Noise rectification in quasigeostrophic forced turbulence

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We study the appearance of large-scale mean motion sustained by stochastic forcing on a rotating fluid (in the quasigeostrophic approximation) flowing over topography. As in other noise rectification phenomena, the effect requires nonlinearity and absence of detailed balance to occur. By application of an analytical coarse-graining procedure we identify the physical mechanism producing such effect: It is a forcing coming from the small scales that manifests in a change in the effective viscosity operator and in the effective noise statistical properties.

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Nonlinear interactions can organize random inputs of energy into coherent motion. This noise-rectification phenomenon has been discussed in several contexts ranging from biology to physics or engineering [1]. Three ingredients are needed to obtain this kind of noise-sustained directed motion: nonlinearity, random noise lacking the property of detailed balance, and some symmetry-breaking feature establishing a preferred direction of motion.

It has been recently shown numerically [2] that directed motion sustained by noise also appears in quasigeostrophic two-dimensional fluid flow over topography. The average flow at large scales approaches a state highly correlated with topography that disappears if noise or nonlinearity are switched-off. The presence of topography provides the symmetry-breaking ingredient needed to fix a preferred direction. The small scales of the flow follow a more irregular behavior.

In this Letter we analytically calculate a closed effective equation of motion for the large scales of the flow, by coarse-graining the small scales. From this effective equation, the forcing of the small scales on the large ones (sustained by noise and mediated by topography) appears clearly as the responsible for the directed currents. The effect is more noticeable for increasing nonlinearity, and appears as a renormalization of the viscosity operator in such a way that it favors relaxation towards a state correlated with topography, instead of towards rest. This forcing vanishes when noise satisfies detailed balance, as in the other kinds of noise-rectification phenomena.

The particular model considered here is the equation describing quasigeostrophic forced turbulence. A large amount of rotating fluid problems concerning planetary atmospheres and oceans involve situations in which vertical velocities are small and slaved to the horizontal motion [3,4]. Under these circumstances flow patterns can be described in terms of two horizontal coordinates, the vertical depth of the fluid becoming a dependent variable. Although the fluid displays many of the unique properties of two-dimensional turbulence, some of the aspects of three-dimensional dynamics are still essential, leading to a quasi two-dimensional dynamics. In particular, bottom topography appears explicitly in the equations. This kind of quasi-twodimensional dynamics is not only of relevance to the case of rotating neutral fluids but there is also a direct correspondence with drift-wave turbulence in plasma physics [5,6].

The streamfunction $\psi(\mathbf{x},t)$, with $\mathbf{x} \equiv (x,y)$, in the quasigeostrophic approximation is governed by the dynamics: [4]:

$$\frac{\partial \nabla^2 \psi}{\partial t} + \lambda \left[\psi, \nabla^2\psi + h\right] = \nu \nabla^4 \psi + F \ , \tag{1}$$

where ν is the viscosity parameter, $F(\mathbf{x},t)$ is any kind of relative-vorticity external forcing, and $h = f\Delta H/H_0$, with f the Coriolis parameter, H_0 the mean depth, and $\Delta H(\mathbf{x})$ the local deviation from the mean depth. λ is a bookkeeping parameter introduced to allow perturbative expansions in the interaction term. The physical case corresponds to $\lambda = 1$. The Poisson bracket or Jacobian is defined as

$$[A,B] = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial y} \ . \tag{2}$$

Equation (1) represents the time evolution of the relative vorticity subjected to forcing and dissipation. In the case of drift-wave turbulence for a plasma in a strong magnetic field applied in the direction perpendicular to $\mathbf{x}$, ψ is related to the electrostatic potential, and $h = \ln(\omega_c/n_0)$, where ω_c and n_0 are the cyclotron frequency and plasma density respectively. Eq. (1) is also the limiting case of the more general Charney-Hasegawa-Mima equation when the scales are small compared to the ion Larmor radius or the barotropic Rossby radius [6–8].

We now stablish how the dynamics of long-wavelength modes in (1), when F is a random forcing, is affected by the small scales. Stochastic forcing has been used in fluid dynamics problems to model stirring forces [9], wind forcing [10], short scale instabilities [3], thermal noise [11,12], or processes below the resolution of computer models [13], among others [14]. A useful choice of F, flexible enough to model a variety of processes, is to assume F to be Gaussian stochastic process with zero mean and correlations given by $\left\langle \hat{F}_{\mathbf{k}}(\omega)\hat{F}_{\mathbf{k}'}(\omega')\right\rangle = Dk^{-y}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')$. $\hat{F}_{\mathbf{k}}(\omega)$ denotes the Fourier transform of $F(\mathbf{x},t)$, $\mathbf{k} = (k_x, k_y)$, and $k = |\mathbf{k}|$. The process is then white in time but has power-law correlations in space. $y = 0$ corresponds to white-noise also in space and, in the absence of topography, it sustains the Kolmogorov spectrum [15,16]. In addition this value of y has been observed for wind forcing on the Pacific ocean [17]. Thermal noise corresponds to $y = -4$ [12,16]. In this case there is a fluctuation-dissipation relation between noise and the viscosity term, so that the fluctuations satisfy detailed balance.

To obtain the desired large-scale closed equation we have applied a coarse-graining procedure to the investigation of the dynamics. For our problem it is convenient to use the Fourier components of the streamfunction $\hat{\psi}_{\mathbf{k}\omega}$ or equivalently the relative vorticity $\zeta_{\mathbf{k}\omega} = -k^2\hat{\psi}_{\mathbf{k}\omega}$. This variable satisfies:

$$\zeta_{\mathbf{k}\omega} = G^0_{\mathbf{k}\omega} F_{\mathbf{k}\omega} + \lambda G^0_{\mathbf{k}\omega} \sum_{\mathbf{p},\mathbf{q},\Omega,\Omega'} A_{\mathbf{kpq}} \left(\zeta_{\mathbf{p}\Omega}\zeta_{\mathbf{q}\Omega'} + \zeta_{\mathbf{p}\Omega} h_{\mathbf{q}}\right) \quad , \qquad (3)$$

where the interaction coefficient is:

$$A_{\mathbf{kpq}} = (p_x q_y - p_y q_x) p^{-2} \delta_{\mathbf{k},\mathbf{p}+\mathbf{q}} \quad , \qquad (4)$$

the bare propagator is:

$$G^0_{\mathbf{k}\omega} = (-i\omega + \nu k^2)^{-1} \quad , \qquad (5)$$

and the sum is restricted by $\mathbf{k} = \mathbf{p} + \mathbf{q}$ and $\omega = \Omega + \Omega'$. $\mathbf{p} = (p_x, p_y)$, $p = |\mathbf{p}|$, and similar expressions hold for $\mathbf{q}$. $0 < k < k_0$, with k_0 an upper cut-off. Following the method in Ref. [18], one can eliminate the modes $\zeta^>_k$ with k in the shell $k_0 e^{-\delta} < k < k_0$ and substitute their expressions into the equations for the remaining low-wavenumber modes $\zeta^<$ with $0 < k < k_0 e^{-\delta}$. To second order in λ, the resulting equation of motion for the modes $\zeta^<$ is:

$$\frac{\partial \nabla^2 \psi^<}{\partial t} + \lambda \left[\psi^<, \nabla^2\psi^< + h^<\right] = \nu' \nabla^4(\psi^< - g h^<) + F' \; , \qquad (6)$$

where

$$\nu' = \nu \left(1 - \frac{\lambda^2 S_2 D (2+y)\delta}{32(2\pi)^2 \nu^3}\right), \qquad (7)$$

$$g(\lambda, D, \delta, \nu, y) = \frac{\lambda^2 D S_2 (y+4)\delta}{16(2\pi)^2\nu^3} \quad . \qquad (8)$$

$F'(\mathbf{x},t)$ is an effective noise which turns out to be also a Gaussian process with mean value and correlations given by:

$$< F'(\mathbf{x},t) > = -\frac{\lambda^2 D S_2 (4+y)\delta}{16(2\pi)^2\nu^2} \nabla^4 h^<, \qquad (9)$$

$$\left\langle \left(\hat{F}'_k(\omega) - \left\langle \hat{F}'_k(\omega)\right\rangle\right) \left(\hat{F}'_{k'}(\omega') - \left\langle \hat{F}'_{k'}(\omega')\right\rangle\right)\right\rangle = D k^{-y} \delta(k+k')\delta(\omega+\omega') \qquad (10)$$

S_2 is the length of the unit circle: 2π. Equations (6)-(10) are the main result in this Letter. They give the dynamics of long wavelength modes $\psi^<$. They are valid for small λ or, when $\lambda \approx 1$, for small width δ of the elimination band. The effects of the eliminated short wavelengths on these large scales are described in the new structure of the viscosity operator and the corrections to the noise term F'. The action of the dressed viscosity term $\nabla^4(\psi^< - gh^<)$ is no longer to drive large scale motion towards rest, but towards a motion state ($\approx gh^<$) characterized by the existence of flow following the isolevels of bottom perturbations $h^<$. This ground state would characterize the structure of the mean pattern. The energy in this ground state is determined by the function $g(\lambda, D, \delta, \nu, y)$ which measures the influence of the different terms of the dynamics (nonlinearity, noise, viscosity). Relation (8) shows that while nonlinearities [19] and noise increase the energy level of the ground state, high values of the viscosity parameter would imply a reduction of the strength of the ground state motion due to damping effect that viscosity exerts over small scales. The other mechanism that reinforces the existence of average directed motion comes from the fact that the dressed noise has got a mean component as a result of the small scale elimination.

A most interesting fact in (8) and (9) is the presence of the factor $y + 4$. It implies that the tendency to form directed currents reverse sign as y crosses the value -4, and that it vanishes if $y = -4$ which is the value for thermal noise satisfying detailed balance. The vanishing of the directed currents, obtained here to second order in λ, is in fact an exact result valid to all orders in the perturbation expansion. This can be seen from the exact solution of the Fokker-Planck equation associated with Eq.(1) for this value of y [20]. This reflects the fact that noise rectification can not occur when detailed balance holds.

As a consistency check we point out that if the term representing the ambient vorticity h is zero, classical results of two-dimensional forced turbulence are recovered [15]. To show that the result implied by the perturbative expressions (6)-(10) is really present for arbitrary λ we check the increasing tendency towards average flow following the topography for increasing λ: Numerical simulations of (1) have been conducted in a parameter regime of geophysical interest: we take $f = 10^{-4}s^{-1}$ as appropriate for the Coriolis effect at mean latitudes on Earth and $\nu = 200 m^2 s^{-1}$ for the viscosity, a value usual for the eddy viscosity in ocean models. We use the numerical scheme developed in [21] on a grid of 128×128 points with a proper inclusion of the stochastic term [2]. The distance between grid points corresponds to 10 km, so that the total system size is $L = 1280$ km. The amplitude of the forcing, $D = 1 \times 10^{-9}\ m^2 s^{-3}$ has been chosen in order to obtain final velocities of several centimeters per second. The topographic field (shown in Fig. 1) is randomly generated from a specific isotropic power spectrum (Fig. 2) with random phases. The model was run for 6×10^5 time steps (corresponding to 247 years) once a statistically stationary state was reached, and some of the results for the mean streamfunction are displayed in Figs. 3 and 4. Currents with a well defined average sense appear. Consistently with our analytical results, the contour levels of the mean streamfunction follow the topographic contours more closely the higher the value of λ is. This is more quantitatively shown in Fig. 5 where the linear correlation coefficient ρ between the mean field $< \psi >$ and the underlying topography is plotted as a function of λ. A spectral analysis of the different resulting fields show that the large scales are better adjusted to topography, as well as the very small scales where no significant motion is present (Fig. 2). Discrepancies are clear for the small but excited scales. This can be understood considering that the effect of viscosity on these small scales is still to drive the system towards rest.

For negative values of y noise acts more strongly on the small scales, where viscosity damping is more important, so that a larger noise intensity is needed to obtain significant large-scale directed currents. More important is the reversal in the sense of the currents when $y < -4$. This can be characterized by the change in sign of ρ. For example, for $y = -6$ and $D = 2 \times 10^{-7}\ m^8 s^{-3}$, $\rho = -0.6$. More detailed results will be presented elsewhere [20].

Concluding, the outcome of this work can be formulated as follows: quasigeostrophic flows develop mean patterns in the presence of noisy perturbations. As relations (6)-(10) show, the origin of these patterns is related with nonlinearity and lack of detailed balance. Nonlinear terms couple the dynamics of small scales with the large ones and provide a mechanism to transfer energy from the fluctuating component of the spectrum to the mean one. This mean spectral component, that is inexistent in purely two-dimensional turbulence [22], is controlled by the shape of the bottom boundary and characterizes the structure of the pattern. The existence of these noise-sustained structures has a wide range of implications in the above mentioned fields of fluid and plasma physics. First because it highlights the important and organizing role that noise can play in these systems. Secondly, it establishes the need to modify not only the value of the parameters (as usually done in eddy-viscosity approaches) when performing large eddy simulations with insufficient small-scale resolution, but also the structure of the equations in a way determined by topography. This last statement has been previously suggested from a heuristic point of view in the context of large-scale ocean models [23,24]. Our results represent a step forward towards the justification of such approaches.

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† URL: `http://www.imedea.uib.es/`

[1] M.O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993); J. Maddox, Nature **369**, 181 (1994); C. R. Doering, W. Horsthemke, J. Riodan, J. Phys. Rev. Lett. **72**, 2984 (1994); J. Maddox, Nature **368**, 287 (1994); J.K. Douglass, Lon Wilkens, Eleni Pantazelou, Frank Moss, Nature **365**, 337 (1993); S. M. Bezrukov, Igor Vodyanoy, Nature **378**, 362 (1995); J. Rousselet, L. Salome, A. Ajdari, J. Prost, Nature **370**, 446 (1994).

[2] A. Álvarez, E. Hernández-García, J. Tintoré, Physica **A 247**, 312 (1997). chao-dyn/9701009 .

[3] G. P. Williams, J. Atmos. Sci. **35**, 1399 (1978).

[4] J. Pedlosky, *Geophysical fluid dynamics*, Springer-Verlag (New York, 1987).

[5] A. Hasegawa, K. Mima, Phys. Fluids **21**, 87 (1978).

[6] A. Hasegawa, C. G. Maclennan, Phys. Fluids **22**, 11 (1979).

[7] A. Hasegawa, Advances in Physics **34**, 1 (1985).

[8] W. Horton, Phys. Rep. **192**, 1 (1990).

[9] A.C. Martí, J.M. Sancho, F. Sagués, and A. Careta, Phys. Fluids **9**, 1078 (1997). chao-dyn/9703015 .

[10] D.S. Battisti, J. Atmos. Sci. **45**, 2889 (1988).

[11] M. Treiber, Phys. Rev. **E 53**, 577 (1996).

[12] L.D. Landau and M. Lifshitz, *Fluid Dynamics, 2nd Edition. Course of Theoretical Physics, vol. 6* (Pergamon, New York, 1987).

[13] P.J. Mason, Q.J.R. Meteorol. Soc. **120**, 1 (1994).

[14] W.D. McComb, Rep. Prog. Phys. **58**, 1117 (1995).

[15] V. Yakhot, S. A. Orszag, J. Sci. Comp.**1**, 1(1986).

[16] It is common to work with the equation for the velocity field instead than with the vorticity one (1). The exponent y' for the correlations of the noise term added to the velocity equation is related to our y by $y = y' - 2$.

[17] M.H. Freilich and D.B. Chelton, J. Phys. Oceanogr. **16**, 741 (1985).

[18] D. Foster, D. Nelson, M. Stephen, Phys. Rev. A **16**, 732 (1977).

[19] It should be noted that the parameter λ in (1) measures not only the size of the nonlinear term, but also of the linear interaction between streamfunction and topography. However it is easy to show, by using energy arguments, that the average streamfunction is zero if the nonlinear term is absent from (1), so that the presence of λ in (8) and (9) is a consequence of nonlinearity, not of linear scattering of the streamfunction by the topography.

[20] A. Álvarez, E. Hernández-García, J. Tintoré, in preparation (1998).

[21] P. F. Cummins, J. Mar. Res. **50**, 545 (1992).

[22] P.D. Thompson, J. Fluid Mech. **55**, 711 (1972).

[23] G. Holloway, J. Phys. Oceanogr. **22**, 1033 (1992).

[24] A. Álvarez, J. Tintoré, G. Holloway, M. Eby, J.M. Beckers, J. Geophys. Res., **99**, 16053 (1994).

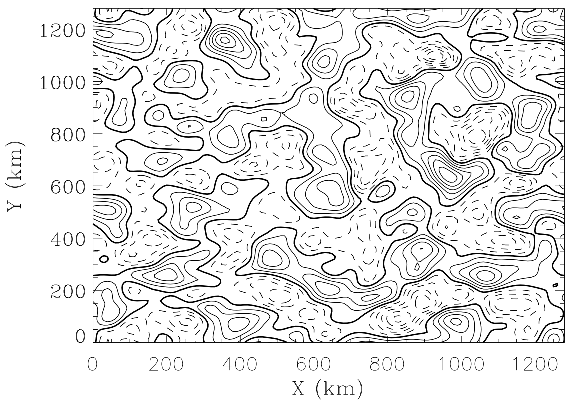

FIG. 1. Depth contours of a randomly generated bottom topography. Maximum depth is $381.8m$ and minimum depth $-381.8m$ over an average depth of $5000m$. Levels are plotted every $63.6m$. Continuous contours are for positive deviations with respect to the mean, whereas dashed contours are for negative ones.

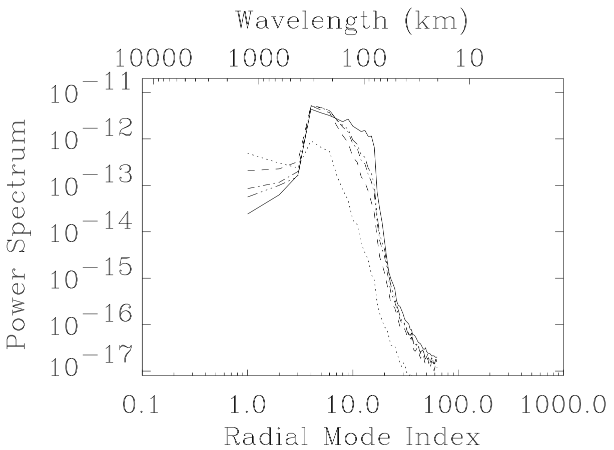

FIG. 2. Comparison, as a function of the radial wavenumber index $Lk^2/2\pi$, of the power spectra of the bottom topography (solid line) and the power spectra of the mean streamfunction obtained for $\lambda = 0.1$ (dotted line), 0.3 (dashed line), 0.6 (dash-dotted line) and 1 (dash-dot-dot-dot line). $y = 0$ and $D = 10^{-9}m^2s^{-3}$. In order to carry out the comparison, the fields have been normalized to have the same maximum value.

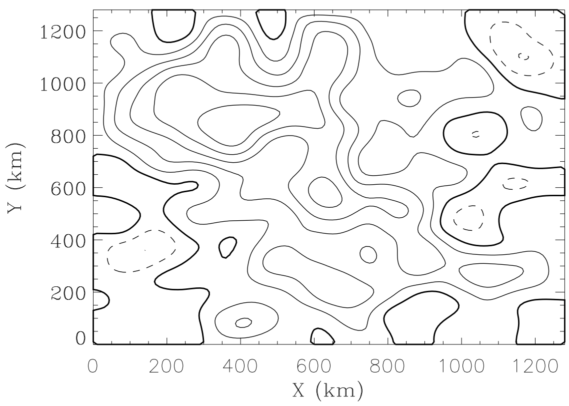

FIG. 3. Mean streamfunction computed by time averaging when a statistically stationary state has been achieved. Continuous contours denote positive values of the streamfunction, whereas dashed contours denote negative ones; $\lambda = 0.1$, $y = 0$, and $D = 10^{-9}m^2s^{-3}$. Maximum and minimum values are 1637.7 and -1637.7 m^2/s,and levels are plotted every 272.95 m^2/s.

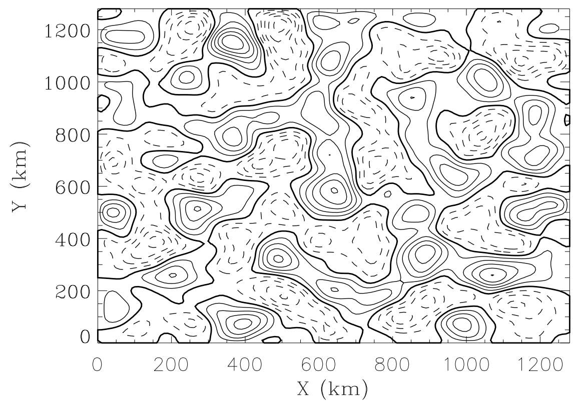

FIG. 4. The same as Fig. 3 but for $\lambda = 1$, maximum and minimum values are 991.864 and -991.864 m^2/s, and levels are plotted every 165.31 m^2/s.

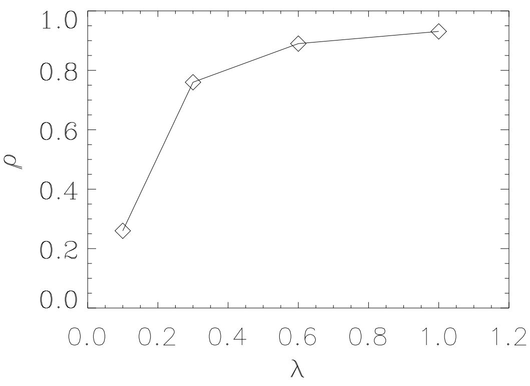

FIG. 5. Linear correlation coefficient ρ between the mean streamfunction and topographic fields as a function of the interaction parameter λ. $y = 0$ and $D = 10^{-9}m^2s^{-3}$