On Modular Smoothing and Scaling Functions for Mode Locking

Nikola Buric, Julyan H. E. Cartwright,
Ian C. Percival & Oreste Piro *
School of Mathematical Sciences
Queen Mary and Westfield College
University of London
Mile End Road
London E1 4NS

August 4, 1991

Abstract

The mode-locking structure of the sine circle map is investigated using the method of modular smoothing. It is shown that the method leads to a scaling function generated by the Gauss transformation. We speculate about a recursive procedure to obtain increasingly smooth descriptions of the fractal structure based on this method.

THE quasiperiodic transition to chaos in families of maps on the circle has been extensively studied in the past [1, 2, 3, 4, 5, 6]. It is known that the transition occurs at a parameter value at which the map ceases to be a diffeomorphism. Typically, this happens with the appearance of a cubic inflection point, and at greater values of the parameter, the map becomes non-invertible.

A typical example of this class of maps is the sine circle map

$$f(\theta) = \theta + \Omega - \frac{k}{2\pi} \sin 2\pi\theta \bmod 1. \tag{1}$$

QMW preprint DYN #91-7, Phys. Lett. A, **163**, 63–67, 1992

*A CONICET (Consejo Nacional de Investigaciones Cientificas y Técnicas de Argentina) Fellow. Present Address: Institut Nonlinéaire de Nice, Université de Nice—Sophia Antipolis, Parc Valrose, 06034 Nice Cédex, France

In this case the transition occurs at the parameter value $k = 1$, when the map is known as a critical circle map.

One useful concept for characterizing the dynamics of circle maps is the rotation number

$$\rho(\Omega) = \lim_{n\to\infty} \frac{f^n(\theta) - \theta}{n}. \qquad (2)$$

It has been proven that this limit exists irrespective of the initial conditions as long as the map is monotonic. For a given value of the nonlinearity (in this case k) the rotation number is non-decreasing with the other parameter (here Ω). If the nonlinearity is greater than zero, for each rational p/q, there is an open interval in Ω of size $\Delta(p/q)$ where $\rho(\Omega) = p/q$. For subcritical maps the measure of these intervals is not complete, leaving a finite measure of Ω for which ρ is irrational. At criticality, the Ω measure for ρ irrational becomes zero. This complicated fractal structure, shown in Fig. 1, is known as the devil's staircase.

Another fractal of similar complexity arises in the study of the Hamiltonian transition to chaos [7, 8, 9]. The phase space of integrable systems is foliated into invariant tori on which quasiperiodic motion with frequency ratio ν occurs. Under the action of a nonlinear perturbation of size k, tori of rational ν disappear for arbitrarily small values of k. However the KAM theorem ensures that for almost all irrational ν, invariant tori persist up to a finite value of k. The Hamiltonian transition to chaos is characterized by the critical function $K(\nu)$ which gives the smallest value for which the torus of frequency ν is destroyed. As an example let us look at the critical function for the semistandard map (see Fig. 2)

$$\begin{aligned} p_{n+1} &= p_n + ike^{2\pi i x_n} \\ x_{n+1} &= x_n + p_{n+1}. \end{aligned} \qquad (3)$$

Although the resemblance of the two objects (Figs. 1 and 2) is not at first sight obvious, the features of one have some echoes in complementary features of the other. In order to show this we can consider a further reorganization of the data in the devil's staircase.

In Fig. 3 we show the widths of intervals with rotation number p/q as a function of p/q. We term this the mode-locking function, $\Delta(p/q)$. Notice that for each rational p/q there are sequences of intervals of decreasing size, corresponding to rationals of increasing denominator, converging to p/q.

We can see somewhat similar behaviour in the critical function; the function value at sequences of irrationals converging to every rational

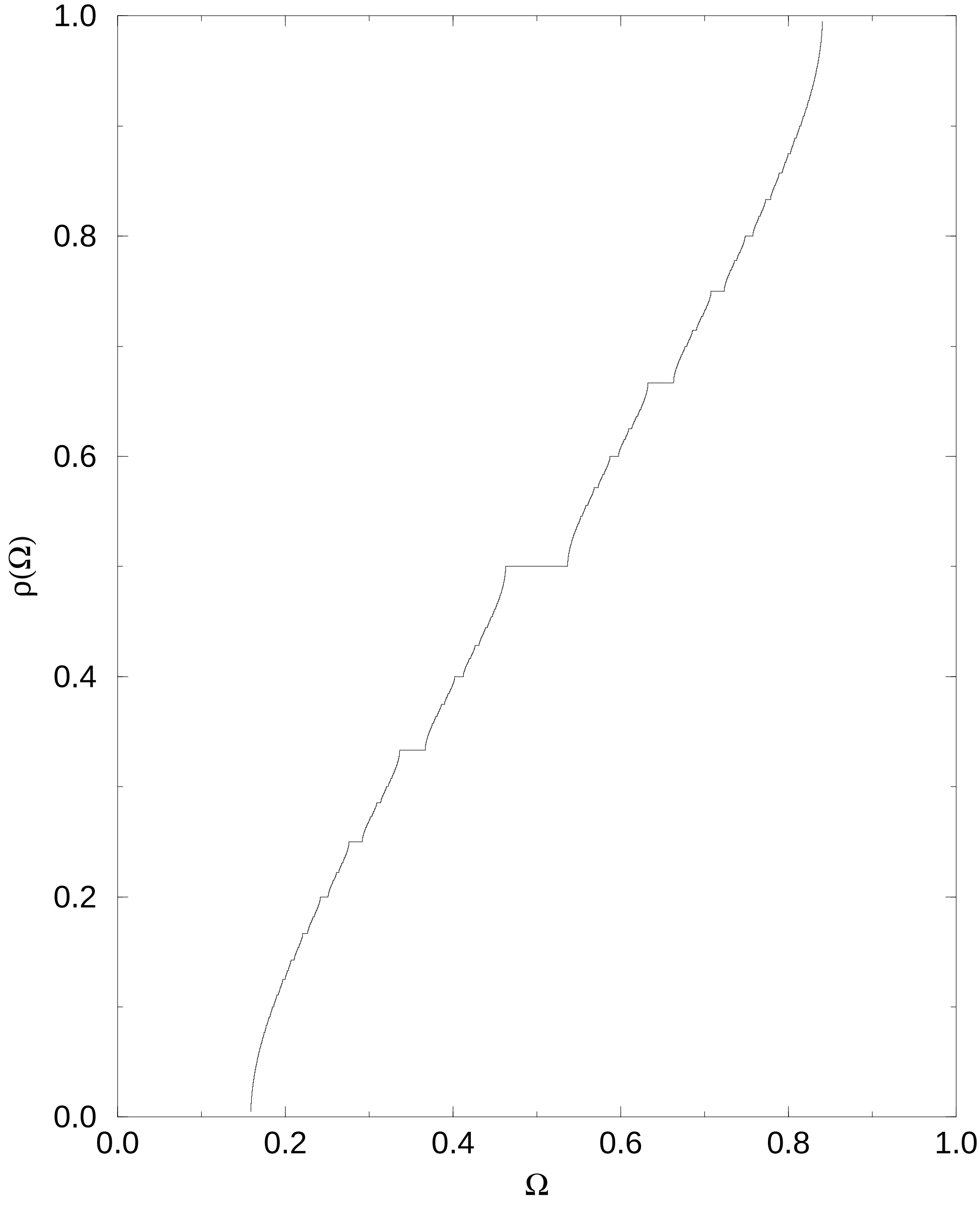

Figure 1: The devil's staircase $\rho(\Omega)$ for the critical sine circle map.

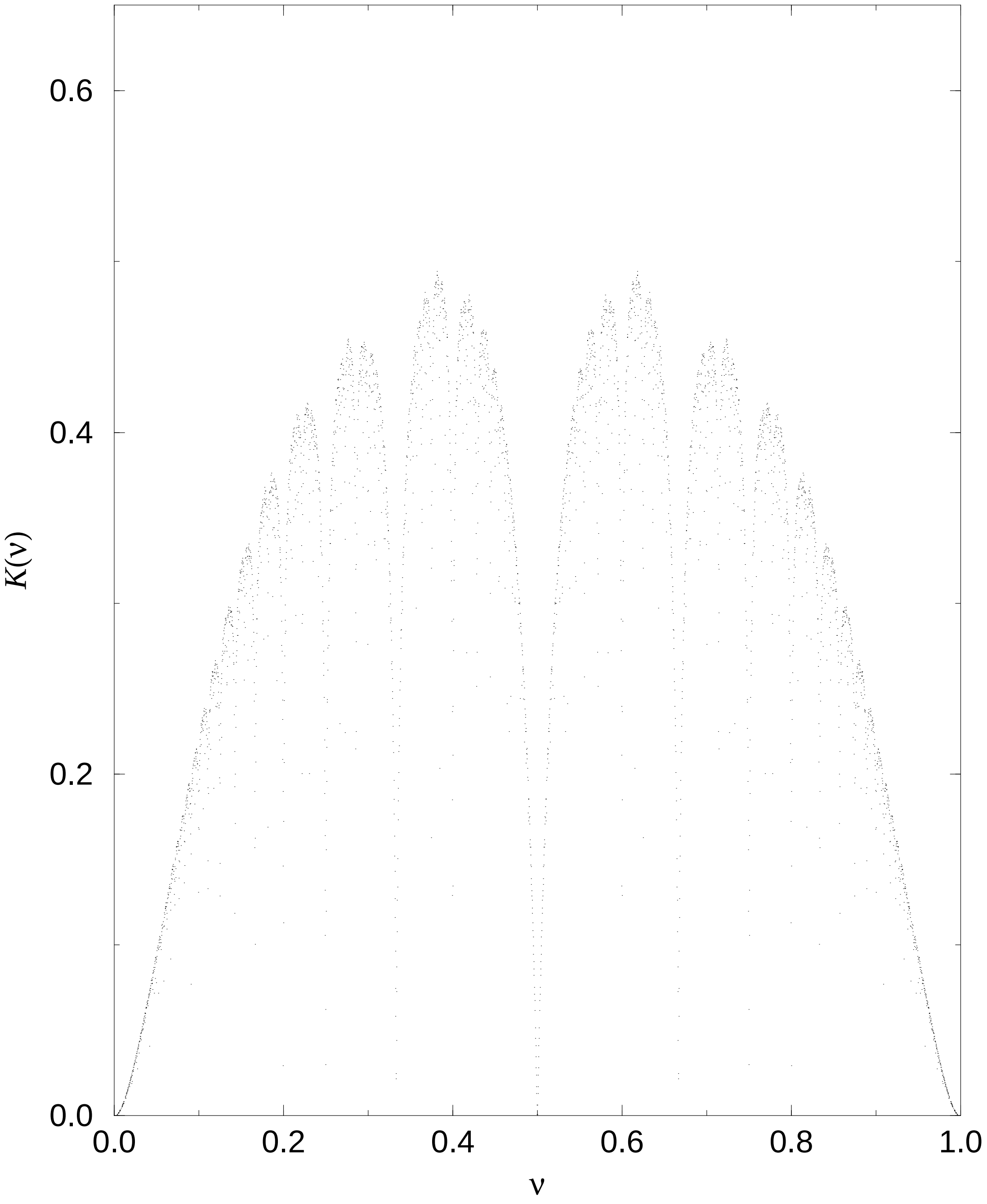

Figure 2: The critical function $K(\nu)$ for the semistandard map.

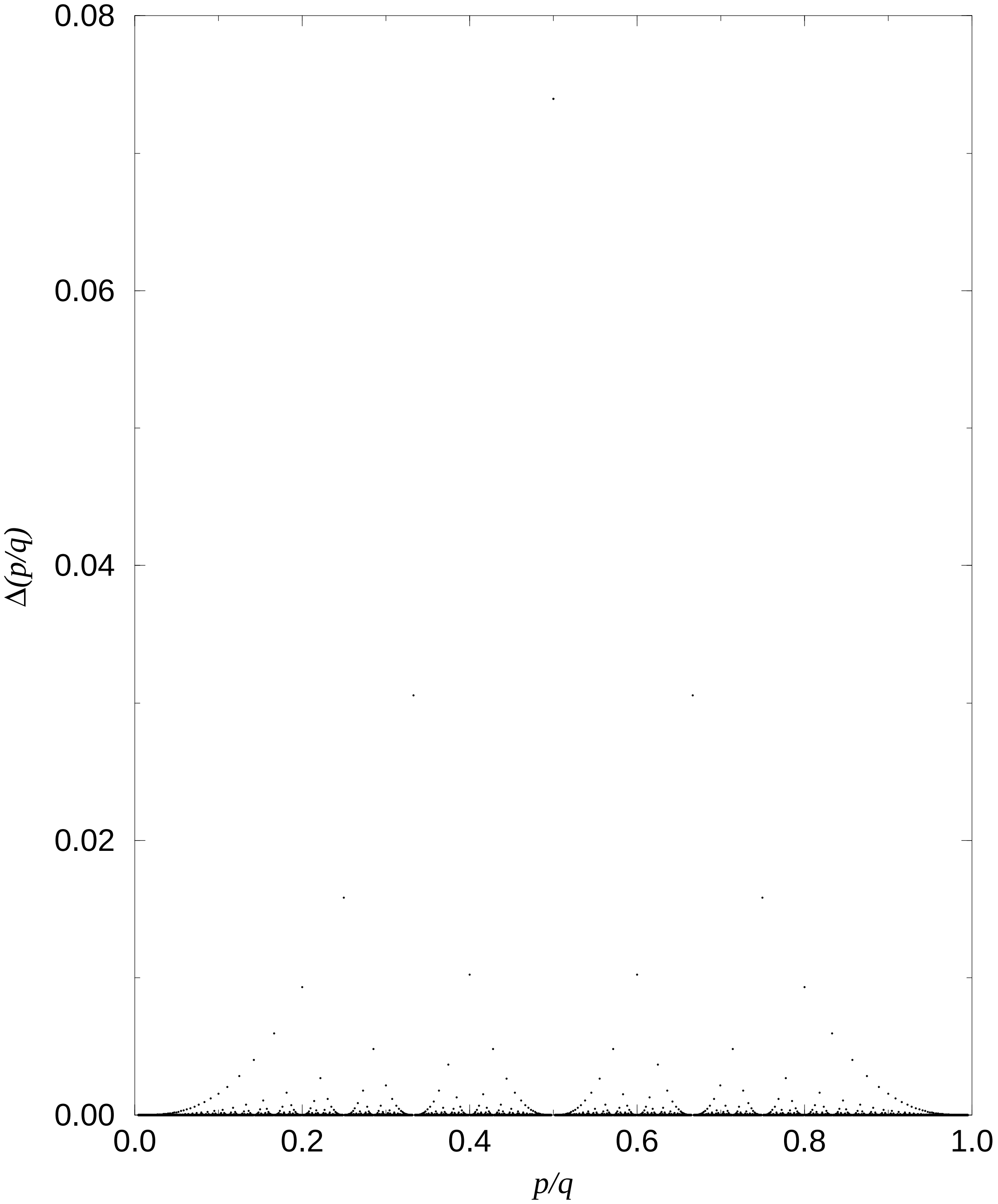

Figure 3: The mode-locking function $\Delta(p/q)$ for the critical sine circle map.

approaches zero. However while the critical function is zero at every rational (and is continuous at every rational), the mode-locking function jumps away from zero at every rational (and is thus discontinuous).

In order to stress the similarities outlined above we introduce the scaled mode-locking function shown in Fig. 4. Here we have multiplied $\Delta(p/q)$ by a scaling factor q^{α}, where $\alpha = -\log\delta/\log\gamma = 2.1644\ldots$. $\gamma = (\sqrt{5}-1)/2 = 0.6180\ldots$ is the golden mean, and $\delta = 2.8336\ldots$ is Shenker's δ [2], the rate of change of $\Delta(p_n/q_n)$ as p_n/q_n tends to the golden mean or another noble number. It turns out that this scaling produces a tangency of envelopes for the main sequences of intervals.

The mode-locking function is defined entirely on the rationals so the singularity structure must be viewed as the variety of asymptotic behaviour of sequences of rationals converging to rationals of smaller denominator. This is to be contrasted with the critical function where the singularities are viewed as the asymptotic behaviour of sequences of irrationals converging to rationals. Critical functions are continuous at the rationals (tori with irrational rotation number arbitrarily close to a rational break up at arbitrarily small values of K, and rational tori break up at $K = 0$) but mode-locking functions are essentially discontinuous. That is to say

$$\Delta\left(\frac{p_n}{q_n}\right) \xrightarrow[\substack{p_n, q_n \to \infty \\ \frac{p_n}{q_n} \to \frac{p}{q}}]{} 0 \tag{4}$$

while $\Delta(p/q) \neq 0$. However, $q^{\alpha}\Delta(p/q)$ converges to a finite value as p/q tends to a noble irrational. Therefore a complete analogy with the critical function can be had by considering the function $\Delta(\nu) = \lim_{p_n/q_n \to \nu} q_n^{\alpha}\Delta(p_n/q_n)$ on the nobles. We shall not take this further in the present discussion since our aim here is to show the links between modular smoothing and scaling functions.

Understanding of these highly singular objects can be achieved by reordering the information they contain into the form of a smooth function. That is, we should like to be able to reconstruct the entire singular structure from a smooth function. Procedures aiming to achieve this goal have been introduced independently in the two cases: modular smoothing for the critical function, and scaling functions for the mode-locking function. We wish to show here how these two procedures are connected.

Modular smoothing consists of cancelling singularities at points related by a modular transformation. In the case of critical functions for systems like Eq.(3), the singularities at the rationals p/q are found to

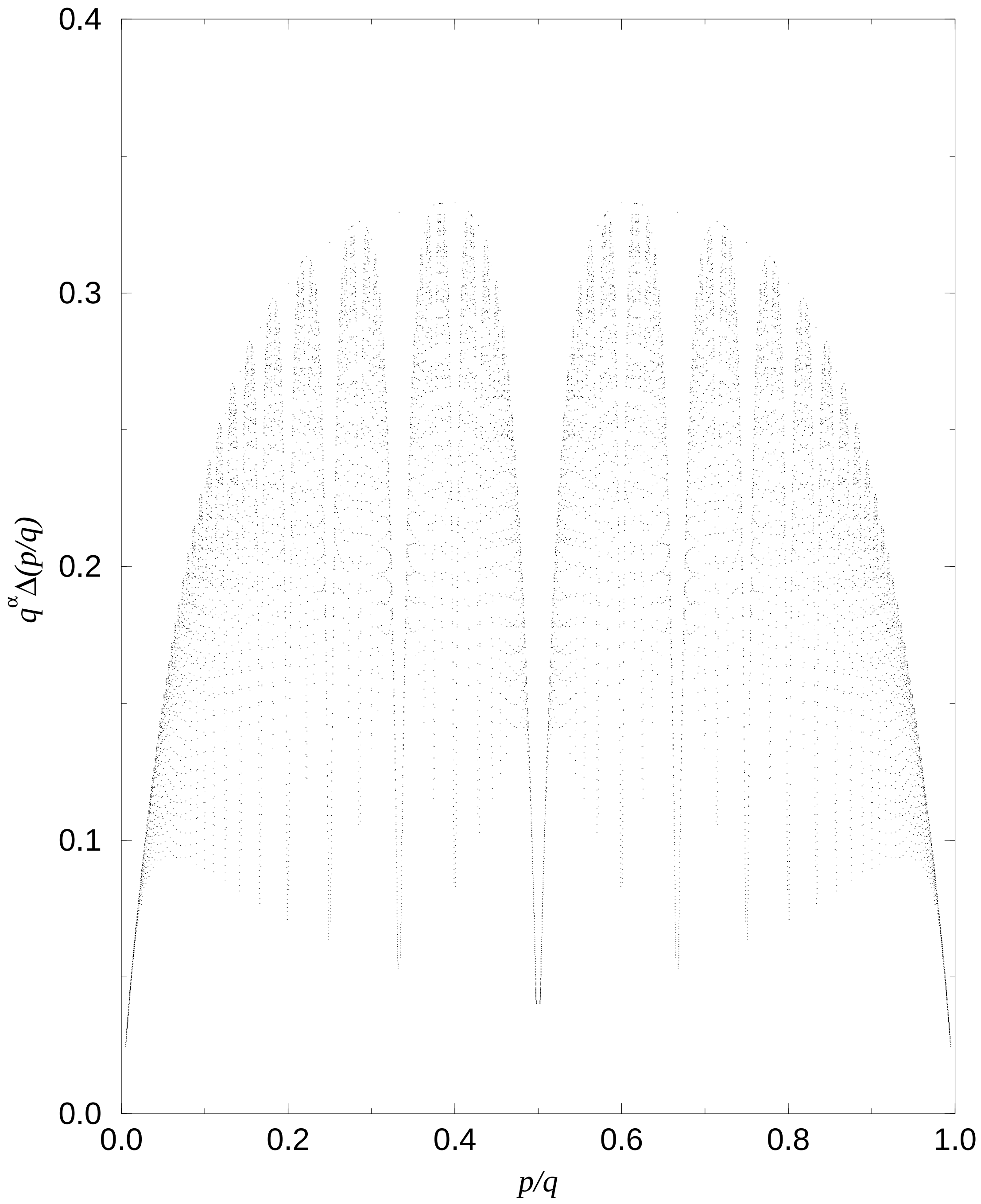

Figure 4: The scaled mode-locking function $q^{\alpha}\Delta(p/q)$ with $\alpha = 2.1644$ for the critical sine circle map.

have the form

$$K(\nu) \sim |\nu - p/q|^{\alpha(\nu)/q} \times \text{other factors}, \qquad (\nu \sim p/q). \tag{5}$$

This suggests that the function $L_0^{CF}(\nu) \equiv -\ln K(\nu)$ can be written as

$$\begin{aligned} L_0^{CF}(\nu) &\sim q^{-1}\alpha(\nu)\ln|\nu - p/q| + \text{correction terms}, \qquad (\nu \sim p/q) \quad (6) \\ &\equiv q^{-1}Q|\nu - p/q| + \text{correction terms}, \qquad (\nu \sim p/q), \quad (7) \end{aligned}$$

where Q has singularities weaker than poles. We further assume

$$L_0^{CF}(\nu^{-1}) \sim p^{-1}Q|\nu - p/q| + \text{correction terms}, \qquad (\nu \sim p/q). \tag{8}$$

Therefore a weakening of singularities can be achieved by taking a linear combination of both functions with appropriate weight factors. Specifically the function

$$L_1^{CF}(\nu) \equiv L_0^{CF}(\nu) - \nu^{-1}L_0^{CF}(\nu^{-1}), \tag{9}$$

near each rational, can be shown to be of the form

$$L_1^{CF}(\nu) \sim p^{-1}(\nu - p/q)\,Q|\nu - p/q| + \text{correction terms}, \qquad (\nu \sim p/q), \tag{10}$$

and the logarithmic singularities will have been smoothed out by the factor $(\nu - p/q)$ (Fig. 5).

The case of the mode-locking function is different because the singularities around any rational are all of the same type. For example, it is well known that for harmonic sequences $\nu_n = (p' + np)/(q' + nq)$ converging to p/q,

$$\Delta(\nu_n) \sim |\nu_n - p/q|^{-3} \qquad \forall p/q. \tag{11}$$

This shows us how to proceed if we wish to apply modular smoothing to the mode-locking function. Let us consider the function $L_0^{ML}(\nu) \equiv -\ln\Delta(\nu)$. Then

$$L_0^{ML}(\nu) \sim 3\ln|\nu - p/q| + \text{correction terms}, \qquad (\nu \sim p/q). \tag{12}$$

The smoothing procedure is obvious in this case and no weight factor is necessary. Hence the function

$$L_1^{ML}(\nu) \equiv L_0^{ML}(\nu) - L_0^{ML}(\nu^{-1}) \tag{13}$$

does not contain any logarithmic singularities. From the invariance of $\Delta(\nu)$ under unit translations, it follows that

$$L_1^{ML}(\nu) = -\ln\frac{\Delta(\nu)}{\Delta\{\nu^{-1}\}}, \tag{14}$$

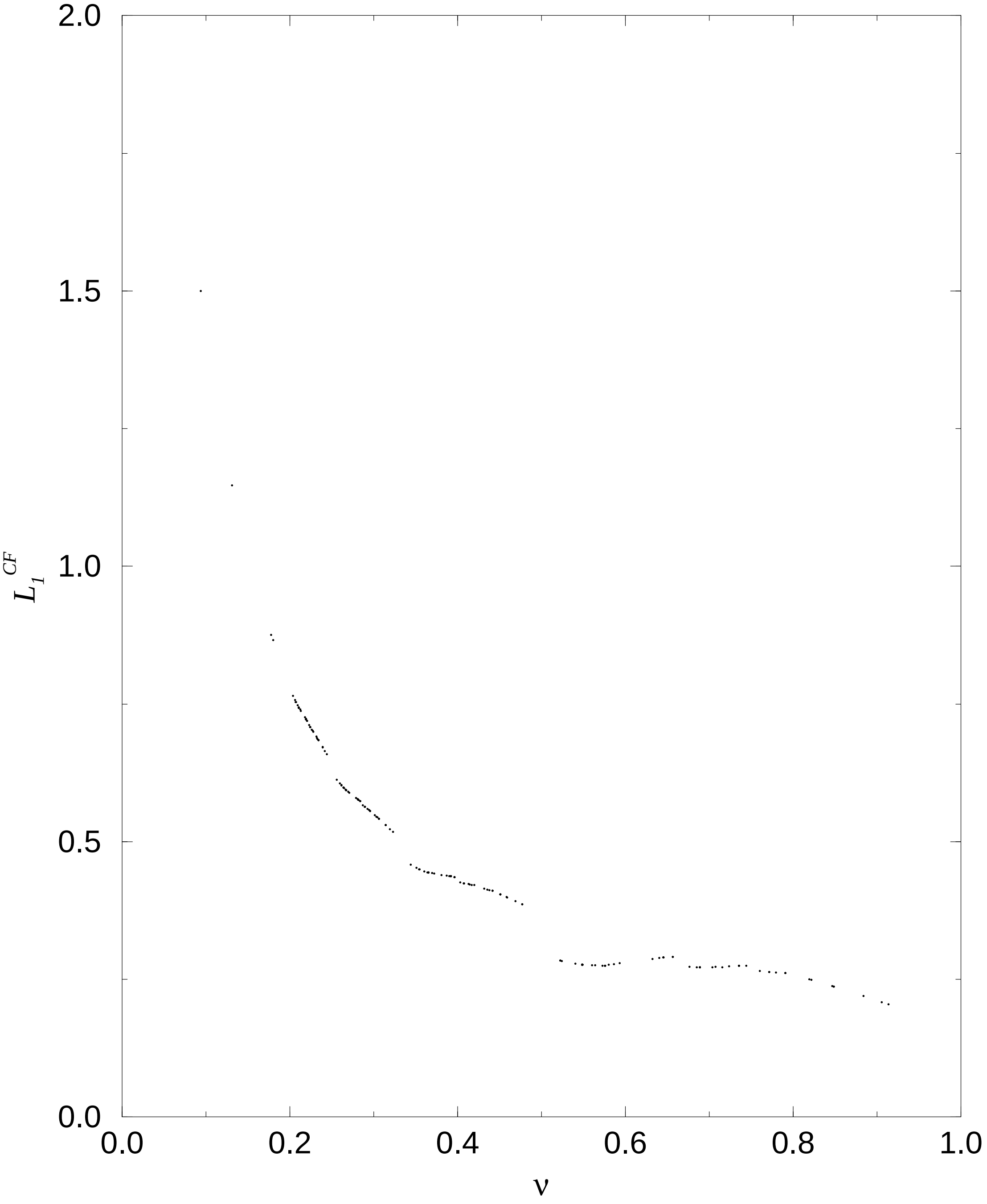

Figure 5: L_1 for the semistandard map.

where $\{\nu\}$ indicates the fractional part of ν. The transformation $G : \nu \to \{\nu^{-1}\}$ is known as the Gauss map.

The functions obtained by taking the ratio of pairs of locking intervals chosen according to different strategies are known as mode-locking scaling functions. Various schemes have been proposed and investigated by several authors [10, 11, 12]. The reported schemes have all been based on the hierarchical organization of the locking intervals provided by the Farey tree [3, 10, 11]. The different strategies mentioned above amount to different ways of choosing pairs of rationals between the last generation and its ancestors in the tree. Since the Gauss transformation maps a generation onto its full ancestry, the ratio in the right hand side of Eq.(14) is one of the possible scaling functions. In other words, the modular-smoothed mode-locking function is just the logarithm of the Gauss-labelled scaling function.

Fig. 6 shows the modular-smoothed mode-locking function, L_1^{ML}. Notice that although L_1^{ML} looks much more regular than L_0^{ML}, it is clearly not smooth. The origin of the discontinuities in L_1^{ML} must be the multiplying factors of the logarithmic singularities in the approximate expression, because the logarithmic singularities themselves have all been removed. The situation with the critical function is different. Here the origin of roughness in L_1^{CF} is taken to be due to singularities not removed but merely weakened in the first step of the smoothing. This weakening process can be carried further in a similar way to the first step. The function

$$L_2^{CF}(\nu) \equiv (\nu+1)L_1^{CF}(\nu+1) - \nu L_1^{CF}(\nu) \tag{15}$$

is then obtained which has only singularities of the type $(\Delta(\nu))^2 \log \Delta(\nu)$ near the rationals, and is thus once differentiable. For the mode-locking function, a further step cannot be expected to improve on L_1^{ML}. We show this in Figs. 7 and 8, which have L_2^{CF} for the critical function and the corresponding L_2^{ML} for the mode-locking function.

The Gauss-labelled scaling function $\sigma^G(\nu) = e^{-L_1^{ML}(\nu)}$ can be compared in smoothness to the dual-labelled scaling function $\sigma^D(\nu) = \Delta(D\nu)/\Delta(LD\nu)$ of Cvitanović et al. [10]. Here $L(p/q) = p'/q'$ is the left parent in the Farey tree with p' and q' given by $pq' = p'q+1$ and $0 < q' < q$. $D(p/q) = q'/q$, where q' is the denominator of the left parent, is the dual. The functions are shown in Figs. 9 and 10 respectively. It can be shown that the overall shape of both functions is approximately $\sigma(\nu) = \nu^2$.

We have shown the connection that exists between modular smoothing and scaling functions when both techniques are applied to mode locking. Both are only partially successful in coding the fractal structure into a smooth function. The standard implementation of modular

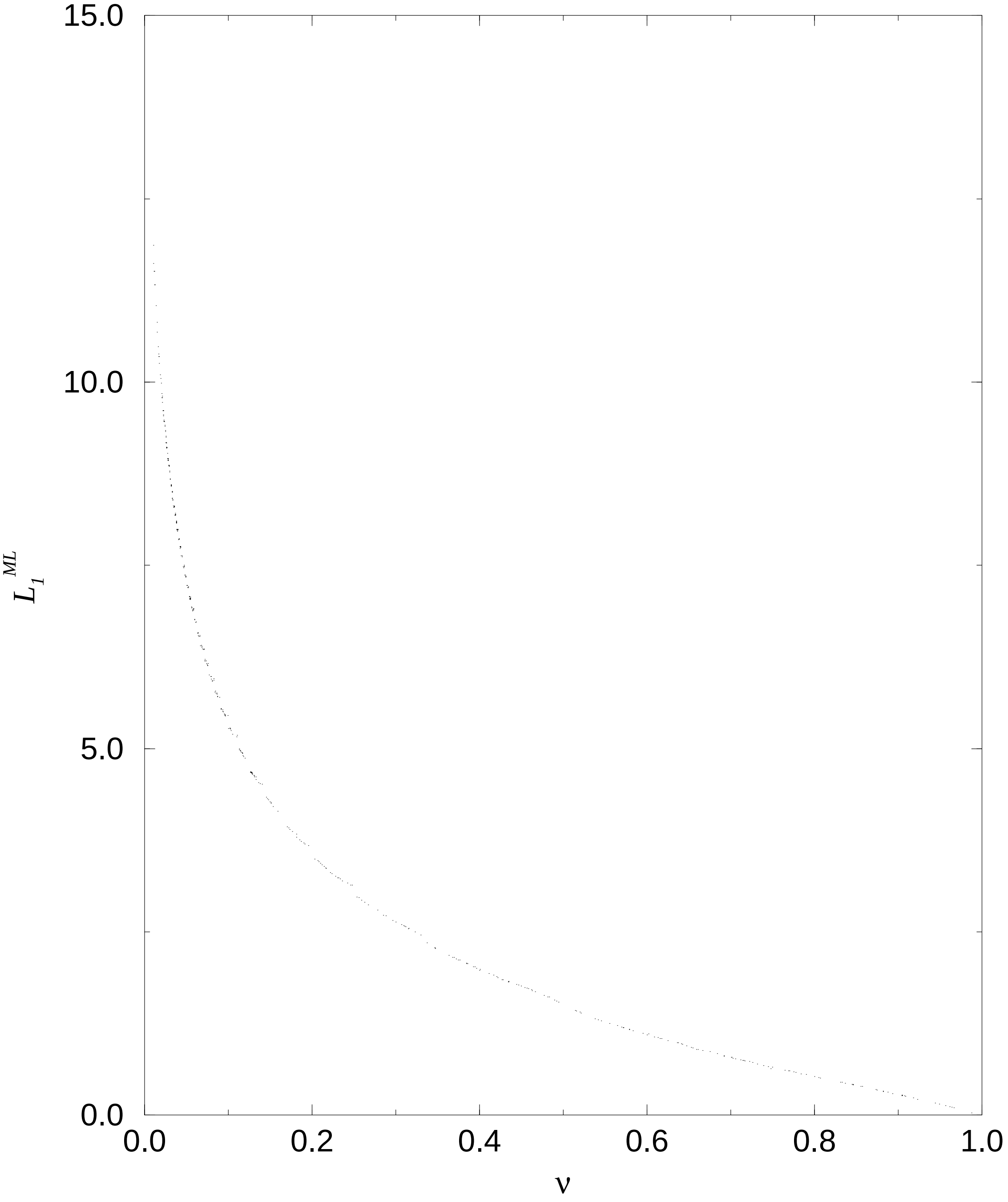

Figure 6: L_1 for the critical sine circle map.

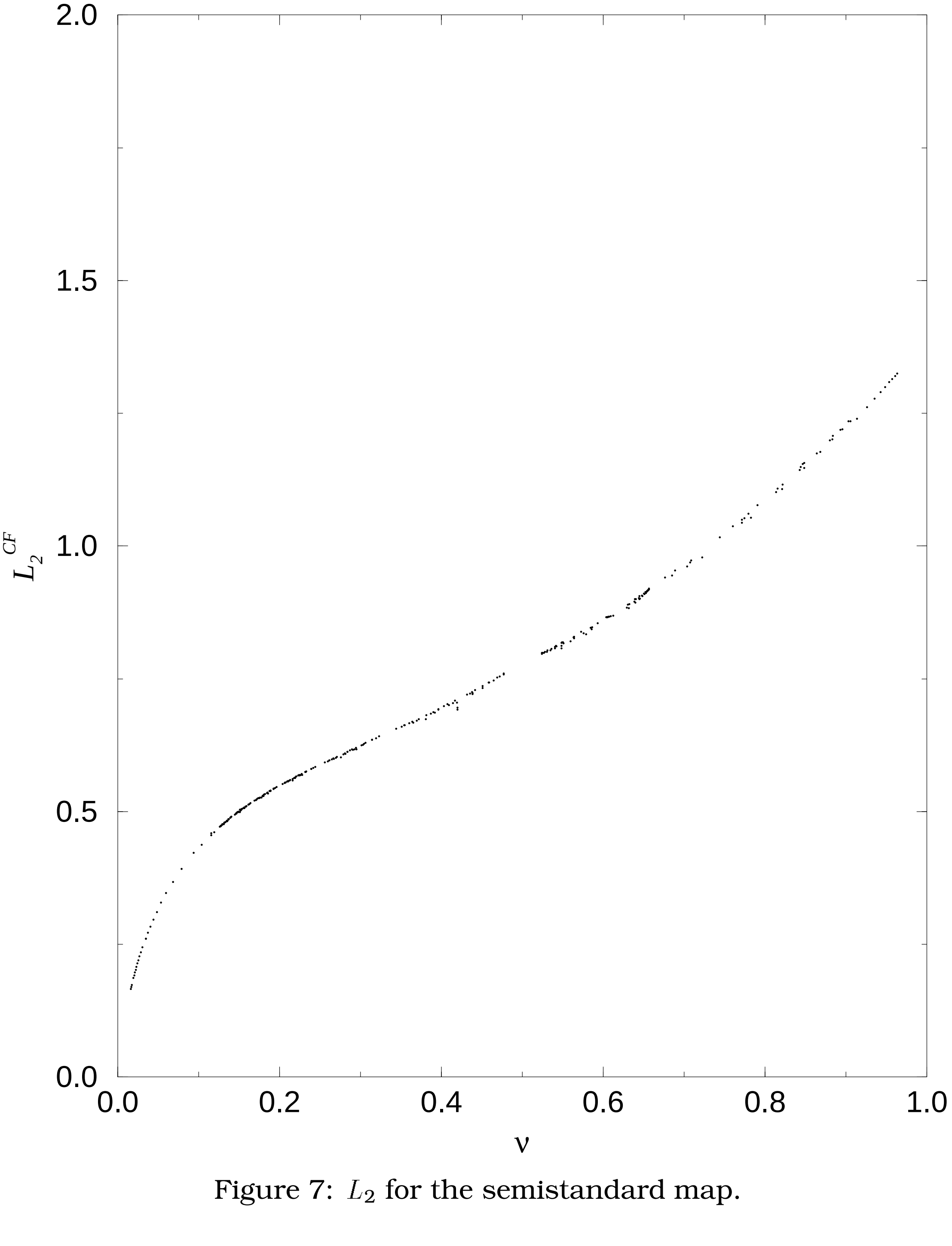

Figure 7: L_2 for the semistandard map.

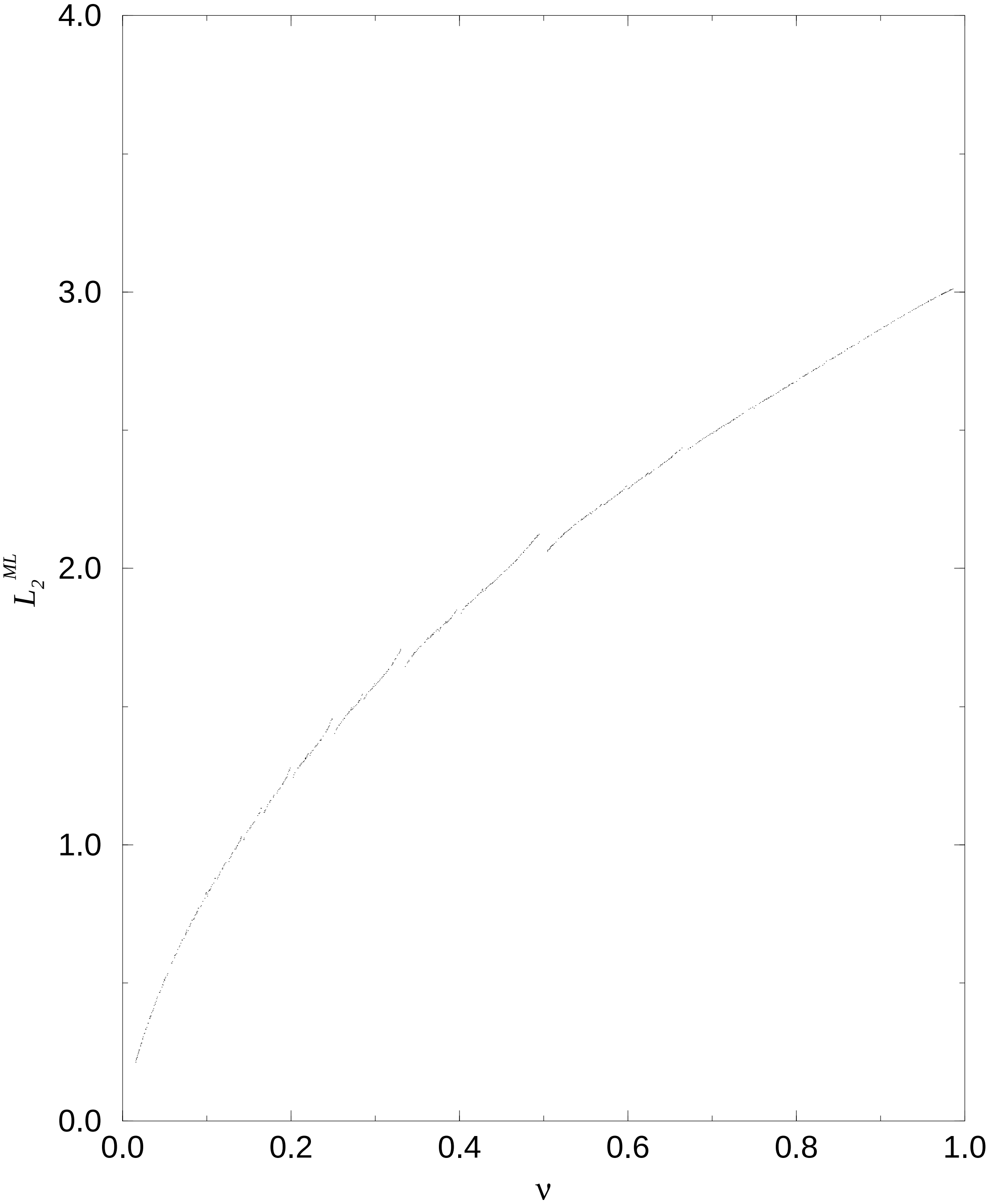

Figure 8: L_2 for the critical sine circle map.

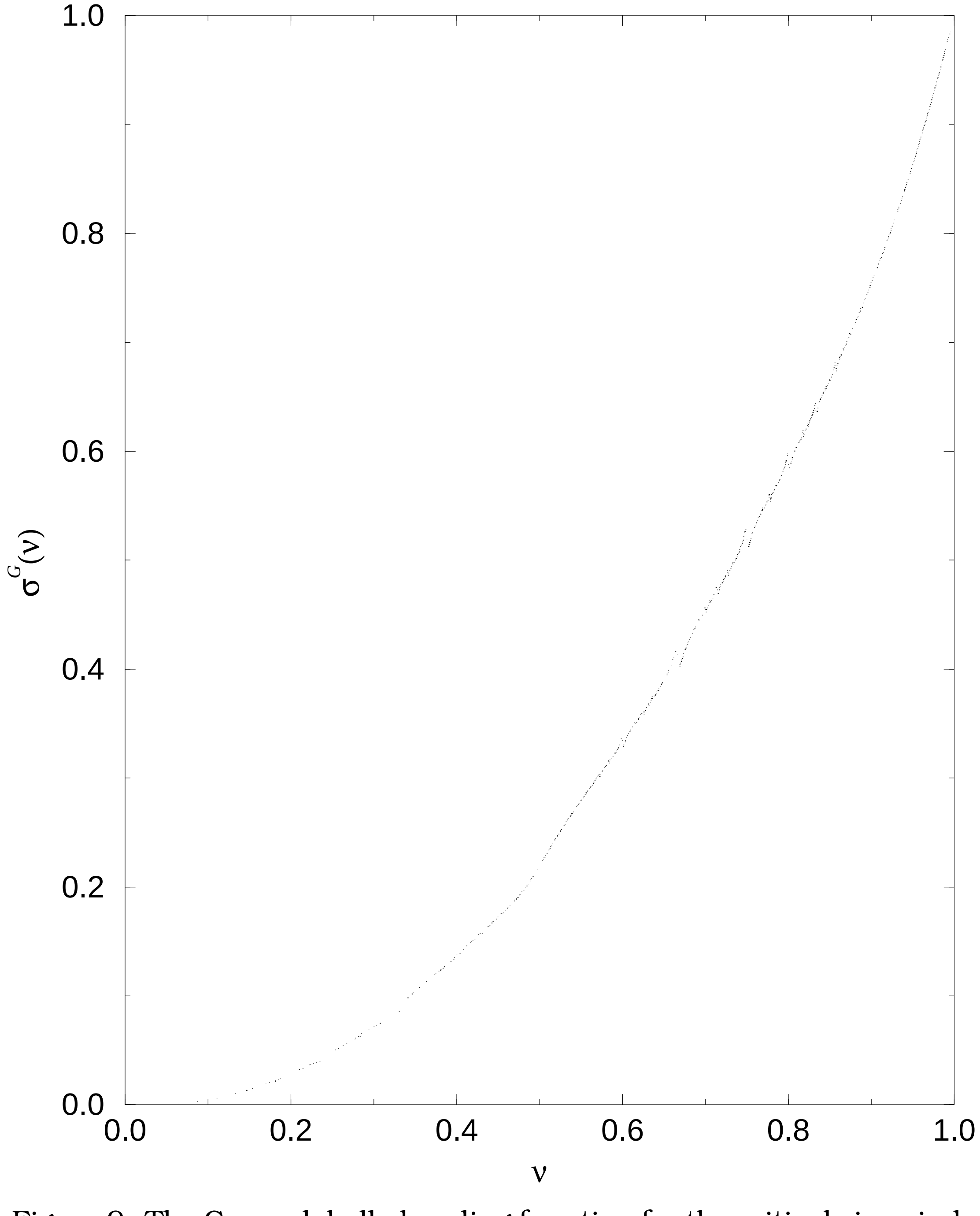

Figure 9: The Gauss-labelled scaling function for the critical sine circle map.

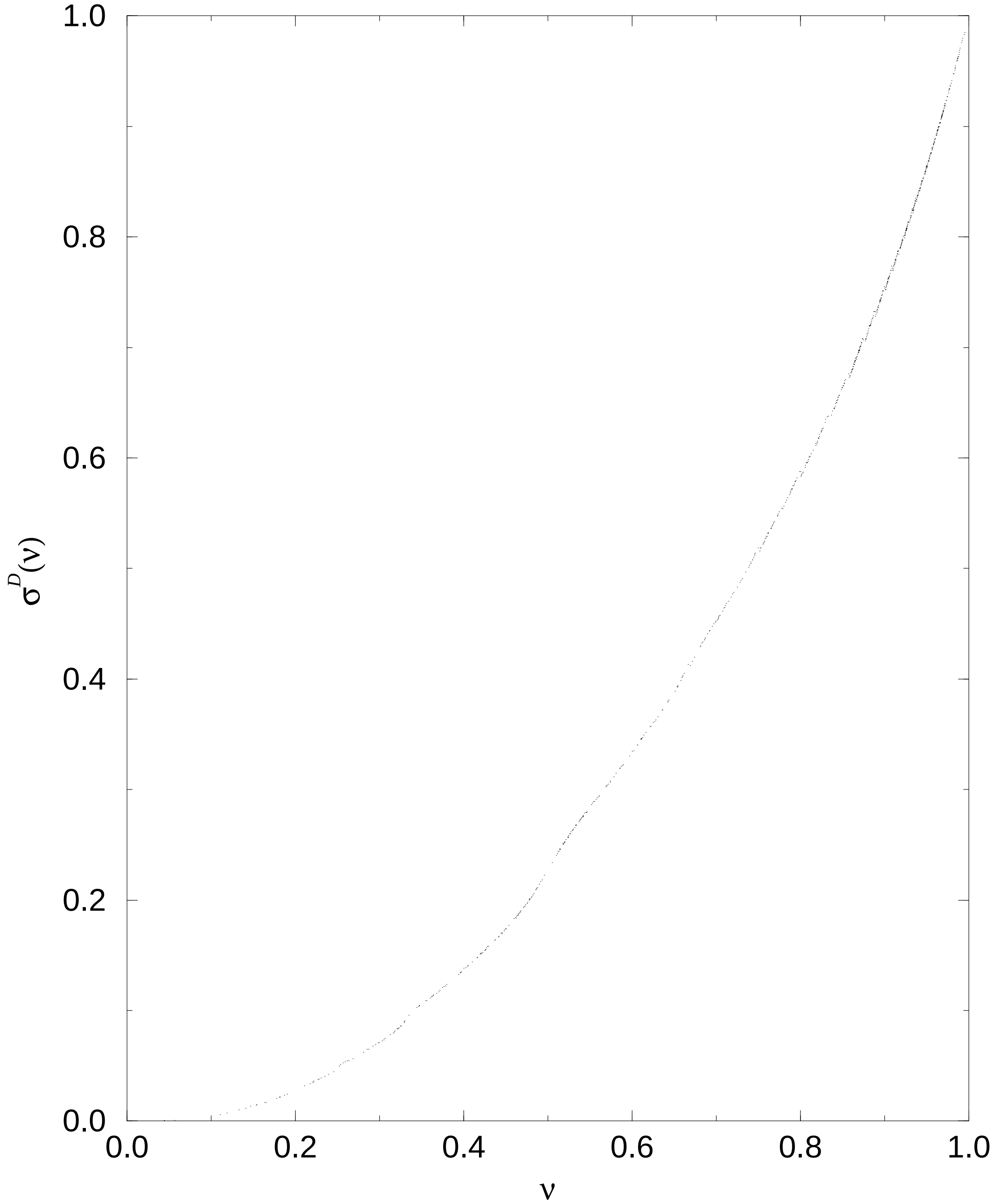

Figure 10: The dual-labelled scaling function for the critical sine circle map.

smoothing gives us an iterative procedure which provides functions of an increasing degree of smoothness. In the present case, such an implementation does not lead to a similar iterative scheme. Nevertheless, the idea of an iterative smoothing algorithm may still be entertained. For example, the Gauss-labelled scaling function shows self-similar features not too different from those of the mode-locking function. One can speculate that the scaling function might itself be described by a smoother function; the scaling function of the scaling function. Work on these lines is currently in progress and will be reported elsewhere.

Acknowledgements

We should like to acknowledge the financial support of the Science and Engineering Research Council (SERC). NB would like to thank the Institute of Theoretical Physics, Belgrade for partial support, JHEC is pleased to thank the AEJMC Foundation and OP would also like to thank the Wolfson Foundation and CONICET.

References

[1] M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker. Quasiperiodicity in dissipative systems: A renormalization group analysis. *Physica*, 5D:370, 1982.

[2] S. J. Shenker. Scaling behavior in a map of a circle onto itself: Empirical results. *Physica*, 5D:405, 1982.

[3] D. L. González and O. Piro. Chaos in a nonlinear driven oscillator with exact solution. *Phys. Rev. Lett.*, 50:870, 1983.

[4] S. Ostlund, D. Rand, D. Sethna, and E. D. Siggia. Universal properties of the transition from quasi-periodicity to chaos in dissipative systems. *Physica*, 8D:303, 1983.

[5] M. H. Jensen, P. Bak, and T. Bohr. Transition to chaos by interaction of resonances in dissipative systems. I. Circle maps. *Phys. Rev. A*, 30:1960, 1984.

[6] T. Bohr, P. Bak, and M. H. Jensen. Transition to chaos by interaction of resonances in dissipative systems. II. Josephson junctions, charge–density waves, and standard maps. *Phys. Rev. A*, 30:1970, 1984.

[7] N. Burić, I. C. Percival, and F. Vivaldi. Critical function and modular smoothing. *Nonlinearity*, 3:21, 1990.

[8] N. Burić and I. C. Percival. Modular smoothing and finite perturbation theory. *Nonlinearity*, 4:981, 1991.

[9] O. Piro, N. Burić, and I. C. Percival. Exact results on the critical function for the motion of an electron driven by two plane waves. *Phys. Lett. A*, 165:320, 1992.

[10] P. Cvitanović, B. Shraiman, and B. Söderberg. Scaling laws for mode lockings in circle maps. *Physica Scripta*, 32:263, 1985.

[11] M. J. Feigenbaum. Unpublished preprint, 1984.

[12] G. Gunaratne. PhD thesis, Cornell, 1986.