I·M·E·D·E·A Institut Mediterrani d'Estudis Avançats

Neutrally Buoyant Particles and Bailout embedding in 3D flows

Idan Tuval and Oreste Piro

Dpto de Física Interdisciplinar, IMEDEA, (CSIC-UIB), Palma, Spain

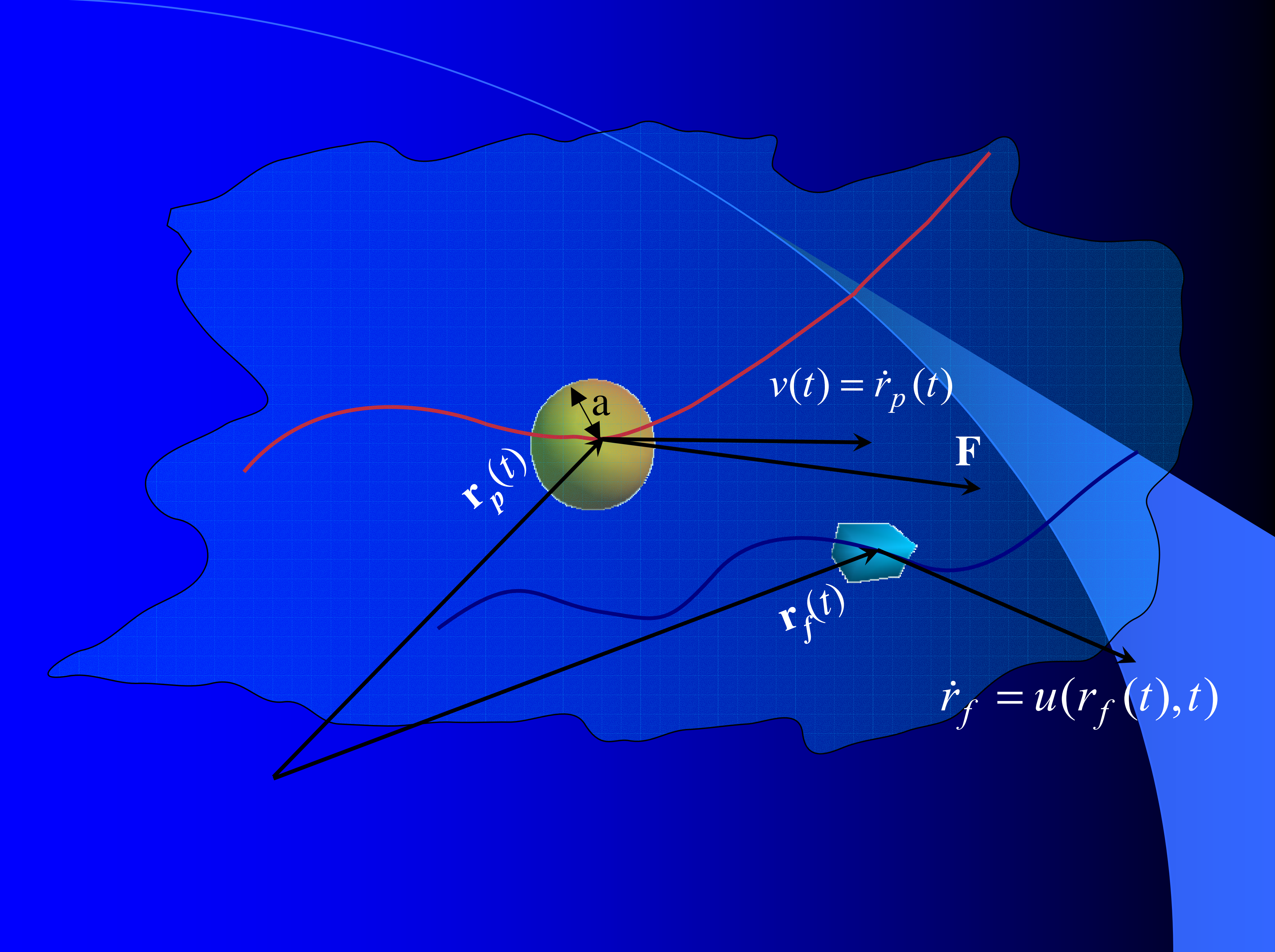
Marcelo Magnasco

Mathematical Physics Lab, Rockefeller University, New York, USA

Julyan Cartwright

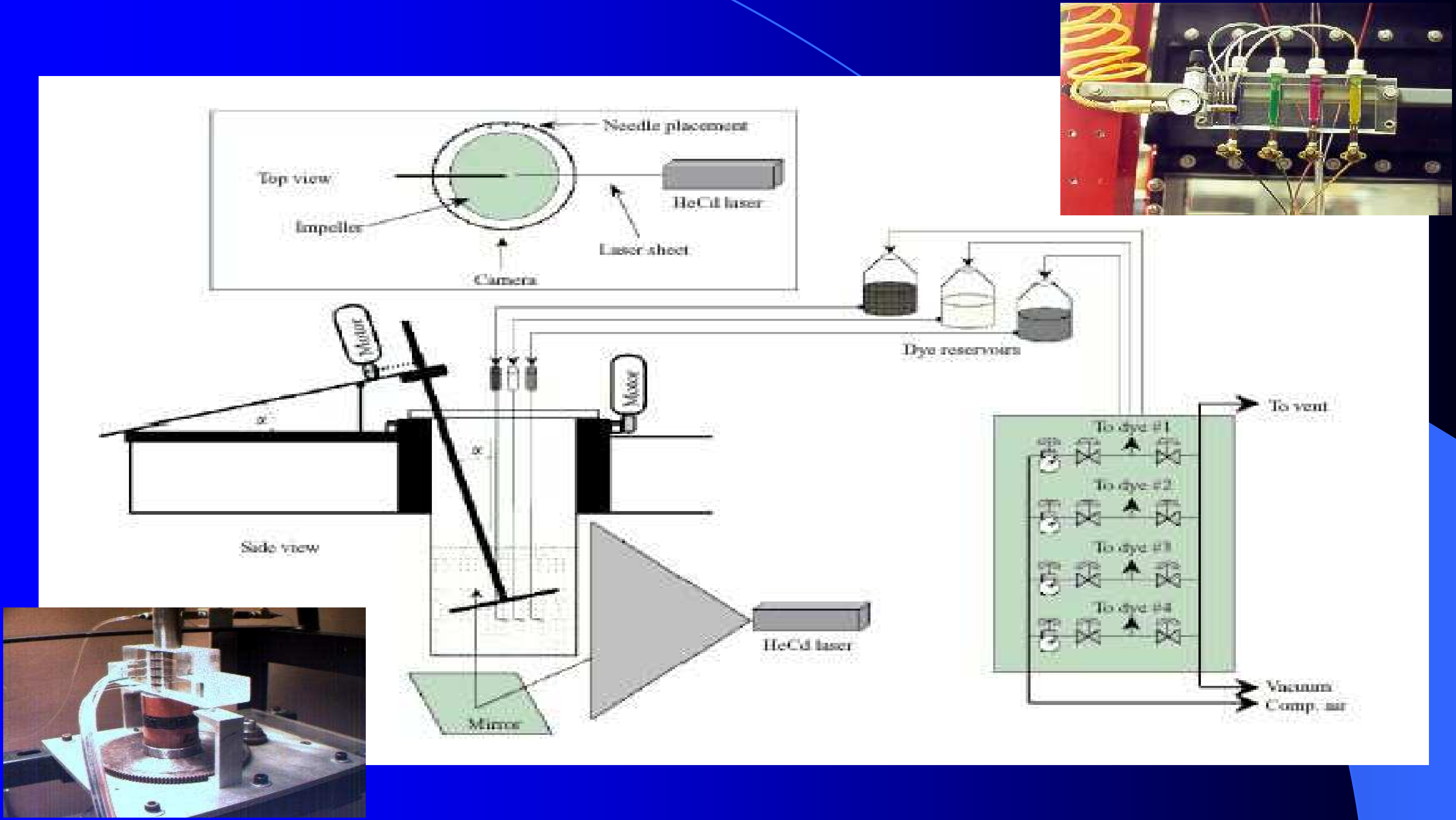
Laboratorio de Estudios Cristalográficos, IACT (CSIC-UGR), Granada, Spain.

Is an ideal tracer Anything else than a dream?



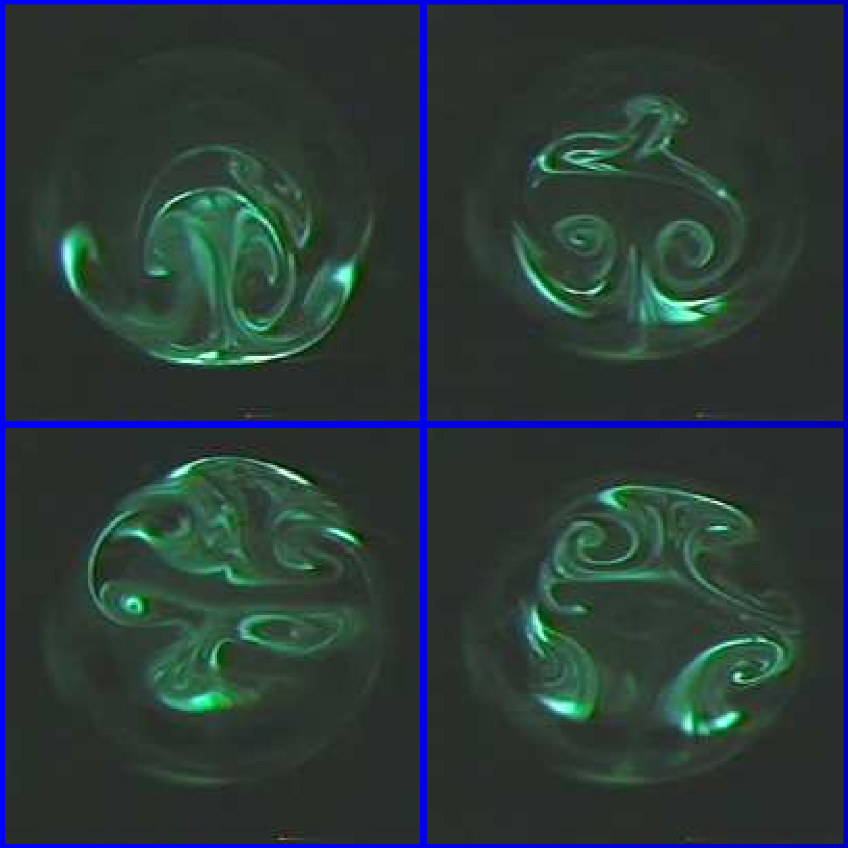
a
$v(t) = \dot{r}_p(t)$
F
$\mathbf{r}_p(t)$
$\mathbf{r}_f(t)$
$\dot{r}_f = u(r_f(t), t)$

Experimental 3D flows

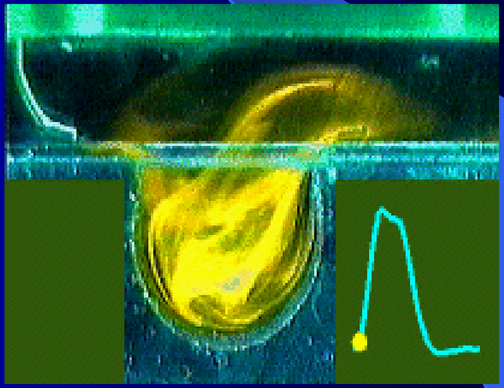


G.O.Fountain, D.V.Khakhar et al., SCIENCE, 281 (1998)

Experimental flow's visualization

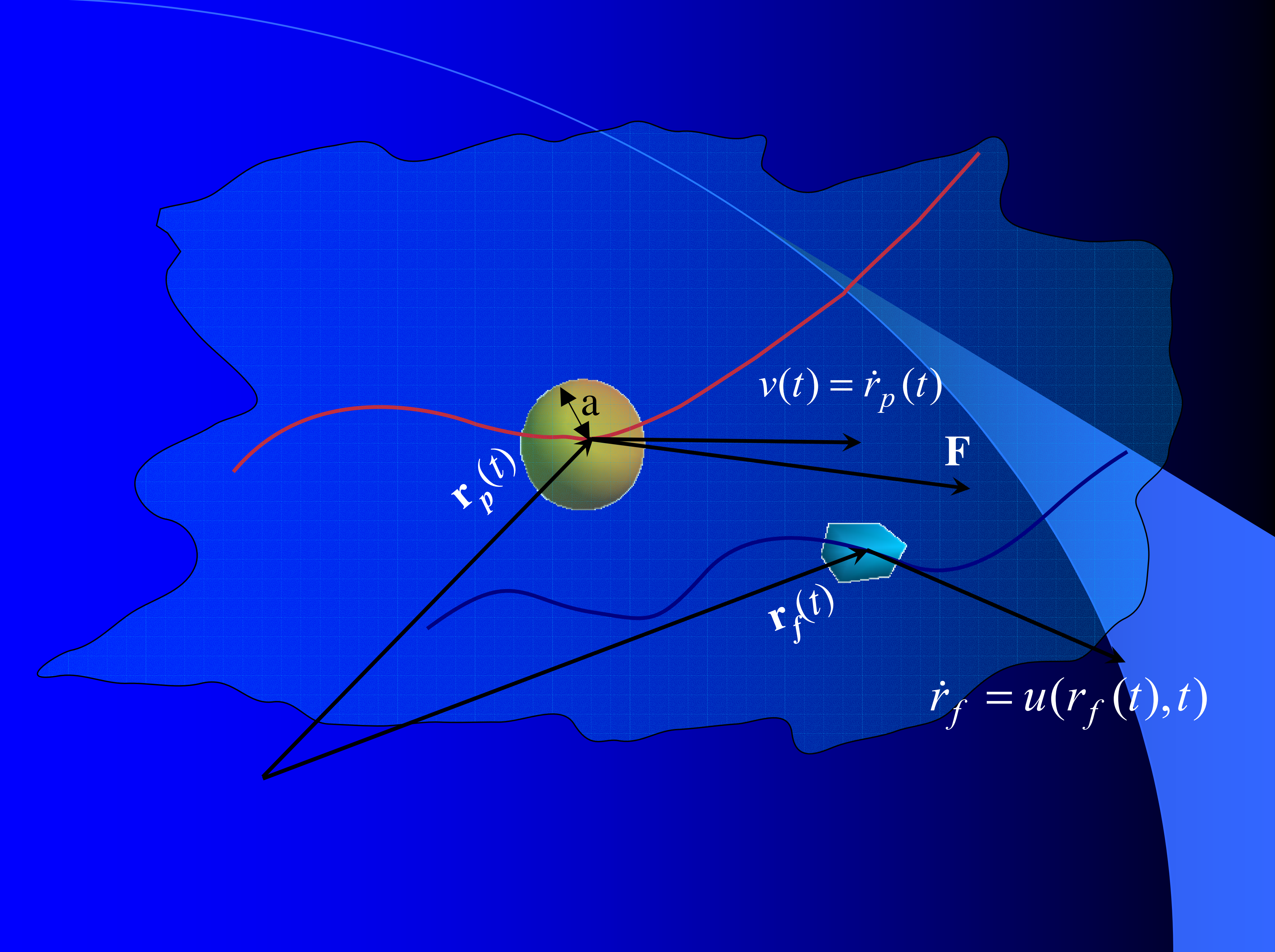


Aerodynamical Properties



Blood vessel ANEURISM

Click the image twice for movie



a
$v(t) = \dot{r}_p(t)$
$\mathbf{F}$
$\mathbf{r}_p(t)$
$\mathbf{r}_f(t)$
$\dot{r}_f = u(r_f(t), t)$

$$\rho_p \frac{dv}{dt} = \rho_f \frac{Du}{Dt} + (\rho_p - \rho_f)\cdot g$$

$$- \frac{9\nu\rho_f}{2a^2}\left(v - u - \frac{a^2}{6}\nabla^2 u\right)$$

$$- \frac{\rho_f}{2}\left(\frac{dv}{dt} - \frac{D}{Dt}\left[u + \frac{a^2}{10}\nabla^2 u\right]\right)$$

$$- \frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{1}{\sqrt{t-\zeta}}\frac{d}{d\zeta}\left(v - u - \frac{a^2}{6}\nabla^2 u\right)\cdot d\zeta$$

v = particle´s velocity
u = fluid velocity
ρ_f = fluid density
ρ_p = particle density
a = radius of the particle
g = gravity

M.R.Maxey, J.J.Riley. Phys.Fluids 26, 883 (1983)

The Maxey-Riley equation for neutrally buoyant particle:

$$\frac{dv}{dt} = \frac{Du}{Dt} - St^{-1}(v-u) - \frac{1}{2}\left(\frac{dv}{dt} - \frac{Du}{Dt}\right)$$

Where:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u$$ Along the path of a fluid element

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (v \cdot \nabla)u$$ Along the trajectory of the particle

Incorrectly $\longrightarrow \frac{Du}{Dt} = \frac{du}{dt} \longrightarrow \frac{d}{dt}(v-u) = -\frac{2}{3}St^{-1}(v-u) \longrightarrow$ **Ideal tracer**

Correctly $\longrightarrow \frac{Du}{Dt} \neq \frac{du}{dt} \longrightarrow \frac{d}{dt}(v-u) = -[(v-u)\cdot\nabla]u - \frac{2}{3}St^{-1}(v-u)$

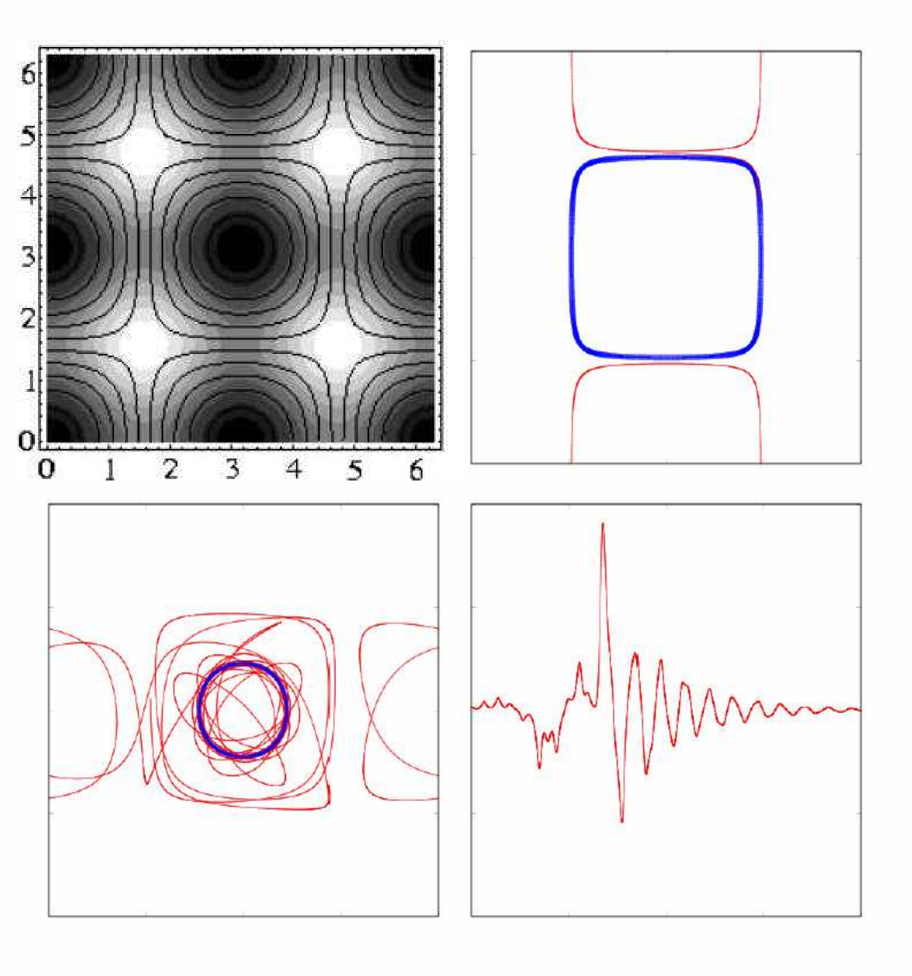
$$\delta \equiv \dot{x} - u$$

$$\delta = 0 \rightarrow \dot{x} = u$$

2D steady flow with hyperbolic fixed points.

$$\psi(x,y) = A\cos(x)\cos(y)$$

$$\begin{cases} U_x = \partial_y \psi \\ U_y = -\partial_x \psi \end{cases}$$



Neutrally buoyant particles:

$$\frac{d}{dt}(v - \mathbf{u}) = -(v - \mathbf{u}) \cdot \left(\frac{2}{3} St^{-1} \mathbf{I} + \nabla \mathbf{u} \right)$$

A.Babiano, J.H.E Cartwright, O.Piro and A.Provenzale, P.R.L, 84 (2000)

Bailout embedding

$$\frac{dx}{dt} = f(x)$$

-Invariant Submanifold.

-Dissipative System.

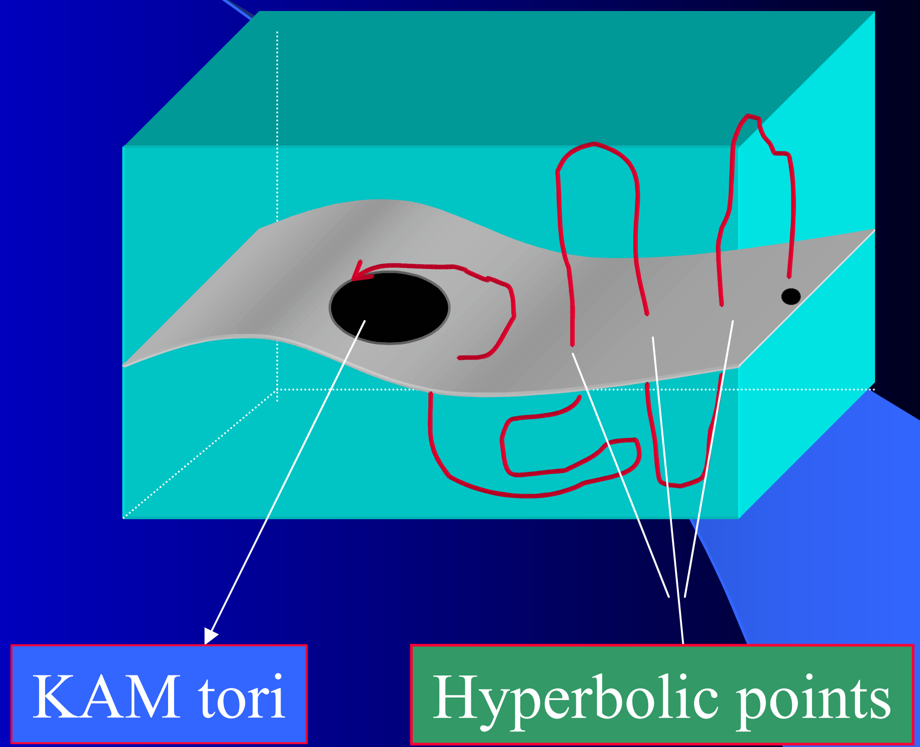
-Asymptotic Set.

Bailout embedding:

$$\frac{d}{dt}\big(\dot{x} - f(x)\big) = -k(x)\big(\dot{x} - f(x)\big)$$

k(x)>0

k(x)<0



Bailout function:

$$k(x) = (\lambda + \nabla u)$$

2D flows:

$$\nabla u = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$$

$\mu \in R$

$\mu \in I$

Map Bailout

The ODE bailout can be extended to maps in the obvious fashion:

$$\boxed{x_{n+1} = f(x_n)}$$

$$(x_{n+2} - f(x_{n+1})) = e^{k(x_n)} \cdot (x_{n+1} - f(x_n))$$

where as before, k(x) is an indicator function, < 0 on undesired orbits. Specializing to the case $k \approx \nabla f - \lambda$ is straightforward:

$$(x_{n+2} - f(x_{n+1})) = e^{-\lambda} \nabla f\big|_{x_n} \cdot (x_{n+1} - f(x_n))$$

Map Bailout at Work in 2D

Let f be the "Standard Map":

$$x_{n+1} = x_n + \frac{k}{2\pi}\sin(2\pi y_n)$$
$$y_{n+1} = y_n + x_{n+1}$$

Its bailout map is a 4D recurrence

$$x_{n+1} = T(x_n)$$

$$\delta_{n+1} = e^{-\lambda} \nabla T \big|_{x_n} \cdot \delta_n$$

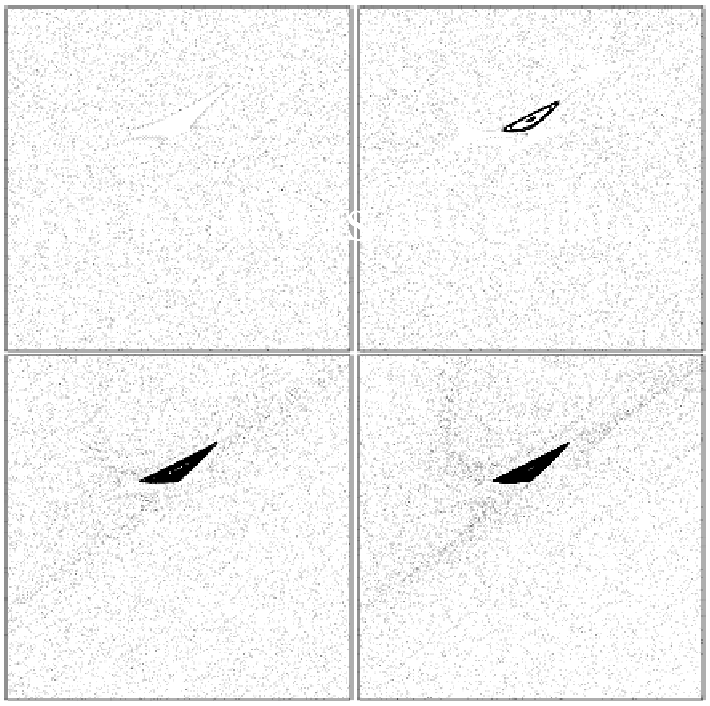


FIG. 3: The standard map for $k = 7$ has a chaotic sea covering almost the entire torus, except for a tiny period-two KAM island near position $0, 2/3$. 1000 random initial conditions were chosen, iterated for 20000 steps, then the next 1000 iterations are shown. The images here are a box $-0.05 < x < 0.05$, $0.61 < p < 0.71$. (a) Original map, (b) $\gamma = 1.4$, (c) $\gamma = 1.3$, (d) $\gamma = 1.2$.

Increasingly chaotic for increasing k

J.H.E Cartwright, M.O. Magnasco, O.Piro, Phys. Rev. E, 65 (2002)

ODE Bailout at Work in a Hamiltonian flow

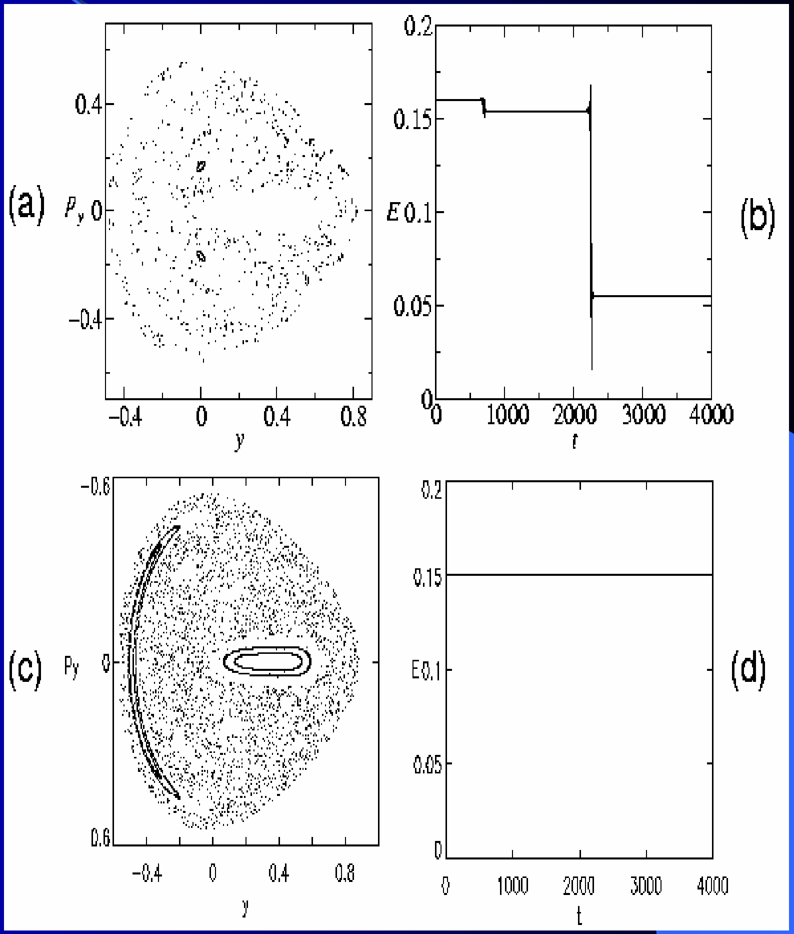
Let f be the Henon-Heiles flow:

$$H(x,y)=\frac{1}{2}(x^2+y^2+p_x^2+p_y^2)+x^2y-\frac{y^3}{3}$$

Constant Energy modification:

$$\begin{cases}\dot{x}=f(x)\\ E\equiv 0\end{cases}\rightarrow f\cdot\nabla E=0$$

$$\ddot{x}\cdot\nabla E+\dot{x}\cdot\nabla\nabla E\cdot\dot{x}=0$$



Noisy dynamics

Invariant densities are powerful...

A lot of progress has been made in dynamical systems theory by studyng the invariant measures (=probability densities) of a dynamical system. The most interesting one is the BRS measure (after Bowen-Ruelle-Sinai). This measure is obtained by histogramming how often the dynamics visits each point in phase space. There is one per attracting set.

There are powerful functional methods associated with this formulation, since the invariant measure satisfies a Liouville transport equation which is Fokker-Planck-like. Thus while they are ‘elementary’ objects coming out straigth from the dynamics, they can be studied with the full might of functional analysis.

But no interesting invariant measures for Hamiltonian systems !!!

Study of Hamiltonian systems and KAM theory generally has been hampered by theuninteresting features of their measures:

• if the system is ergodic, then it's uniform (Lebesgue) automatically due to the Liouville theorem.

• if it isn't, then it does not have a unique invariant measure to begin with: in the case of KAM systems, the measure disgregates into a millefeuille of KAM lamins and ergodic regions.

Adding a white noise (=a thermal bath) makes the system ergodic and the Lebesgue measure is the unique invariant measure. Boring!

We will hack a framework within which we get interesting invariant measures for Hamiltonian systems.

Noisy bailout dynamics

$$\delta_{n+1} = e^{-\lambda} \nabla T\big|_{x_n} \cdot \delta_n + \xi_n$$

Under the assumption that the δ are infinitesimally small, then we get the classical orbits $x_{n+1}=T(x_n)$ and then we can explicitly write down the solution for the δ. And evaluate the expectation value of δ^2 :

$$\tau(x) = \frac{\langle \delta^2 \rangle}{\langle \xi^2 \rangle} = \sum \left(e^{-j\lambda} \cdot \prod \nabla T\big|_{x_{n-k}} \right)^2$$

Where the < > are averages over the ξ process.

We can compute explicitly the above expression which depends only on the current value of the position, defining a sort of "temperature" amplitude τ(x) for the fluctuations δ.

Detachment

Thus, when λ equals the local Lyapunov exponent at x, the series defining τ(x) stops being absolutly convergent at x and may blow up. As λ is lowered further, more and more points x have local Lyapunov exponents greater than λ and so τ(x) formally diverges at more and more points x.

Where $\tau(x) = \infty$ it means that $< \delta^2 >$ is finite even if $< \xi^2 >$ is infinitesimally small. Thus the embedding trajectories have detached from the actual trajectories, and the approximation given above break down.

Detachment is the process that we first naively envisioned as defining bail-out embeddings.

Avoidance

But when λ is large enough

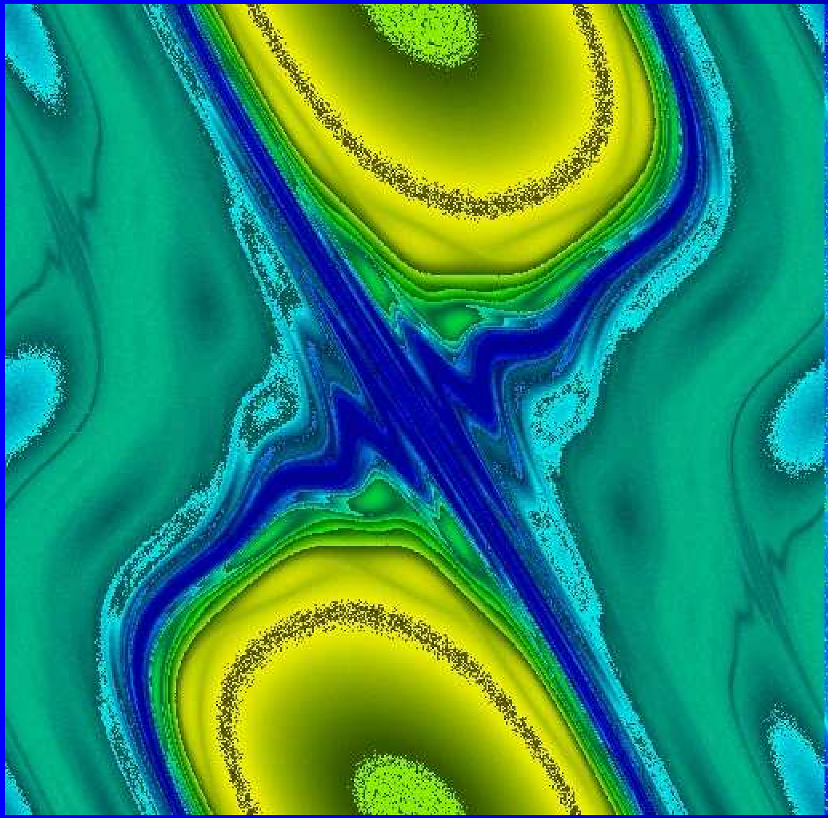
$$\tau(x) = \frac{\langle \delta^2 \rangle}{\langle \xi^2 \rangle} = \sum \left(e^{-j\lambda} \cdot \prod \nabla T \big|_{x_{n-k}} \right)^2$$

converges to a well defined function $\tau(x)$ of the space that modulates the fluctuations of δ. Particles tend to avoid the “hot” regions (large local Lyapunov exponents) of the dynamics while “cold” regions are visited more often.

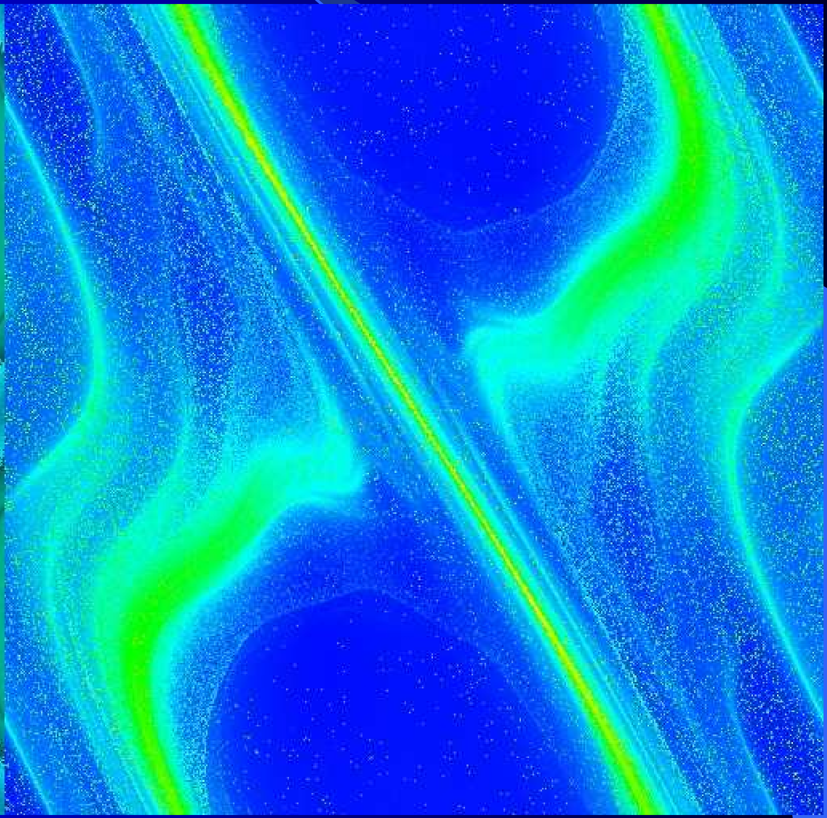
“Temperature” and Distributions

2D (Standard Map)

Particle Distribution



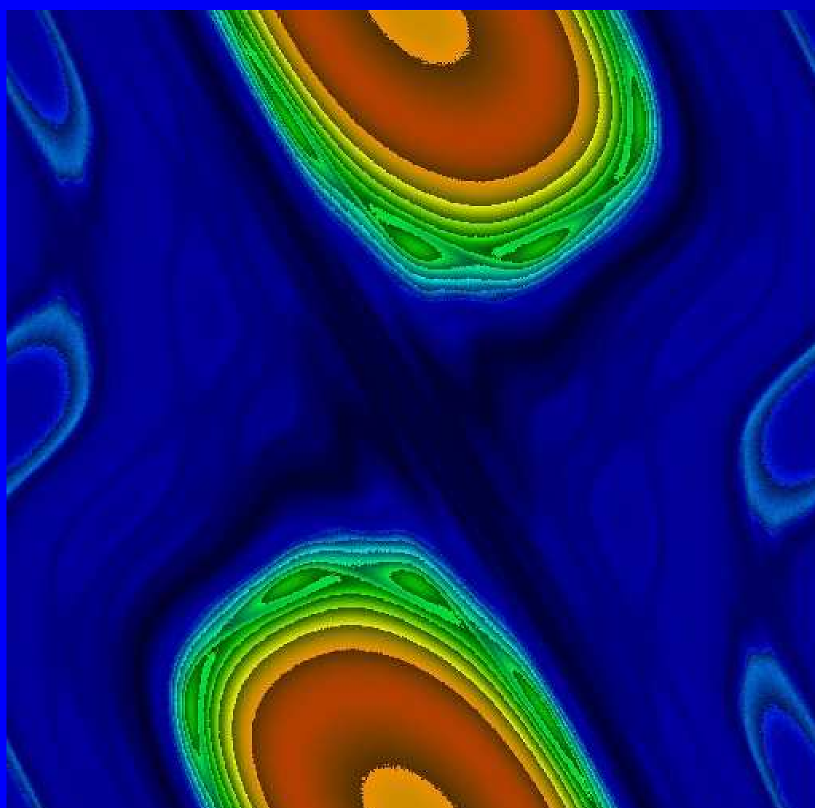
Temperature



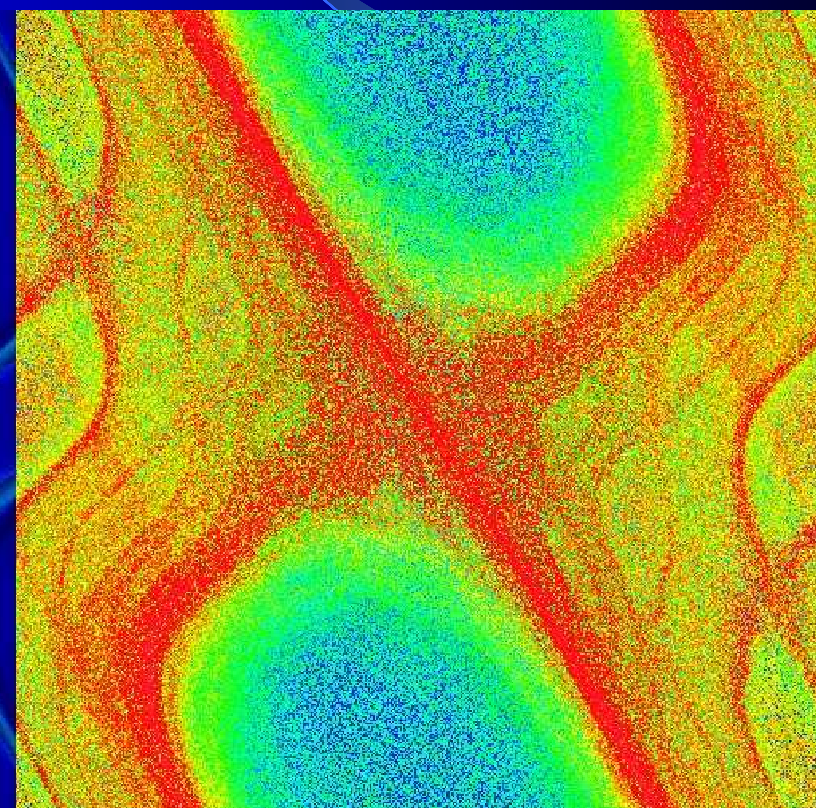
“Temperature” and Distributions

2D (Standard Map)

Particle Distribution

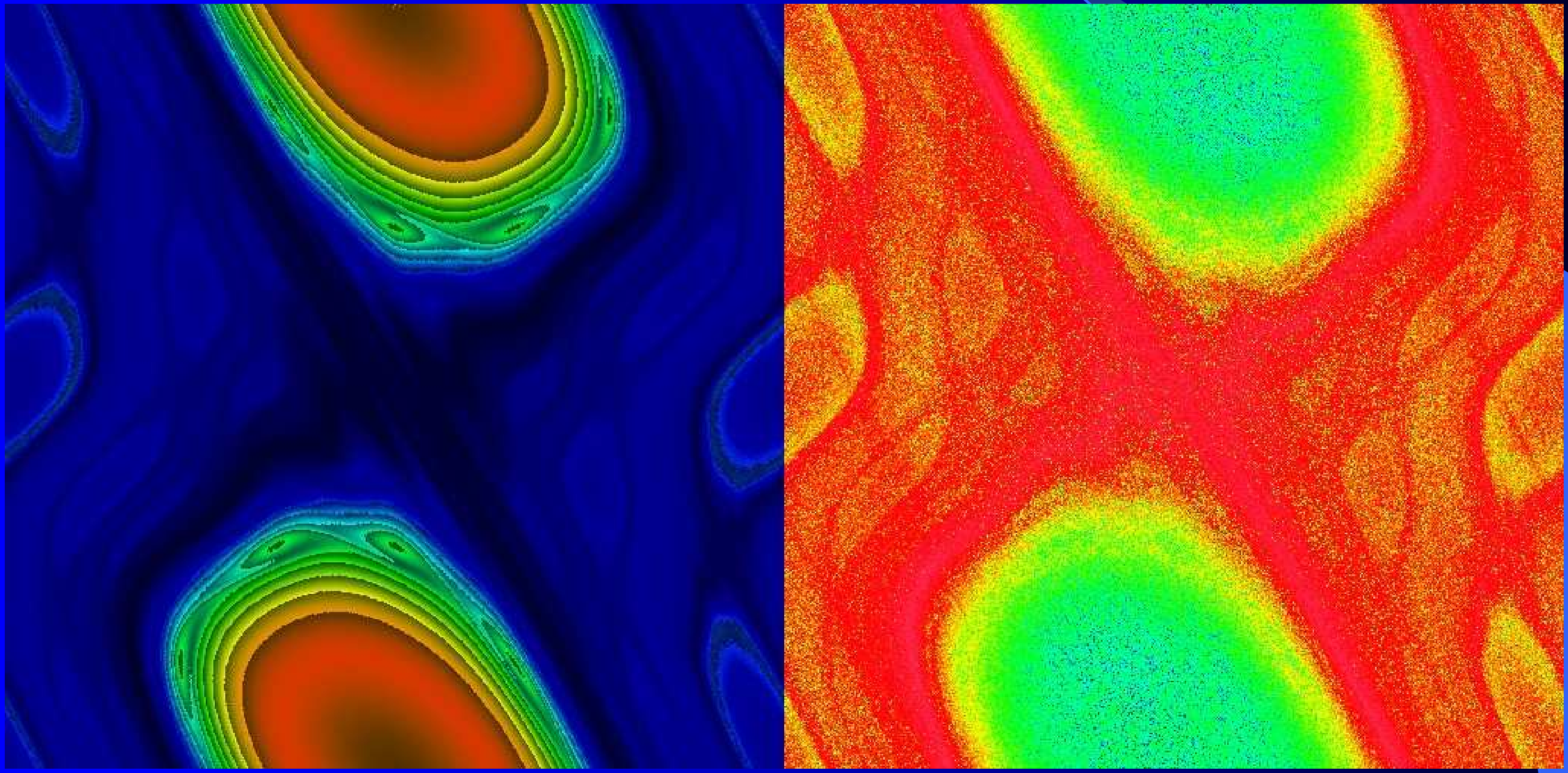


Temperature



"Temperature" and Distributions
2D (Standard Map)

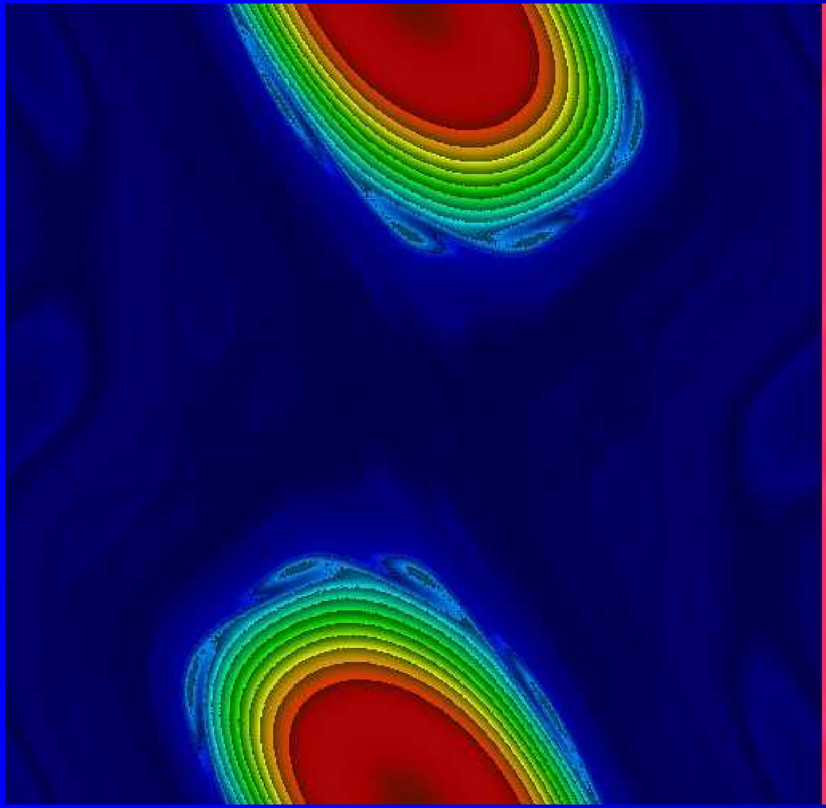
Particle Distribution   Temperature



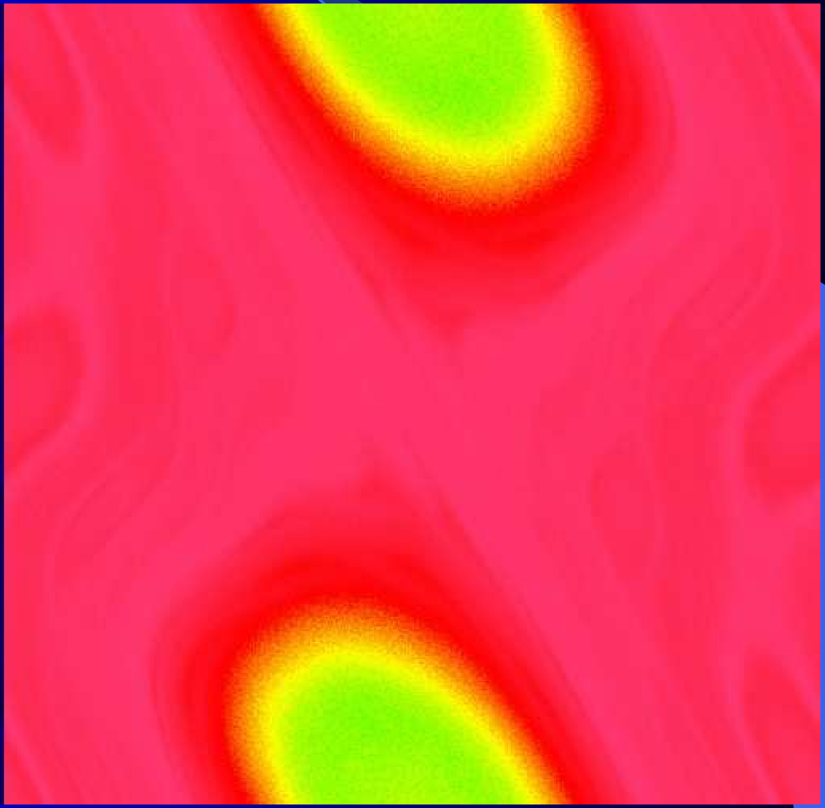
“Temperature” and Distributions

2D (Standard Map)

Particle Distribution



Temperature



3D flows

- Obvious importance from the point of view of applications.
- Very few simple time-dependent 3D incompressible flow models there exist. Alternative (qualitative) approach: to model the flow by iterated 3D volume preserving maps.
- ABC maps, that display many of the basic features of the evolution of the fluid flows of our interest.

ABC Maps

Integrable map: $I' = I$

$\theta' = \theta + \omega(I)$

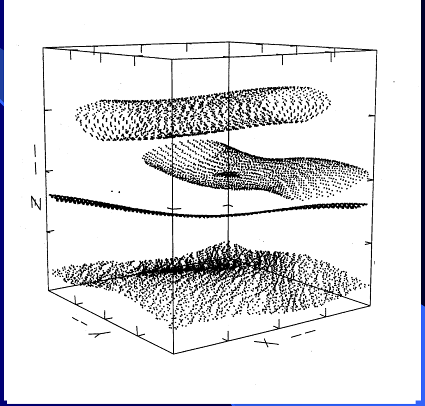
$$x' = x + A_1 \operatorname{sen} z + C_2 \cos y$$
$$y' = y + B_1 \operatorname{sen} x' + A_2 \cos z$$
$$z' = z + C_1 \operatorname{sen} y' + B_2 \cos x'$$

Action-angle-angle: motion on two dimensional Tori

Resonant Tori break-up into tubes

Chaotic volumes around hyperbolic lines

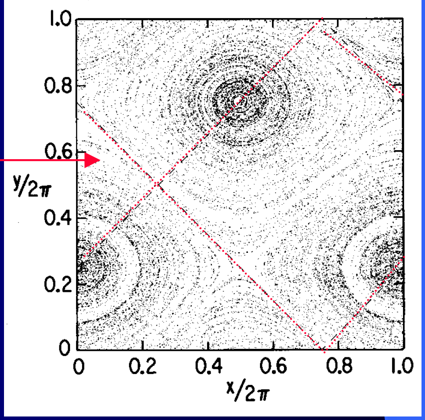
Non-resonant tori persist



Action-action-angle: invariant circles that merge into adiabatic invariant surfaces

Resonant surfaces

$\omega(I_1, I_2) = 2\pi k/n$



M.Feingold, L.P.Kadanoff, O.Piro, J.Statist.Phys, 50, 529 (1988)

ABC Maps

Integrable map: I'=I

θ'=θ+ω(I)

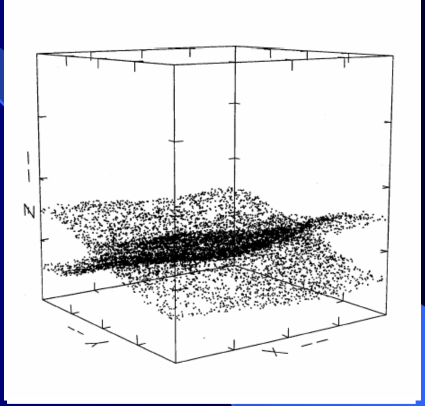
$$x' = x + A_1 \,\mathrm{sen}\, z + C_2 \cos y$$
$$y' = y + B_1 \,\mathrm{sen}\, x' + A_2 \cos z$$
$$z' = z + C_1 \,\mathrm{sen}\, y' + B_2 \cos x'$$

Action-angle-angle: motion on two dimensional Tori

Resonant Tori break-up into tubes

Chaotic volumes around hyperbolic lines

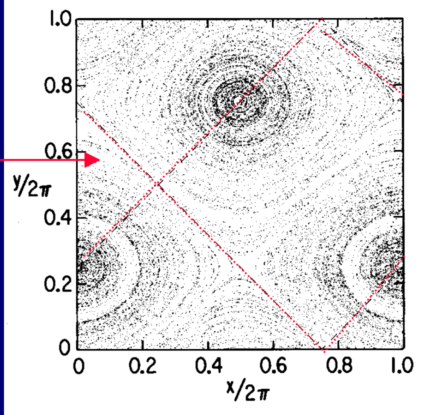
Non-resonant tori persist



Action-action-angle: invariant circles that merge into adiabatic invariant surfaces

Resonant surfaces

$\omega(I_1, I_2) = 2\pi k/n$



M.Feingold, L.P.Kadanoff, O.Piro, J.Statist.Phys, 50, 529 (1988)

Bailout embedding for the ABC map

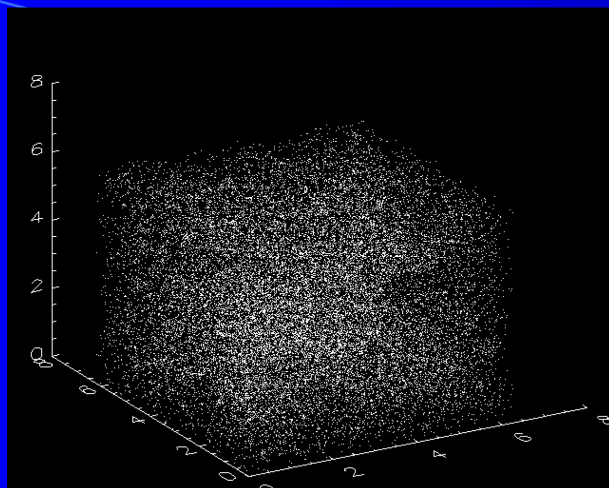
$$T(\vec{x}):(x_n, y_n, z_n) \rightarrow (x_{n+1}, y_{n+1}, z_{n+1})$$

$$\begin{cases} x_{n+1} = x_n + A_1 sen z_n + C_2 \cos y_n \\ y_{n+1} = y_n + B_1 sen x_{n+1} + A_2 \cos z_n \\ z_{n+1} = z_n + C_1 sen y_{n+1} + B_2 \cos x_{n+1} \end{cases}$$

$$\boxed{(x_{n+2} - T(x_{n+1})) = e^{-\lambda} \nabla T \big|_{x_n} \cdot (x_{n+1} - T(x_n))}$$

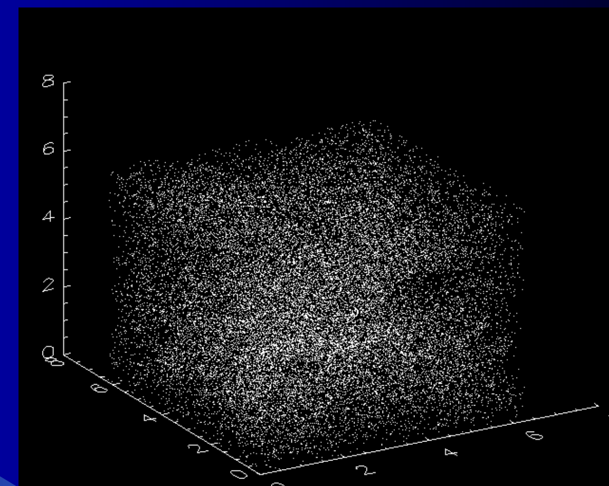
One action case

Neutrally buoyant particle



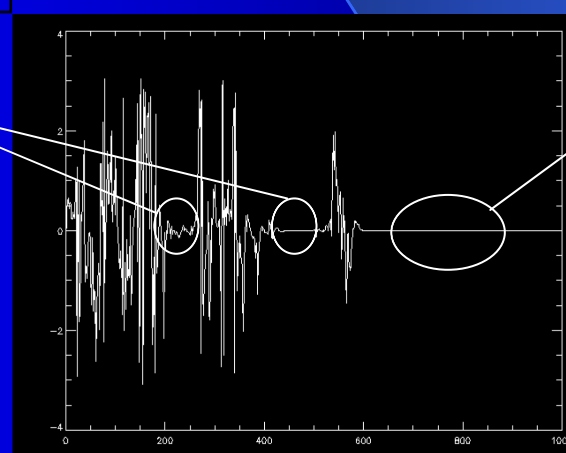
Click the image twice for movie

Fluid parcel



Click the image twice for movie

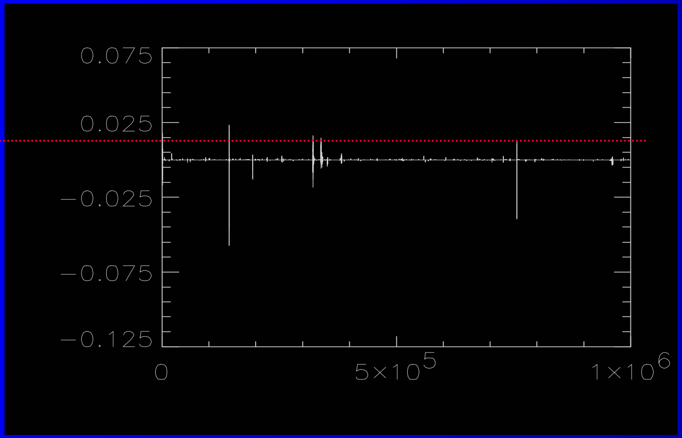
Near hyperbolic points



Colapse to the KAM tori

Two actions case

δ_0

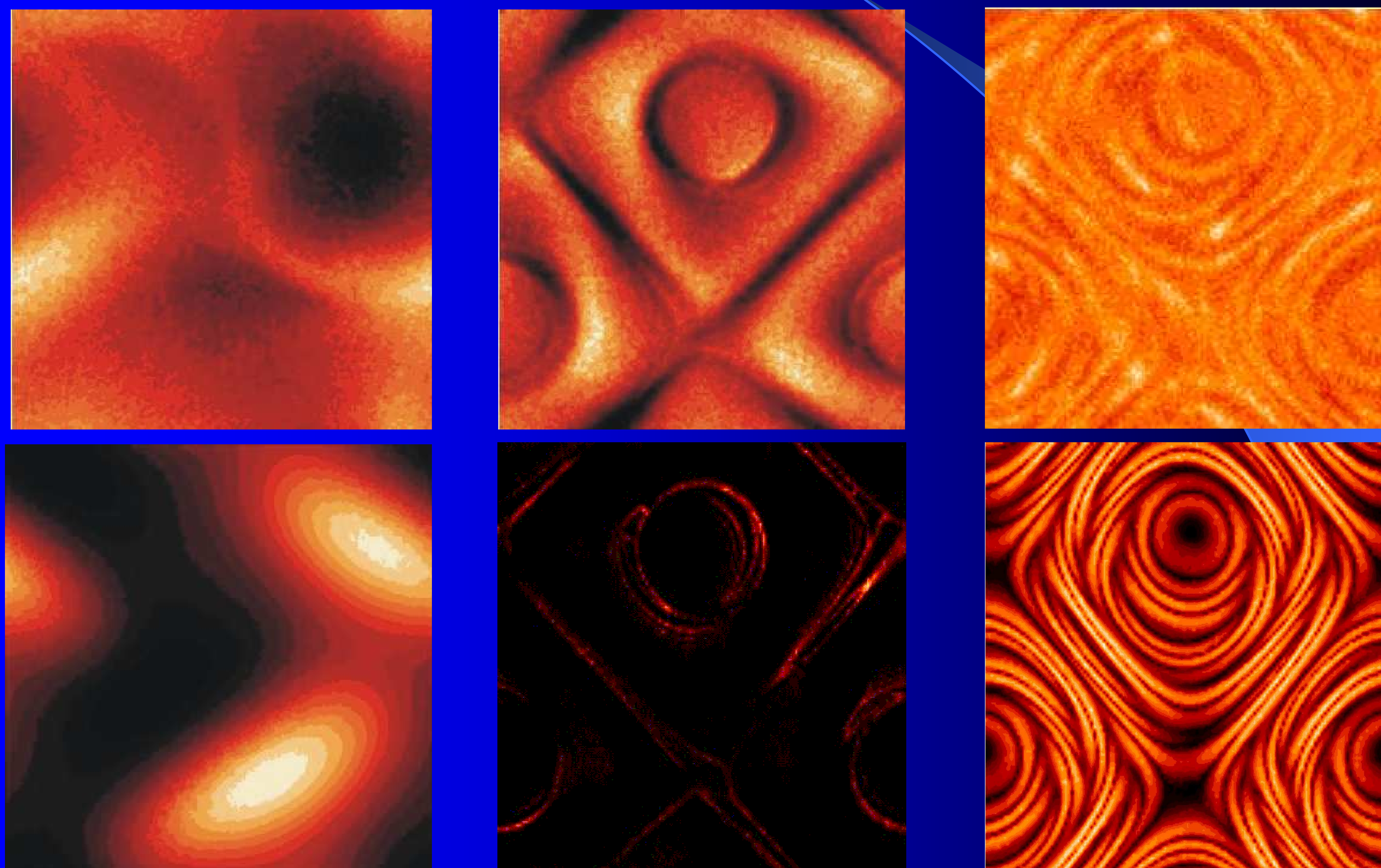


Resonant Structure

"Temperature" and Distributions

3D slices (ABC Map)

1 action | 2 actions | Higly chaotic



Continuous time verification.

Time-periodic ABC flow

$$\frac{dx}{dt} = (1+\sin\, 2\pi t)\cdot(A\sin z + C\cos y),$$
$$\frac{dy}{dt} = (1+\sin\, 2\pi(t+\frac{1}{3}))\cdot(B\sin x + A\cos z),$$
$$\frac{dz}{dt} = (1+\sin\, 2\pi(t+\frac{2}{3}))\cdot(C\sin y + B\cos x).$$

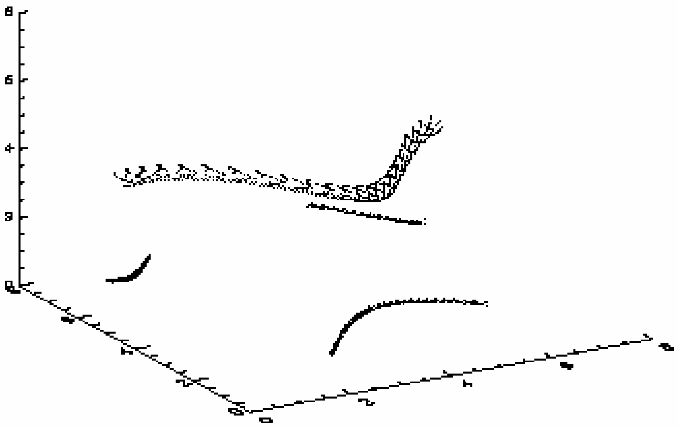


FIG. 4: Stroboscopic sampling (Δt=1) of the position of 10 particles initially distributed at random in a flow described by Eq. (11) with $A = 2$, $B = 0.4$, and $C = 1.2$. The dots represent the positions of these particles at the strobing periods 1000 to 2000.

Summary

- We present a method to control and target chaos in nonlinear dynamical systems: Bailout Embedding.
- We make evident its usefulness in the description of the phase space structure in complex dynamical systems.
- In particular, we have shown its power to make qualitative predictions for the suspended impurities dynamics in 3D time-periodic incompressible flows, and to investigate the structures of these flows as well.