of the field $|A|$ are shown in the left column and (e) the phase is shown in the right column and (f). (a–b): square, parameter values $\alpha = 0$, $\beta = 1.8$; (c–d): circle, parameter values $\alpha = 2$, $\beta = -1.0$; (e–f): stadium, parameter values $\alpha = 2$, $\beta = -0.75$. System sizes and color scale as in Figs. 1 and 2.

For other parameter values, Dirichlet boundary conditions lead eventually to a dynamics characterized by the coexistence of regions dominated by defect turbulence and regions dominated by plane waves (constant $|A|$) whose shape and position normally evolve in time. We have found this behavior in all the domain shapes studied except for the circular case.

For these parameter values, Neumann boundary conditions do not produce a dynamics sensibly different from the one induced by periodic boundary conditions. The only noticeable difference is that in the Neumann case the shock lines are forced, as pointed out before, to orthogonally meet the boundaries.

3. Conclusions

In this paper, we have presented important features of the dynamics of the CGL equation which depend strongly on the type of boundary conditions imposed, as well as on the geometrical shape of the boundaries.

Dirichlet boundary conditions play a double rôle. On one hand, the walls naturally behave as sources (or sinks) of defects. On the other hand, a wall with null Dirichlet conditions shows a tendency to emit plane waves. The interplay between these two properties of the boundaries gives rise to interesting behavior.

In the case of frozen states, the character of the walls as wave emitters dominates. Some geometrical features of the boundaries have a strong influence on the details of the phase synchronization. Corners, for instance, tend to act as pacemakers. In circular domains, on the other hand, the emission is definitively dominated by the internal spirals. Correspondingly, the internal structure of the frozen states is also influenced by the shape of the boundaries. In a square, defects form a chain which is anchored to the boundaries by a set of shock lines; in a circle, on the contrary, the asymptotic state is usually a bound state disconnected from the boundaries.

Neumann boundary conditions seem to have a much weaker influence on the overall dynamical behavior of the CGL equation. However some differences are evident: One is the orientation of the shock lines, perpendicular to the boundaries. The other is that defects can be ejected through the boundaries, thus favoring states dominated by a single spiral in situations where under periodic boundary conditions a *glassy* state with several spiral domains would be formed.

Since the CGL equation appears naturally in a variety of contexts, we believe that the phenomena found in our preliminary explorations are likely to be relevant in many theoretical and experimental situations. Some of the phenomena reported here have intrinsic interest and deserve further analysis.

Acknowledgments

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