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Polarization state selection and stability in a laser with a polarization-isotropic resonator; an example of no lasing despite inversion above threshold

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Abstract

Different models are considered for the study of dynamical instabilities of lasers with a transversally isotropic resonator operating on a single longitudinal mode coupled to a homogeneously broadened $J = 1 \rightarrow J = 0$ material transition. We devise a model of intermediate complexity which retains the dynamics of the atomic variables and which has the experimentally required linearly or circularly polarized steady state solutions with their relative stability near threshold depending on the difference of certain collisionally enhanced relaxation rates. Stability analysis of these steady states in this model provides second thresholds above which the polarization state is time dependent. The stable operation of polarized output states is interpreted in terms of coherence induced absorption or transparency even though the population inversion exceeds the lasing threshold on the transition which drives the orthogonally polarized field. A more general model of the field-atom dynamics than those previously considered is also constructed and analyzed for the stability of its steady states. The fundamental physical condition on the ratio of certain relaxation rates of previous models for stability of circular or linearly polarized solutions is unchanged.

1. Introduction

Lasers with transversally isotropic resonators require use of the vector nature of the transverse electric field for a full description of their stability or dynamics. The behavior of such lasers clearly depends on the angular momentum values of the two levels (which determine the number and couplings of the sublevels) of the medium in resonance with the field [1–3] and on the relative decay rates of the different material variables [2–8]. In particular, for an isotropic resonator, $J \rightarrow J+1$ transitions for $J > 0$ have been shown to lead to stable linearly polarized emission while for $J = 0 \rightarrow J = 1$ and $J = 1 \rightarrow J = 0$ transitions the choice of circularly polarized or linearly polarized emission depends on the relative sizes of the population equilibration rate and the coherence decay rate for the active sublevels of the $J = 1$ state. Stable circularly polarized emission on the 1.523 μm line of a HeNe laser [9] was explained in this manner [2–8].

While some early studies noted that temporal pulsations in the total intensity (going beyond simple beating between two orthogonally polarized modes) might result in these lasers, recently there has been

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renewed interest in the nonlinear dynamics of vector lasers beyond the range of stable polarized emission [3,10–18] and in the possibilities of more complex transverse pattern formation and alternation [19,20]. Among these studies, a new branch was begun by Puccioni et al. [11] with the investigation of the importance of dynamics involving the material variables. However, the particular model used by Puccioni et al. [11] as a modification of the Sargent, Scully and Lamb model [21] and studied in detail by Matlin et al. [15] does not permit the stable circularly polarized solutions which have been observed experimentally and explained in other models. We demonstrate how a simple modification in their model recaptures this feature while providing a model (similar to the general equations of Lenstra [2]) which can be used for more complete studies of steady states, their stability, and time-dependent solutions. While the atomic coupling transition $J = 1 \rightarrow J = 0$ is well known to be neutral with respect to the polarization of the two modes (having equal self and cross saturation for pairs of modes that are either circularly, linearly or elliptically polarized) [1–3,5,9,21], collisional processes have been found to lead to strong coupling between circularly polarized modal amplitudes that favors circularly polarized emission for some lasers of this type [9]. However, a difference in the collision rates within what seem to be a physically allowable range from consideration of collision cross sections [2] would lead to weak coupling favoring linearly polarized emission within our generalized model. We present the results for this model for the stability of the circularly polarized steady-state solutions and the linearly polarized steady state solutions including instabilities involving material dynamics whch are neglected in third-order amplitude equations of previous studies [1–10]. (Although the model specifically applies for $J = 1 \rightarrow J = 0$ transitions, the range of parameters covers strong and weak coupling (cross saturation) and may thus offer hints of the effects of material dynamics on lasers with other transitions. However, application of the dynamical results from the full models considered here to transitions with more complicated sublevel structures, such as the $J = 1 \rightarrow J = 2$ transitions or $J = 2 \rightarrow J = 3$ of the experiments in [1,7,9,17,18] requires a separate analysis.) We also review the isomorphism between representations of the Hamiltonian dynamics for the $J = 1 \rightarrow J = 0$ transition in terms of linearly or circularly polarized fields and Cartesian or Zeeman atomic states, respectively, which leads to the neutral coupling and how this polarization state neutrality is broken by relaxation rates. This leads us to contruct a more general model than those used heretofore [1,2] by including the spontaneous transitions between the two levels. A stability analysis of this general model is readily possible near threshold and it reveals the same conditions on the collisional relaxation rates for stability of the circularly or linearly polarized solutions, indicating that the ratio of the magnetic dipole and electric quadrupole relaxation rates of the $J = 1$ sublevels is the crucial factor in determining the strong or weak coupling.

In section 2 we show how to construct a model of intermediate complexity that generalizes the model studied by Puccioni et al. [11] and Matlin et al. [15]. We demonstrate how the relaxation rates govern the selection of stable polarized states in this model as in previously studied simpler models and we complete a stability analysis of these steady state solutions. In section 3 the stability of these states is reinterpreted in the language of coherent field-matter phenomena in multilevel systems which has previously led to the observation of such phenomena as lasing without inversion, electromagnetically induced transparency, and inversion with absorption [22–25], and a relatively new member of this class, inversion without lasing [26]. In section 4 we review the symmetry of the representations of the dynamics in terms of either circularly polarized fields and Zeeman basis states or linearly polarized fields and Cartesian basis states. We then construct a more general model of these lasers incorporating a full set of collisional decay rates and couplings of the upper and lower level via spontaneous emission. We discuss the relative impact of these parameters on the selection and stability of polarized emission. In section 5 we analyze close to threshold the general model introduced in section 4 and we show that the fundamental condition of previous models for stability of circularly or linearly polarized solutions is unchanged.

2. Improved $J = 1 \rightarrow J = 0$ model of intermediate complexity

The basic Hamiltonian dynamics for a $J = 1 \rightarrow J = 0$ transition does not couple the $M = 0$ sublevel of the J=1 level to any other sublevel, if the axis of quantization is taken along the direction of propagation of the single mode field (see, for example, Refs. [2,21]). This gives a model essentially equivalent to the Maxwell–Bloch model for single mode lasers with the generalization to two field amplitudes (of orthogonal circularly polarized emission), two dipole moments, two population inversions, and an induced coherence between the radiatively active upper sublevels. Puccioni et al. [11] added the feature that collisions might increase the decay rate of the coherence above the minimum value given by the spontaneous decay rate of the upper levels as assumed by Sargent, Scully and Lamb [21] and they added circular phase and loss anisotropies. Their model for isotropic cavity losses and isotropic cavity detunings reads,

$$
\begin{aligned}
\mathrm{d}E_{\mathrm{R}}/\mathrm{d}t &= -\kappa E_{\mathrm{R}} - \mathrm{i}\delta E_{\mathrm{R}} + \kappa P_{\mathrm{R}} , \\
\mathrm{d}E_{\mathrm{L}}/\mathrm{d}t &= -\kappa E_{\mathrm{L}} - \mathrm{i}\delta E_{\mathrm{L}} + \kappa P_{\mathrm{L}} , \\
\mathrm{d}P_{\mathrm{R}}/\mathrm{d}t &= -\gamma_{\perp} P_{\mathrm{R}} + \gamma_{\perp} E_{\mathrm{R}} D_{\mathrm{R}} + \gamma_{\perp} E_{\mathrm{L}} C , \\
\mathrm{d}P_{\mathrm{L}}/\mathrm{d}t &= -\gamma_{\perp} P_{\mathrm{L}} + \gamma_{\perp} E_{\mathrm{L}} D_{\mathrm{L}} + \gamma_{\perp} E_{\mathrm{R}} C^* , \\
\mathrm{d}C/\mathrm{d}t &= -\gamma_c C - (\gamma_{||}/4)[E_{\mathrm{L}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{L}}^*] , \\
\mathrm{d}D_{\mathrm{R}}/\mathrm{d}t &= -\gamma_{||}(D_{\mathrm{R}} - \sigma) - (\gamma_{||}/2)[E_{\mathrm{R}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{R}}^* \\
&\quad + (1/2)(E_{\mathrm{L}}^* P_{\mathrm{L}} + E_{\mathrm{L}} P_{\mathrm{L}}^*)] \\
\mathrm{d}D_{\mathrm{L}}/\mathrm{d}t &= -\gamma_{||}(D_{\mathrm{L}} - \sigma) - (\gamma_{||}/2)[(E_{\mathrm{L}}^* P_{\mathrm{L}} + E_{\mathrm{L}} P_{\mathrm{L}}^*) \\
&\quad + (1/2)(E_{\mathrm{R}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{R}}^*)] , \qquad (1)
\end{aligned}
$$

where the $E_{\mathrm{R,L}}$ are the suitably rescaled slowly varying field amplitudes, $P_{\mathrm{R,L}}$ are the two (rescaled and slowly varying amplitudes of the) dipole moments on the circularly polarized transitions, $D_{\mathrm{R,L}}$ are the rescaled differences between the populations of the upper sublevels and the lower level, and C is the coherence between the upper sublevels. κ is the cavity decay rate for the two modes, δ is the cavity detuning for the two modes, and the γ's are the decay rates of the associated atomic variables, where it is assumed that the collision-induced decay rate of the inter-sublevel coherence, γ_c, may be larger than the population decay rate, $\gamma_{||}$, just as the dipole moment decay rate, $\gamma_{\perp}$, may be collision broadened above the population decay rate. As in the construction of the two-level Maxwell–Bloch models, it is assumed for simplicity that the decay rates of the upper and lower levels are equal.

Since in this model $\gamma_c > \gamma_{||}$, it suffers from the fact that linearly polarized states are always stable near threshold and circularly polarized states are always unstable near threshold, in disagreement with experimental results [9] and with modifications of earlier models to explain this effect [2–8]. The missing feature is collisional processes which may also reorient the atoms, effectively mixing the populations of the upper sublevels. By adding a collisional enhancement of the decay rate of the difference of the populations of the upper sublevels to Eqs. (1), we obtain a generalized model which recaptures the missing features of stability of circularly polarized solutions for certain parameter values.

$$
\begin{aligned}
\mathrm{d}E_{\mathrm{R}}/\mathrm{d}t &= -\kappa E_{\mathrm{R}} - \mathrm{i}\delta E_{\mathrm{R}} + \kappa P_{\mathrm{R}} , \\
\mathrm{d}E_{\mathrm{L}}/\mathrm{d}t &= -\kappa E_{\mathrm{L}} - \mathrm{i}\delta E_{\mathrm{L}} + \kappa P_{\mathrm{L}} , \\
\mathrm{d}P_{\mathrm{R}}/\mathrm{d}t &= -\gamma_{\perp} P_{\mathrm{R}} + \gamma_{\perp} E_{\mathrm{R}} D_{\mathrm{R}} + \gamma_{\perp} E_{\mathrm{L}} C , \\
\mathrm{d}P_{\mathrm{L}}/\mathrm{d}t &= -\gamma_{\perp} P_{\mathrm{L}} + \gamma_{\perp} E_{\mathrm{L}} D_{\mathrm{L}} + \gamma_{\perp} E_{\mathrm{R}} C^* , \\
\mathrm{d}C/\mathrm{d}t &= -\gamma_c C - (\gamma_{||}/4)[E_{\mathrm{L}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{L}}^*] , \\
\mathrm{d}D_{+}/\mathrm{d}t &= -\gamma_{||}(D_{+} - 2\sigma) \\
&\quad - (3\gamma_{||}/4)[E_{\mathrm{L}}^* P_{\mathrm{L}} + E_{\mathrm{L}} P_{\mathrm{L}}^* + E_{\mathrm{R}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{R}}^*] \\
\mathrm{d}D_{-}/\mathrm{d}t &= -\gamma_J D_{-} - (\gamma_{||}/4)[(E_{\mathrm{R}}^* P_{\mathrm{R}} + E_{\mathrm{R}} P_{\mathrm{R}}^*) \\
&\quad - (E_{\mathrm{L}}^* P_{\mathrm{L}} + E_{\mathrm{L}} P_{\mathrm{L}}^*)] , \qquad (2)
\end{aligned}
$$

where D_{+} and D_{-} are the sum and difference, respectively, of D_{R} and D_{L}.

We next consider the steady states and their stability on resonance ($\delta = 0$). The steady state solutions in resonance are

(1) *The off-state*

$$C = E_{\mathrm{R}} = E_{\mathrm{L}} = P_{\mathrm{R}} = P_{\mathrm{L}} = 0, \quad D_{\mathrm{R}} = D_{\mathrm{L}} = \sigma.$$

(2) *The circularly polarized states*

(a) Right circular

$$E_{\mathrm{R}} = P_{\mathrm{R}} = [4\gamma_J(\sigma - 1)/(3\gamma_J + \gamma_{||})]^{1/2} ,$$

$E_L = P_L = C = 0, \quad D_R = 1,$

$D_L = 1 + 2\gamma_{\|}(\sigma - 1)/(3\gamma_J + \gamma_{\|}).$

(b) Left circular

$E_L = P_L = [4\gamma_J(\sigma - 1)/(3\gamma_J + \gamma_{\|})]^{1/2},$

$E_R = P_R = C = 0, \quad D_L = 1,$

$D_R = 1 + 2\gamma_{\|}(\sigma - 1)/(3\gamma_J + \gamma_{\|}),$

with total output power $4\gamma_J(\sigma - 1)/(3\gamma_J + \gamma_{\|})$.

(3) *The linearly polarized states*

$$|E_R| = |E_L| = |P_R| = |P_L| = \{2\gamma_c(\sigma - 1)/(3\gamma_c + \gamma_{\|})\}^{1/2},$$

$$D_R = D_L = 1 + \{\gamma_{\|}(\sigma - 1)/(3\gamma_c + \gamma_{\|})\},$$

$$|C| = \gamma_{\|}(\sigma - 1)/(3\gamma_c + \gamma_{\|}),$$

with total output power $4\gamma_c(\sigma - 1)/(3\gamma_c + \gamma_{\|})$.

The phase shift between E_R and E_L (and the consequent phase shifts for P_R and P_L and C) determines the particular linear polarization. There is an infinity of solutions (on a circle with the relative phase between E_R and E_L varied between 0 and 2π) of this type.

When $\gamma_c = \gamma_J$ the total output power for this state is the same as for the circularly polarized states above. The intensity emitted by the laser in the linearly polarized state is greater for $\gamma_c > \gamma_J$ while the intenstiy emitted by the laser in the circularly polarized state is greater for $\gamma_c < \gamma_J$.

For a specific example, let us take E_R and E_L purely real and positive to create the horizontally polarized state. Then

$$E_R = E_L = P_R = P_L = \{2\gamma_c(\sigma - 1)/(3\gamma_c + \gamma_{\|})\}^{1/2},$$

$$D_R = D_L = 1 + \{\gamma_{\|}(\sigma - 1)/(3\gamma_c + \gamma_{\|})\},$$

$$C = -\gamma_{\|}(\sigma - 1)/(3\gamma_c + \gamma_{\|}).$$

A rotation of this state in the clockwise direction by an angle $\psi/2$ is accomplished by the transform $E'_R = P'_R = E_R \exp(i\psi/2), E'_L = P'_L = E_L \exp(-i\psi/2), C' = C \exp(-i\psi)$.

For both the circularly polarized and linearly polarized solutions, the extracted power is less than would be obtained from two independent two-level transitions each pumped by rate σ. In that case the total intensity would be $2(\sigma - 1)$. Here instead less energy is extracted and the population inversion left to interact with the suppressed state of polarization exceeds the usual value needed to bring a laser with a two-level medium above the lasing threshold ($D = 1$). The physics of this phenomenon is discussed further in section 3.

There are five complex variables in the problem (C, Es, Ps). The global phase (the sum of the phase angles of these variables in the complex plane) is never determined by the criterion of a steady state solution for autonomous conditions, as is true for any autonomous laser. For the linearly polarized solutions, the relative phase between E_R and E_L is not fixed and so there is the larger family of solutions described previously which is parameterized by this relative angle ψ. And in the case that $\gamma_c = \gamma_J$, the angle which determines the degree of ellipticity of the solutions is undetermined in the steady state solution and there is an even larger family of solutions for the variation of these two angles. This was the condition considered in the earliest models [1,21] from which it was predicted that there would be neutral stability of $J = 1 \to J = 0$ lasers with respect to polarization states preferred by the atomic processes. In contrast with single mode lasers, lasers of this type have larger families of solutions parametrized by different phase angles. These are associated zero eigenvalues in the stability analysis. Amplitude fluctuations and laser linewidhs are thus different when the system is driven by noise [27–31]. These effects are enhanced when there is also neutral stability for the degree of ellipticity of the solutions.

Stability analysis

The circularly polarized states are stable in resonance just above the lasing threshold (except for neutral stability of the phase of the emitted field) if $\gamma_J > \gamma_c$ and for values of the pump parameter between threshold and a critical value $\sigma_{cr-circular}$ given by

$$\sigma_{cr-circular} = 1 + \frac{2(\kappa + 1)\gamma_c(\kappa + 1 + \gamma_c)}{\kappa(\gamma_{\|} + \kappa + 1)}.$$

At this critical value there is a double Hopf bifurcation involving the onset of the orthogonally polarized field at a shifted optical frequency. The frequency shift, Ω_c, in the vicinity of the instability threshold is given by

$$\Omega^2_{c-\text{circular}} = \frac{\kappa(\gamma_J - \gamma_c)(\sigma_c - 1)\sigma_c}{2(\sigma_c + \kappa + 1)}.$$

Although there are no oscillations of the total intensity at the instability threshold because the perturbation involves the addition of an orthogonally polarized field, above the threshold nonlinear coupling causes intensity oscillations to develop at twice this frequency.

The linearly polarized states are stable (except for neutral stability of the phase of the emitted field and neutral stability of the orientation of polarization of the emitted field) if $\gamma_J < \gamma_c$ and for values of the pump parameter between threshold and a critical value $\sigma_{cr-\text{linear}}$ given by

$$\sigma_{cr-\text{linear}} = 1 + \frac{(\kappa + 1 + \gamma_J)(\kappa + 1)\gamma_J(3\gamma_c + \gamma_{||})}{\gamma_{||}[2\kappa^2 + 2\kappa + \gamma_c(\kappa - \gamma_J - 1)]}.$$

This instability involves a modulation of the ellipticity at a frequency

$$\Omega^2_{c-\text{linear}} = \frac{2\kappa\gamma_{||}(\sigma_l - 1)(\gamma_c - \gamma_J)}{(3\gamma_c + \gamma_{||})(\kappa + \gamma_J + 1)},$$

which to lowest order does not cause a modulation of the total intensity. The orientation of the linear polarization is neutrally stable, indicating that it is free to diffuse. A second pair of complex conjugate eigenvalues of the stability analysis of the linearly polarized steady state appears for even higher values of the pump parameter if $2\kappa > \gamma_c$, and they correspond to a phase instability of the linearly polarized solution. However, except for the unique case of $\gamma_c = \gamma_J = \gamma_{||}$ where the pump value for the instability of this second pair of complex conjugate eigenvalues coalesces with the first, these eigenvalues do not affect the basic stability of the steady state solution. However, the relatively large modulation instabilities that appear well above the instability threshold clearly involve phase instabilities (frequency shifts and frequency modulation) which are perhaps heralded by these other eigenvalues.

3. No lasing despite inversion above threshold

The stability of a specific polarized state of emission is intuitively surprising since in each case the saturated inversion exceeds that needed to bring single mode emission above the lasing threshold on the transition for the suppressed field. In the case of circularly polarized fields, in the zone of stability the result indicates that all perturbations are suppressed, and this corresponds to attenuation of weak probe fields of any polarization. This differs, from other forms of bistability between two single mode solutions in a multimode model such as discussed in [21]. Bistability and hysteresis in the crossover from one mode to another is observed in third-order Lamb theory models of orthogonally polarized fields in a Vector laser. Unlike third-order Lamb theory where the only possible statement is that the operation of one mode suppresses the gain of the other below threshold, here we can distinguish different contributions to the saturated gain of the suppressed transition. Specifically, because we retain the dynamics of the populations, dipole moments, and coherences, we can see that even in the presence of one single mode solution, the inversion for the other transition remains above threshold. When the dipole moment that provides gain for a mode arises only from the interaction of a field perturbation with the inversion, then having one mode stable requires that the suppressed transition have its inversion suppressed below threshold. What we observe here is somewhat like the mode competition and bistability observed in multi-longitudinal-mode Maxwell–Bloch equations [31] where a detuned mode can suppress the operation of another mode that would be less detuned, although there one cannot identify distinct inversions for the modes (longitudinal modes of different frequencies) since they interact with the same homogeneously broadened atoms. Here by contrast there is a distinct inversion for each of the two modes. Although there is inversion in excess of threshold ($D > 1$ in these units) for the suppressed transition, the absorption from the induced coherence C reduces the amplification on the suppressed transition to less than 1, so that the cavity losses exceed the net gain. There remains net amplification, just less than that needed to sustain lasing. This phenomenon is a reminder that the source term for the electromagnetic field is the dipole moment density and for multilevel systems interlevel coherences contribute to the value of this term. This effect joins the general class of multi-level coherence effects such as lasing without inversion [22,23], electromagnetically induced transparency [23,24] and inversion with absorption [25]. No lasing despite inversion above threshold was also observed recently in a model for $J=1 \rightarrow J=0$ far infrared lasers for which the upper level was pumped coherently by an intense, linearly-polarized laser field [26]. By contrast, our case appears as a spontaneously organized process for an incoherently pumped upper level.

We illustrate the origin of this phenomenon by considering the dynamics of the perturbation of the right circularly polarized solution (writing lower case variables for the perturbations that lead to the instability above the second threshold):

$$\begin{aligned}
dE_R/dt &= -\kappa E_R + \kappa P_R ,\\
de_L/dt &= -\kappa e_L + \kappa p_L ,\\
dP_R/dt &= -\gamma_\perp P_R + \gamma_\perp E_R D_R + \gamma_\perp e_L c ,\\
dp_L/dt &= -\gamma_\perp p_L + \gamma_\perp e_L D_L + \gamma_\perp E_R c^* ,\\
dc/dt &= -\gamma_c c - (\gamma_{||}/4)[e_L^* P_R + E_R p_L^*] ,\\
dD_+/dt &= -\gamma_{||}(D_+ - 2\sigma)\\
&\quad - (3\gamma_{||}/4)[e_L^* p_L + e_L p_L^* + E_R^* P_R + E_R P_R^*] ,\\
dD_-/dt &= -\gamma_J D_- - (\gamma_{||}/4)[(E_R^* P_R + E_R P_R^*)\\
&\quad - (e_L^* p_L + e_L p_L^*)] .
\end{aligned} \tag{3}$$

Taking adiabatic elimination of the material variables, we have

$$c = -(\gamma_{||}/4\gamma_c)[e_L^* P_R + E_R p_L^*]$$

and

$$\begin{aligned}
p_L &= e_L D_L + E_R c^*\\
&= e_L D_L - (\gamma_{||}/4\gamma_c)[e_L E_R P_R^* + |E_R|^2 p_L] ,
\end{aligned}$$

so

$$p_L = \frac{e_L[D_L - (\gamma_{||}/4\gamma_c) E_R P_R^*]}{1 + (\gamma_{||}/4\gamma_c)|E_R|^2} ,$$

which leads to

$$\begin{aligned}
\frac{de_L}{dt} &= -\kappa e_L + \kappa p_L\\
&= -\kappa e_L + \kappa e_L \left(\frac{D_L - (\gamma_{||}/4\gamma_c) E_R P_R^*}{1 + (\gamma_{||}/4\gamma_c)|E_R|^2} \right) .
\end{aligned}$$

This shows explicitly that the gain for e_L may be less than that provided by the saturated population inversion D_L alone. The reduction comes from the coherence c created by the radiating field E_R and dipole moment P_R. Neglecting c amounts to taking γ_c to infinity, in which case the gain for e_L is simply as provided by D_L alone.

The small signal net gain for the circularly polarized field that is off (e_L) is provided by the term $\kappa(\{...\} - 1)$ which can be rewritten using the steady state solutions for the variables as

$$\kappa(\{...\} - 1) = \frac{\kappa([2(\gamma_c - \gamma_J)\gamma_{||}(\sigma - 1)]}{\gamma_c(3\gamma_J + \gamma_{||}) + \gamma_{||}\gamma_J(\sigma - 1)} . \tag{4}$$

The net gain for the left circularly polarized field that is initially off is positive if $\gamma_c > \gamma_J$ and this net gain is negative if $\gamma_c < \gamma_J$. That is, a circularly polarized solution is stable with respect to the perturbation of the other circularly polarized field if $\gamma_c < \gamma_J$ even though the population inversion on the suppressed transition provides more gain than the cavity losses. This is because the absorption resulting from the interlevel coherence is strong enough in this case to reduce the overall gain from the medium to less than the cavity losses.

4. Constructing a more general model

The rather ad hoc method of creating Eqs. (2) from Eqs. (1) suggests that it may be useful to revisit the derivation of such models, the most systematic of which is that of Lenstra [2]. In completing this derivation, we note the symmetry of two representations for the Hamiltonian portion of the dynamics, the breaking of that symmetry by the dissipation rates, the inconsistency of ignoring the contribution of the $M=0$ sublevel of the upper level to the dynamics, and a previously overlooked coupling between this level and the lower level.

We consider a transition between an upper level with $J=1$ and a lower level with $J=0$ in the absence of a magnetic field so that the magnetic sublevels of the $J=1$ level are degenerate in energy. Taking the quantization axis of the upper level along the direction of the field propagation (in a unidirectional ring cavity) the dipole interaction couples the levels as follows. Right circularly polarized fields drive the transition between the $M=1$ sublevel of the upper level and the lower level, while left circularly polarized fields drive the transition between the $M=-1$ upper sublevel and the lower level as indicated schematically in

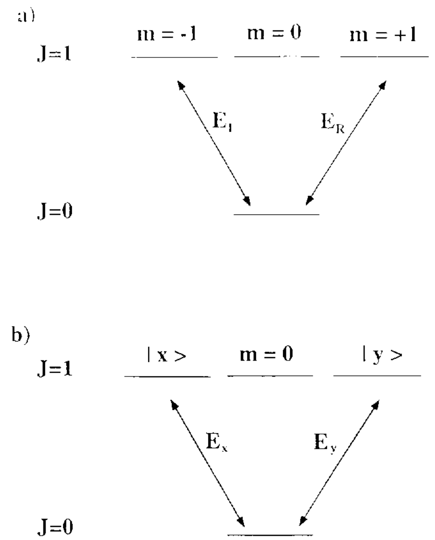

Fig. 1. Schematic diagram of the fields interacting with atomic sublevels. (a) Circularly polarized field amplitudes and Zeeman basis states. (b) Linearly polarized field amplitudes and Cartesian basis states.

Fig. 1a. The $M = 0$ sublevel of the upper state is not coupled to any of the other level by the electric dipole Hamiltonian.

A comprehensive analysis of a model of this system was undertaken by Lenstra, albeit with certain restrictions on the full generality of the dissipative rates and with a limitation of his stability analysis of the lasing steady states to near threshold conditions that could be described by third-order (Lamb) equations for orthogonally polarized fields. While this analysis gives the stability or instability relative to selection of linearly, circularly, or elliptically polarized states near threshold, it does not permit an analysis of the equations for a further bifurcation (loss of stability) at higher pump rates such as we completed for Eqs. (2) in the previous sections.

5. Two representations of the interaction Hamiltonian

The interaction Hamiltonian in the basis of the four atomic states $\{|J,M\rangle\} = \{|1,-1\rangle, |1,0\rangle, |1,+1\rangle, |0,0\rangle \equiv |b\rangle\}$ is given by

$$H_{\text{int}}^{\text{RL}} = \begin{pmatrix} 0 & 0 & 0 & gE_{\text{L}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & gE_{\text{R}} \\ gE_{\text{L}}^* & 0 & gE_{\text{R}}^* & 0 \end{pmatrix},$$

where E_{R} and E_{L} are the left and right circularly polarized components, respectively, of the electric field amplitude and g is a coupling constant incorporating the dipole matrix element of the transitions.

However, if we change the basis states for the density matrix representation by taking orthogonal contributions of the states $|1,-1\rangle$ and $|1,1\rangle$, namely $|1,x\rangle \equiv (|1,-1\rangle + |1,1\rangle)/\sqrt{2}$ and $|1,y\rangle \equiv \mathrm{i}(|1,-1\rangle - |1,1\rangle)/\sqrt{2}$, so that the four basis states are $|1,x\rangle, |1,0\rangle, |1,y\rangle$ and $|b\rangle$, the interaction Hamiltonian takes on an equivalently simple form

$$H_{\text{int}}^{XY} = \begin{pmatrix} 0 & 0 & 0 & gE_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & gE_y \\ gE_x^* & 0 & gE_y^* & 0 \end{pmatrix},$$

where E_x and E_y are the components of the electric field amplitude polarized in the x and y-directions, respectively, given by $E_x \equiv (E_{\text{R}} + E_{\text{L}})/\sqrt{2}$ and $E_y \equiv \mathrm{i}(E_{\text{R}} - E_{\text{L}})/\sqrt{2}$. The pictorial coupling of these levels by the field is shown in Fig. 1b. Note that the full symmetry of these two expressions (and figures) for linear and circular states and field amplitudes also results for $J = 0 \to J = 1$ and $J = 1 \to J = 1$ transitions [33]. For higher angular momentum values the Clebsch Gordan coefficients give different strengths to the different transitions and having more than two active sublevels prevents the definition of $|x\rangle$ and $|y\rangle$ states.

6. Effects of damping and excitation

In adding damping rates to these equations, we have the following physical phenomena to include: spontaneous decay from each level including the fact that some of the spontaneous emission from the upper level involves transitions to the lower level, isotropic collisional enhancement of the decays of the off-diagonal elements of the density matrix (dipole moments and coherence terms) which may differ, and collisional redistribution of the upper state sublevel populations. One may not simply write different decay rates for each of the elements of the atomic density matrix under the assumption of isotropic collisions, since rotation of the coordinate system transforms certain elements into combinations of others. One can, however, separate the elements of the upper state density matrix into groupings of different tensorial order which transform into other members of the group. For the lower level there is only the scalar population. For the upper sublevels there is a scalar (the total population), a vector (involving the population difference of the radiatively active sublevels, which is sometimes referred to as the magnetic dipole of the system), and a rank-two tensor (involving the off diagonal coherence between the radiatively active sublevels and the population difference between the $|M| = 1$ sublevels and the $M = 0$ sublevel, sometimes referred to as the electric quadrupole of the system) (see e.g., Ref. [2]). The assumption is made that the isotropic nature of the collisions is unaffected by the presence of the lasing field or by the dipole and quadrupole moments and coherences created by this field. For weak fields near threshold, no experimental evidence has been observed of a breaking of this isotropy.

We illustrate these limitations as follows. Let $N_{1,1}, N_{1,0}$ and $N_{1,-1}$ be the populations of the three upper state sublevels. We define N_a, N_- and Q, the population contributions to the different tensorial orders, as follows:

$$N_a = N_{1,1} + N_{1,0} + N_{1,-1}\,,$$

$$N_- = N_{1,1} - N_{1,-1}\,,$$

$$Q = N_{1,1} + N_{1,-1} - 2N_{1,0}\,.$$

The contributions of decays, which may be different for different tensorial orders, require

$$\mathrm{d}N_a/\mathrm{d}t = -\gamma_a N_a\,,$$

$$\mathrm{d}N_-/\mathrm{d}t = -\gamma_J N_-\,,$$

$$\mathrm{d}Q/\mathrm{d}t = -\gamma_c Q\,,$$

where γ_c is the decay rate of the interlevel coherence (called C in the previous section), γ_a is the spontaneous decay rate of the upper level which we assume is not collisionally enhanced (characteristic of electronic transitions in atomic gas lasers), and $\gamma_J = \gamma_a + \gamma'_J$ and $\gamma_c = \gamma_a + \gamma'_c$, where the primes denote the collisional contributions.

Including these decay rates and a sublevel independent incoherent excitation mechanism (R) the non-Hamiltonian part of the equations for the populations of the different sublevels and of the other atomic variables may be written as follows:

$$\begin{aligned}
\mathrm{d}N_{1,1}/\mathrm{d}t &= -\gamma_a N_{1,1} + R - \gamma_{J'}(N_{1,1} - N_{1,-1})/2 \\
&\quad - \gamma_{c'}(N_{1,1} - 2N_{1,0} + N_{1,-1})/6\,, \\
\mathrm{d}N_{1,0}/\mathrm{d}t &= -\gamma_a N_{1,0} + R \\
&\quad + 2\gamma_{c'}(N_{1,1} - 2N_{1,0} + N_{1,-1})/3\,, \\
\mathrm{d}N_{1,-1}/\mathrm{d}t &= -\gamma_a N_{1,-1} + R - \gamma_{J'}(N_{1,-1} - N_{1,1})/2 \\
&\quad - \gamma_{c'}(N_{1,1} - 2N_{1,0} + N_{1,-1})/6\,, \\
\mathrm{d}N_b/\mathrm{d}t &= -\gamma_b N_b + \gamma_{ab}(N_{1,1} + N_{1,-1} + N_{1,0})\,, \\
\mathrm{d}P_{R,L}/\mathrm{d}t &= -[(\gamma_a + \gamma_b)/2 + \gamma'_\perp] P_{R,L}\,, \\
\mathrm{d}C/\mathrm{d}t &= -(\gamma_a + \gamma'_c) C\,, \qquad (5)
\end{aligned}$$

where γ_{ab} is the portion of the upper level spontaneous decay rate which is directed to the lower level and γ_b is the decay rate of the lower level population which we assume is not collisionally enhanced. γ_a and γ_b appear in the minimum decay rates (in the absence of collisions) for C and the Ps since these are formed from spontaneously decaying amplitudes. $\gamma'_\perp$ is the collisionally induced rate of decay (dephasing) of the transition dipole moments.

7. General model for circularly polarized fields and Zeeman basis states

Incorporating the dissipation and excitation rates and the Hamiltonian dynamics into a model for the evolution of the system, we obtain the following equations:

$$\begin{aligned}
\mathrm{d}E_R/\mathrm{d}t &= -\kappa E_R - \mathrm{i}\delta E_R + \kappa P_R\,, \\
\mathrm{d}E_L/\mathrm{d}t &= -\kappa E_L - \mathrm{i}\delta E_L + \kappa P_L\,, \\
\mathrm{d}P_R/\mathrm{d}t &= -\gamma_\perp P_R + \gamma_\perp E_R(D_{11} - D_b) + \gamma_\perp E_L C\,, \\
\mathrm{d}P_L/\mathrm{d}t &= -\gamma_\perp P_L + \gamma_\perp E_L(D_{1-1} - D_b) + \gamma_\perp E_R C^*\,, \\
\mathrm{d}C/\mathrm{d}t &= -\gamma_c C - (\gamma_a/4)[E_L^* P_R + E_R P_L^*]\,, \\
\mathrm{d}D_a/\mathrm{d}t &= -\gamma_a D_a + 3\gamma_a \sigma \\
&\quad - (\gamma_a/4)(E_R^* P_R + E_R P_R^* + E_L^* P_L + E_L P_L^*)\,,
\end{aligned}$$

$$\mathrm{d}Q_D/\mathrm{d}t = -\gamma_a Q_D - \gamma_c' Q_D - (\gamma_a/4)(E_R^* P_R + E_R P_R^* + E_L^* P_L + E_L P_L^*) ,$$

$$\mathrm{d}D_-/\mathrm{d}t = -\gamma_a D_- - \gamma_J' D_- - (\gamma_a/4)[E_R^* P_R + E_R P_R^* - (E_L^* P_L + E_L P_L^*)] ,$$

$$\mathrm{d}D_b/\mathrm{d}t = -\gamma_b D_b + \gamma_{ab} D_a + (\gamma_a/4)(E_L^* P_L + E_L P_L^* + E_R^* P_R + E_R P_R^*), \quad (6)$$

where the Es and Ps are variables rescaled as in the previous model for the slowly varying amplitudes of the circularly polarized components of the electric fields and the atomic dipole moment densities, Ds and Q_D are the suitably rescaled combinations of the population densities of the various sublevels corresponding to the Ns and Q defined in Eqs. (4) and (5), C is the rescaled coherence, and σ is the rescaled pump parameter. We have also used the definition $\gamma_\perp = (\gamma_a + \gamma_b)/2 + \gamma_\perp'$. These equations differ in structure from those of Lenstra [2] in the presence of the γ_{ab} term in the evolution of D_b which represents spontaneous decay of population into the lower level. Anisotropies in the losses or detunings (so-called amplitude or phase anisotropies) can be easily added to these equations in the manner discussed elsewhere [2,11,15]. Because the cavity loss rate is used in renormalizing the field variables more is involved in such an addition than simply changing the terms in the field equations. In addition, anisotropies in κ or δ in one basis (linear or circular) cause mixing of the fields in the other basis.

Representation of the equations in the Cartesian basis

The equations can be rewritten in terms of the similarly rescaled slowly varying amplitudes of the linearly polarized fields E_x and E_y using $E_{R,L} = (E_x \mp \mathrm{i}E_y)/\sqrt{2}$. Rewriting the atomic variables in terms of equivalent expressions for the amplitudes of the states $|1,x\rangle$ and $|1,y\rangle$ defined previously, we find

$$D_{1x} = (D_{1,1} + D_{1,-1})/2 + \mathrm{Re}\, C ,$$

$$D_{1y} = (D_{1,1} + D_{1,-1})/2 - \mathrm{Re}\, C ,$$

$$P_x = (P_L + P_R)/\sqrt{2} ,$$

$$P_y = \mathrm{i}(P_L - P_R)/\sqrt{2} ,$$

$$C_{xy} = -\mathrm{i}(D_{1,1} - D_{1,-1})/2 - \mathrm{Im}\, C ,$$

which implies

$$-\mathrm{Im}\, C = \mathrm{Re}\, C_{xy} ,$$

where Ds are the populations of the specified upper sublevel, Ps are the dipole moments between the specified upper state sublevel and the lower state, and C_{xy} involves the product of the amplitude of the $|1,x\rangle$ state and the complex conjugate of the amplitude of the $|1,y\rangle$, state in complete analogy with the definition of C for the amplitudes of the upper state Zeeman sublevels involved in the transition. As a consequence we have as equations for these variables:

$$\mathrm{d}E_x/\mathrm{d}t = -\kappa E_x - \mathrm{i}\delta P_x + \kappa P_x ,$$

$$\mathrm{d}E_y/\mathrm{d}t = -\kappa E_y - \mathrm{i}\delta P_y + \kappa P_y ,$$

$$\mathrm{d}P_x/\mathrm{d}t = -\gamma_\perp P_x + \gamma_\perp E_x (D_{1x} - D_b) + \gamma_\perp E_y C_{xy},$$

$$\mathrm{d}P_y/\mathrm{d}t = -\gamma_\perp P_y + \gamma_\perp E_y (D_{1y} - D_b) + \gamma_\perp E_x C_{xy}^*,$$

$$\mathrm{d}C_{xy}/\mathrm{d}t = -\gamma_J C_{xy} - (\gamma_c' - \gamma_J')\mathrm{Re}\, C_{xy} - (\gamma_a/4)[E_y^* P_x + E_x P_y^*],$$

$$\mathrm{d}D_{1,x}/\mathrm{d}t = -\gamma_c D_{1,x} + \gamma_a \sigma + (\gamma_c'/3)(D_{1,x} + D_{1,0} + D_{1,y}) - (\gamma_a/4)(E_x^* P_x + E_x P_x^*),$$

$$\mathrm{d}D_{1,0}/\mathrm{d}t = -\gamma_a D_{1,0} + \gamma_a \sigma + (\gamma_c'/3)(D_{1,x} - 2D_{1,0} + D_{1,y}) ,$$

$$\mathrm{d}D_{1,y}/\mathrm{d}t = -\gamma_c D_{1,y} + \gamma_a \sigma + (\gamma_c'/3)(D_{1,y} + D_{1,0} + D_{1,x}) - (\gamma_a/4)(E_y^* P_y + E_y P_y^*) ,$$

$$\mathrm{d}D_b/\mathrm{d}t = -\gamma_b D_b + \gamma_{ab}(D_{1,x} + D_{1,y} + D_{1,0}) + (\gamma_a/4)(E_y^* P_y + E_y P_y^* + E_x^* P_x + E_x P_x^*). \quad (7)$$

$P_x(P_y)$ is not only driven by $E_x(E_y)$ but also by $E_y(E_x)$. This is a consequence of the atomic sublevel basis of the couplings (which give the coherence term C) and provides the most distinctive difference in these realistic models from the "isotropic oscillator" model used, for example, by Siegman [34] and Gil [19]. If one is working with a reduced set of equations (by adiabatic elimination of the material variables), these differences lead to different couplings of the orthogonally polarized field amplitudes through different cross saturation terms and different ranges for the parameters.

The equations for the variables using the states $|x\rangle$ and $|y\rangle$ and linearly polarized fields have the same structure as those for the Zeeman basis states and the circularly polarized fields with the interchange of the roles of the decay rates γ_c and γ_J in the dissipation of the upper level coherence and the population difference of the radiatively active upper sublevels. We show in the next section that the difference of these rates governs the stability near threshold of the circularly polarized and linearly polarized solutions with respect to amplitude perturbations of the orthogonally polarized field, as is true for the model of Eqs. (2) and in prior models [2–8]. Hence if the circularly polarized solutions are stable (unstable) with respect to amplitude instabilities because of the ratio of these decay rates, the linearly polarized solutions are unstable (stable).

8. Recovering Eqs. (2) from the more general model

Eqs. (2) can be obtained from the more general set of Eqs. (6) by taking $\gamma'_c = 0$ and $\gamma_{ab} = 0$ which is the only way to fully decouple the variable $D_{1,0}$ from the dynamics in the presence of dissipation. These steps have several consequences.

(i) The inference from the Hamiltonian that the $|1,0\rangle$ level is decoupled from the dynamics is inconsistent with the requirements of isotropic collisions which give the same rate to the mixing of this level with the radiatively coupled sublevels (which is ignored in Eqs. (1) and (2)) as to the decay of the coherence (which is included in those same equations). In addition the conditions $\gamma_a = \gamma_b$ and $\gamma_{ab} = 0$, which are required if one is to deal only with population differences between the upper and lower states rather than the populations themselves, and $\gamma'_c = 0$ require that $\gamma_c = \gamma_a$.

(ii) For these parameter values there can only be stable circularly polarized solutions near threshold (except for elliptically polarized solutions when $\gamma_a = \gamma_c = \gamma_J$). The whole result is rather unphysical if collisional effects are included in nonzero values of γ'_J and $\gamma'_\perp$ while $\gamma'_c = 0$. In addition, γ_{ab} is usually of order γ_a and this requires that $\gamma_b > 3\gamma_a$ if there is to be any inversion at all.

However, one can recover the structure of Eqs.(2) if one takes several less severe approximations. Rather than "ignoring" $D_{1,0}$, one can assume it adiabatically follows the dynamics which may be slower than the γ_c rate for its relaxation. Solving for $D_{1,0}$ gives the expression

$$D_{1,0} = [3\gamma_a\sigma + \gamma'_c(D_{1,1} + D_{1,-1})]/(3\gamma_a + 2\gamma'_c),$$

which can be substituted into the equations for $D_{1,x}$, and $D_{1,y}$ or $D_{1,1}$ and $D_{1,-1}$. For example,

$$\begin{aligned} dD_{1,1}/dt = &-\gamma_a D_{1,1} \\ &+ 3\gamma_a\sigma[(\gamma_a + \gamma'_c)/(3\gamma_a + 2\gamma'_c)] \\ &- \gamma'_J(D_{1,1} - D_{1,-1})/2 \\ &- (\gamma'_c/6)(D_{1,1} + D_{1,-1})[3\gamma_a/(3\gamma_a + 2\gamma'_c)] \\ &- (\gamma_a/4)(E^*_R P_R + E_R P^*_R). \end{aligned}$$

The result is a higher pump rate for the radiatively coupled levels (by a factor of up to 3/2 if $\gamma_a \ll \gamma'_c$) and reduced dependence on the sum of the upper level populations (approaching zero if $\gamma_a \ll \gamma'_c$). Making a variant of the further approximation that is commonly made to reduce a two-level system to the structure of the Bloch equations (namely that the lower level population decay rate is equal to one half the decay rate of the sum of the populations of the radiatively active levels) and ignoring the spontaneous transitions that carry population from the upper states to the lower states by setting $\gamma_{ab} = 0$ (the most difficult to justify of these assumptions, though one also made by Lenstra [2]), the equations are given as follows for the renormalized variables describing the Zeeman atomic states and the amplitudes of the circularly polarized fields:

$$\begin{aligned} dE_R/dt &= -\kappa E_R - i\delta E_R + \kappa P_R, \\ dE_L/dt &= -\kappa E_L - i\delta E_L + \kappa P_L, \\ dP_R/dt &= -\gamma_\perp P_R + \gamma_\perp E_R D_R + \gamma_\perp E_L C, \\ dP_L/dt &= -\gamma_\perp P_L + \gamma_\perp E_L D_L + \gamma_\perp E_R C^*, \\ dC/dt &= -\gamma_c C - (\gamma_a/4)[E^*_L P_R + E_R P^*_L], \\ dD_+/dt &= -\gamma_{||}(D_+ - 2\sigma) \\ &\quad - (3\gamma_a/4)[E^*_L P_L + E_L P^*_L + E^*_R P_R + E_R P^*_R], \\ dD_-/dt &= -\gamma_J D_- - (\gamma_a/4)[(E^*_R P_R + E_R P^*_R) \\ &\quad - (E^*_L P_L + E_L P^*_L)], \end{aligned}$$

D_R, D_L, D_+ and D_- have the same meanings as in Eqs. (2); $\gamma_{||}(= \gamma_a[3(\gamma_a + \gamma'_c)/(3\gamma_a + 2\gamma'_c)])$ is the decay rate of the population inversions modified by the correction factor from the adiabatic elimination of $D_{1,0}$. One further rescaling of Es and Ps by $(\gamma_{\|}/\gamma_a)^{1/2}$ brings these equations to the form of the modified model of Section 2. In reaching this set of equations by adiabatic elimination of $D_{1,0}$ we find the same terms (the same structure of the equations in terms of couplings of the variables) as in the model of Section 2, though with rescaled variables and coefficients and with different accessible limits for the parameters. Because the modified $\gamma_{\|}$ is bigger than γ_a, it is possible in systems with low collision rates for γ_c and γ_J to be less than $\gamma_{\|}$ which opens several previously not considered parts of the parameter space to exploration for potentially physically observable behavior. However, since in most laser systems $\gamma_b \gg \gamma_a$ and $\gamma_{ab} \neq 0$, it is more reasonable to explore the complete set of equations for novel forms of possible observable behavior.

9. Reduction of the general model to third-order Lamb theory

The analysis of our general model close to threshold gives a further justification for Eq. (2), since the essential stability features of linear versus circularly polarized solutions are shown to be well described by the model introduced in section 2. In addition, this analysis identifies the role of the dissipative rate γ_{ab}, which was not included in the general analysis of Lenstra [2] close to threshold.

A third-order Lamb theory valid close to threshold is obtained by adiabatic elimination of the material variables in Eqs. (6). We first eliminate the variables associated with population of the different sublevels obtaining:

$$\begin{aligned} D_{1,1} - D_b &= (2D_a + 3D_- + Q)/6 - D_b \\ &= \sigma' - \{[\alpha + (\gamma_a/\gamma_J)](E_R^* P_R + E_R P_R^*) \\ &\quad - [\alpha - (\gamma_a/\gamma_J)](E_L^* P_L + E_L P_L^*)\}/8 , \\ D_{1,-1} - D_b &= (2D_a - 3D_- + Q)/6 - D_b \\ &= \sigma' - [(\alpha - (\gamma_a/\gamma_J))(E_R^* P_R + E_R P_R^*) \\ &\quad - (\alpha + (\gamma_a/\gamma_J))(E_L^* P_L + E_L P_L^*)]/8 , \end{aligned}$$

where

$$\sigma' = \sigma(1 - 3\gamma_{ab}/\gamma_b) ,$$

$$\alpha = [2 + (\gamma_a\gamma_c)]/3 + 2(\gamma_a - \gamma_{ab})/\gamma_b .$$

When $\delta = 0$, the coherence C and dipole polarizations can be further adiabatically eliminated easily because there is a common phase for E_R and P_R and also a common phase for E_L and P_L. The usual third order expansion yields the following set of reduced equations for the field amplitudes:

$$dE_R/dt = (\sigma' - 1)E_R - \beta[|E_R|^2 + \gamma|E_L|^2]E_R ,$$

$$dE_L/dt = (\sigma' - 1)E_L - \beta[|E_L|^2 + \gamma|E_R|^2]E_L ,$$

where time has been rescaled with the inverse cavity lifetime κ, and the self and cross-saturation parameters are given by

$$\beta = (\sigma'/4)[\alpha + (\gamma_a/\gamma_J)] ,$$

$$\gamma = 1 + [2\gamma_a(\gamma_J - \gamma_c)/\gamma_J\gamma_c]/[\alpha + (\gamma_a/\gamma_J)] .$$

For this reduced model circularly polarized solutions with either $E_R = 0$ or $E_L = 0$, are stable for $\gamma > 1$, while linearly polarized solutions are stable for $\gamma < 1$, and elliptically polarized solutions occur for $\gamma = 1$.

The spontaneous decay of the population into the lower level, as taken into account by γ_{ab}, implies the existence of a higher effective threshold $\sigma' = 1$. This is a small modification since typically $\gamma_b \gg \gamma_a > \gamma_{ab}$. In addition, for a these values of parameters, it is clear that $\alpha > 0$ and the crossover from linearly to circularly polarized light occurs for $\gamma_J = \gamma_c$, independent of γ_{ab}.

A third-order Lamb theory of Eqs. (2) yields the same form of reduced equations with σ' replaced by σ, γ_a replaced by $\gamma_{\|}$ and $\alpha = 3$. As a consequence, our more general model and the one in section 2 differ in a small shift of the threshold value due to the nonvanishing γ_{ab}, but have the same stability boundary between circular and linearly polarized light.

10. Summary and conclusions

We have demonstrated how a more approximate model [11] for the combined dynamics of material variables, intensity and polarization state in lasers with isotropic cavities interacting with media with $J = 1 \rightarrow J = 0$ transitions can be generalized by keeping important effects discussed in Ref. [2] and by adding the full effects of spontaneous emission transitions between the two lasing levels as well. We have shown that this general model can be used to explore these complex dynamics in both what are called "weak coupling" and "strong coupling" limits (by adjustments of the decay rates of different tensorial orders of the $J = 1$ sublevel density matrix), thereby providing an opportunity to go beyond the limitations imposed by third-order Lamb theory models. While similar models were constructed previously (see Ref. [2] for an example and a detailed review), their prompt reduction to third-order Lamb theory models for coupled mode amplitudes prevented the study of more complex dynamics made possible by the evolution of the material variables. We have also illustrated the neutrality of the coupling of the polarized modes by the $J = 1 \to J = 0$ atomic transition by explicitly constructing a Cartesian basis pair of sublevels of the $J = 1$ state which are radiatively coupled to the lower level by linearly polarized fields.

As in previously analyzed models [2–9], near the lasing threshold the stability of circularly or linearly polarized solutions is determined by whether the ratio of the decay rates of the magnetic dipole and the electric quadrupole (coherence) of the sublevels of the $J = 1$ level is greater or less than unity, respectively.

We have also found the thresholds for instabilities which may cause pulsations of the total intensity and/or the ellipticity (such as were found by Puccioni et al. in their model [11] which we have generalized) involving the dynamics of the material variables. These instabilities are neglected in models of only coupled field amplitudes, such as those models which use third-order Lamb theory, because of the critical involvement of the material variables. Thresholds for these instabilities may be relatively close to the lasing threshold and they exist regardless of whether it is the linearly polarized or circularly polarized solutions which are stable just above the lasing threshold. With the growing interest in material dynamics as part of polarization dynamics in various lasers, our result provides several models for some of those studies. We have also shown that the approximate model of Puccioni et al. has substantial qualitative similarity to our more general model so that detailed analyses of it are likely to provide generic insight for dynamics that may be observed for $J = 1 \to J = 0$ lasers (and perhaps for lasers with other transitions for which material dynamics are important).

We have discussed how polarization state selection in a laser is determined by quantum coherences. Explicit consideration of such coherences permits identification of the physics of "no lasing in spite of inversion above threshold". This provides a second example (after Ref. [26]) in which lasing in multilevel systems involves the coherence phenomena associated with lasing without inversion, electromagnetically induced transparency, and inversion with absorption.

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