Variational principle for the Pareto power law

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collaboration

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Anirban Chakraborti and Marco Patriarca, PRL 103, 228701 (2009)

Motivation

- The goal of the work is the study of plausible mechanisms of appearance of power-law distributions, such as Pareto's power law of income distribution and Zipf law for the rate of occurrence of words.

- Heterogeneity is known in general to be a main feature of complex systems and be responsible the emergence of some collective counterintuitive behaviors such as diversity-induced resonance.

- Here it is shown that heterogeneity in the number of degrees of freedom of the units composing a complex system may lead distributions with power laws.

- Known mechanisms leading to power law distributions are
  - Avalanche processes, e.g. in Self Organized Criticality
  - Multiplicative stochastic process
  - Non-extensive thermodynamics/entropy (C. Tsallis, J.Stat.Phys. 52, 479 (1988); E.M.F. Curado and C. Tsallis, J.Phys.A 24, L69 (1991))
  - Generalized Gibbs distribution (R.A. Treumann and C.H. Jaroschek, Phys. Rev. Lett. 100, 155005 (2008))
  - Superstatistics (C. Beck, Physica A 365, 96 (2006)).

Outline

(homogeneous) Kinetic Exchange Models can appear in

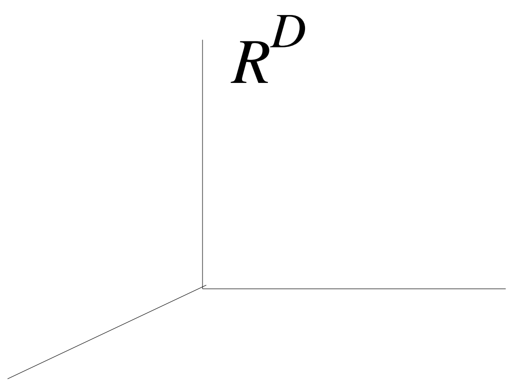
- Kinetic theory in *D* dimensions
- Study of diffusion in a network
- Kinetic Wealth Exchange Models

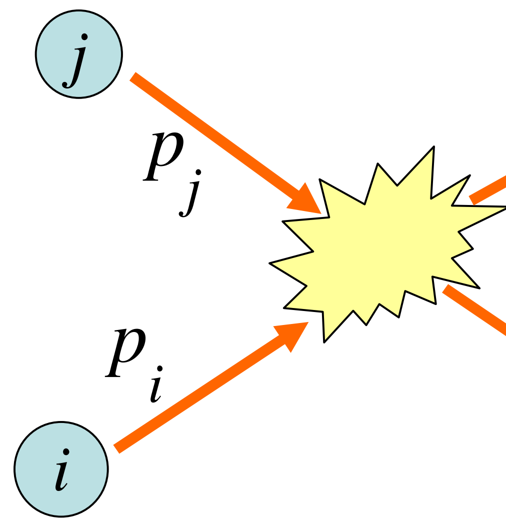
with an identical formulation.

In their heterogeneous versions, they can reproduce power laws, e.g.

- Power law in load distribution of scale-free networks
- Zipf's law from the Random walk in the semantic network
- Pareto's Law from heterogeneous Kinetic Wealth Exchange Models

1. Kinetic Theory in *D* dimensions

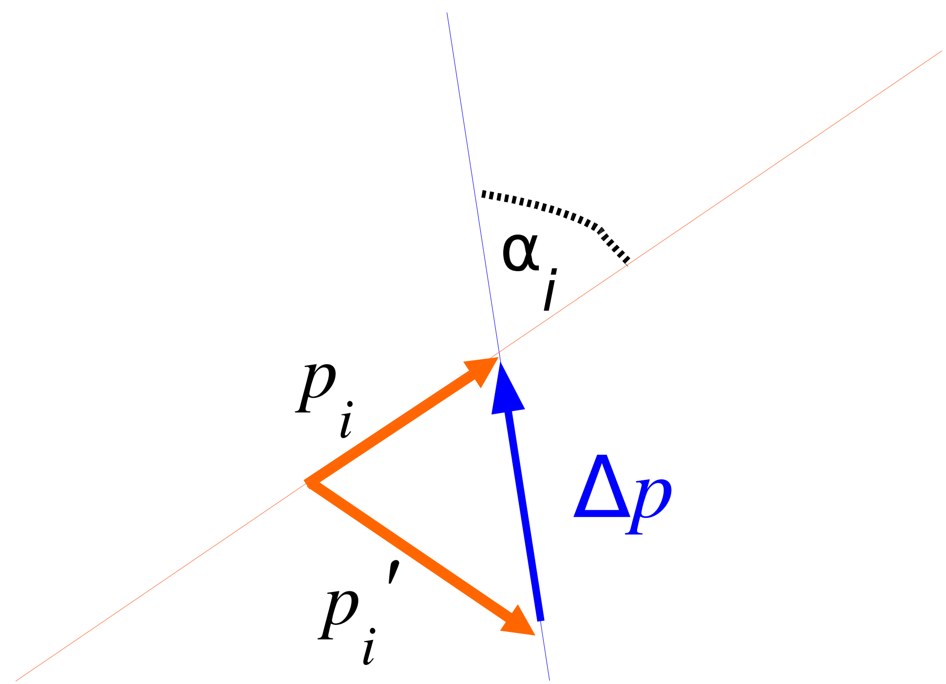




If the initial particles momenta are p_i and p_j, introduce the momentum transfer

$$\Delta p = p_j' - p_j = p_i - p_i'$$

and the angles α_i and α_i respect to the initial momenta p_i and p_j, respectively.



Using energy and momentum conservation, one obtains for the kinetic energies $x_i = ½\,(p_i)^2$ and $x_j = ½\,(p_j)^2$ of particles *i* and *j*

$$x_i \rightarrow x_i - \widetilde{\omega}_i x_i + \widetilde{\omega}_j x_j$$
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where the ω's coefficients are related to the cosines squared,

$$0 \le \widetilde{\omega}_i = (\cos\alpha_i)^2 \le 1$$

$$0 \le \widetilde{\omega}_j = (\cos\alpha_j)^2 \le 1$$

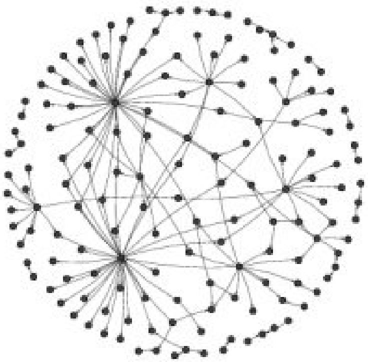
In *D* dimensions it can be shown that assuming initial random directions,

$$\langle \widetilde{\omega} \rangle = \langle (\cos\alpha)^2 \rangle = 1/D$$

For the equipartition theorem,

$$\langle x_i \rangle = D\,k_B T/2 \sim D$$

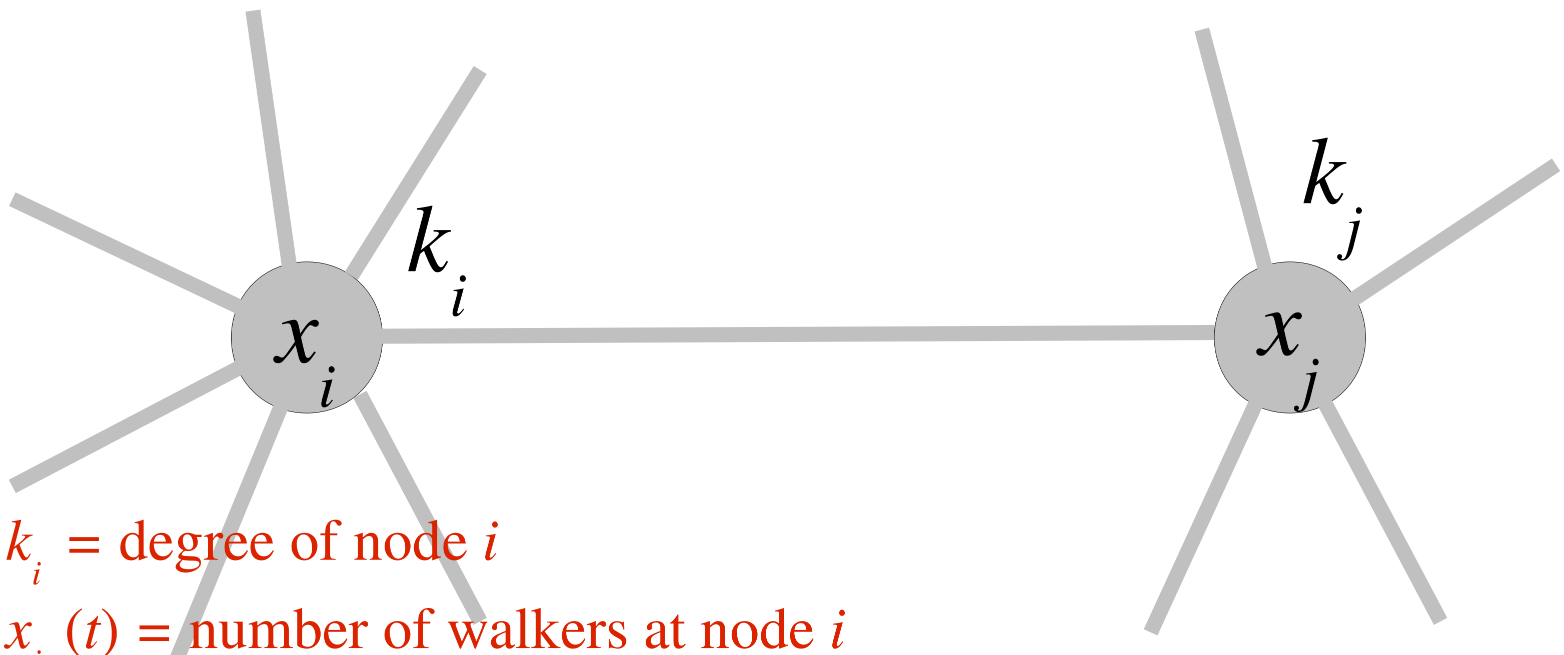
2. Random Walk across a network



Consider N walkers moving across a network.
A generic node i has k_i links and $x_i(t)$ walkers at time t.

The update rules for the flow between nodes i and j, assuming homogeneous diffusion, is

$$x_i(t+1) = x_i(t) - \varpi_i x_i(t) + \varpi_j x_j(t)$$
$$x_j(t+1) = x_j(t) + \varpi_i x_i(t) - \varpi_j x_j(t)$$



k_i = degree of node i

$x_i(t)$ = number of walkers at node i

Here the ω's are random coefficients in the range

$$O \leq \tilde{\varpi}_i \leq 1$$

$$O \leq \tilde{\varpi}_j \leq \text{١}$$

The average values are

$$\langle \tilde{\varpi}_i \rangle = \text{١}/k_i$$

$$\langle \tilde{\varpi}_j \rangle = 1/k_j$$

It can be shown that in the stationary state

$$\langle x_i \rangle \sim k_i$$

3. Kinetic wealth exchange model with saving propensity (*)

Definition of the model

- N agents interacting randomly in pairs, characterized by the saving parameters $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $0 < \lambda_i < 1$.
- The state of the system is specified through the agent wealths $(x_1, x_2, \ldots, x_N)$.
- At each time step t two agents i and j are extracted randomly and exchange wealth according to

$$x_i' = \lambda_i x_i + \epsilon_1 (1 - \lambda_i) x_i + \epsilon_2 (1 - \lambda_j) x_j$$

$$x_j' = \lambda_j x_j + (1 - \epsilon_1)(1 - \lambda_i) x_i + (1 - \epsilon_2)(1 - \lambda_j) x_j$$

(*) J. Angle, Social Forces 65, 293 (1986),
A. Chakraborti and B.K. Chakrabarti, Eur. Phys. J. B 17, 167 (2000).

Here ε_1 and ε_2 are uniform random numbers in (0,1), independent or possibly the same random number, depending on the model.

The update rule can be rewritten as

$$\begin{aligned} x_i(t+1) &= x_i(t) - \tilde{\omega}_i x_i(t) + \tilde{\omega}_j x_j(t) \\ x_j(t+1) &= x_j(t) + \tilde{\omega}_i x_i(t) - \tilde{\omega}_j x_j(t) \end{aligned}$$

with

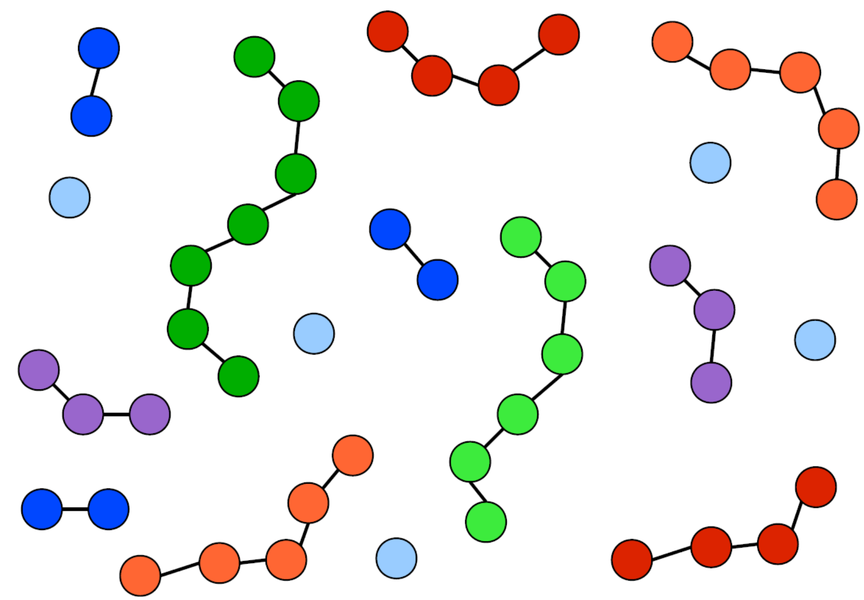
$$\begin{aligned} \tilde{\omega}_i &= (1-\epsilon_1)(1-\lambda_i) \equiv (1-\epsilon_1)\omega_i \\ \tilde{\omega}_j &= \epsilon_2(1-\lambda_j) \equiv \epsilon_2\omega_j \end{aligned}$$

In a heterogeneous model the average value is $\langle x_i \rangle \sim 1/(1-\lambda_i)$

Model system of a perfect gas with heterogeneous dimensions

The model system can represent a perfect gas with heterogeneous dimensions (each particle lives in a space with a different dimension) or a heterogeneous mixture of polymers, each polymer having a different number of degree of freedoms.

The heterogeneity is described by the probability $P(n)$ that a sub-system has a certain number $D = 2\,n$ of degrees of freedom.



For a fixed n, the equilibrium probability density of a D-dimensional harmonic oscillator is the gamma-distribution of order n,

$$f(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} \mathrm{e}^{-\beta x}$$

Then, for a general $P(n)$, the equilibrium distribution is the aggregate density,

$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x) = \int_1^\infty dn\, P(n) \frac{\beta^n}{\Gamma(n)} x^n e^{-\beta x}$$

This can be obtained by varying the Boltzmann entropy of the heterogeneous system.

Variational principle for heterogeneous dimensions

Given the dimension density $P(n)$, $1<n<\infty$,
one can define the entropy functional as follows.

Entropy Functional	$S[f]=\int dn\,P(n)\int dx\,f_n(x)\left\{\ln\left[\frac{f_n(x)}{x^{n-1}}\right]+\mu_n+\beta x\right\}$
Constraints on probability conservation	$I[f]=\int_0^\infty dx\,f_n(x)=1$
(Single) constraint on energy conservation	$X_{tot}[f]=\int dn\,P(n)\int_0^\infty dx\,x\,f_n(x)=1$

By variation of S, one obtains the aggregate density, i.e. the probability density to obtain a certain value *x* of the energy, independently of the corresponding number $2n$ of degrees of freedom,

$$f(x)=\int_1^\infty dn\,P(n)\,\beta\,\gamma_n(\beta x)=\int_1^\infty dn\,P(n)\frac{\beta^n}{\Gamma(n)}x^n e^{-\beta x}$$

Result for the aggregate distribution

The aggregate density can be rewritten as

$$f(x) = \int dn\, P(n) \beta \gamma_n(\beta x) = \beta \exp(-\beta x) \int dm \exp(-\phi(m))$$

where $m = n - 1$. The integrand function has a maximum at $\beta x \sim 1$.
Then using the Stirling approximation, one can write

$$\phi(m) \approx -\ln[P(m+1)] - m\ln(\beta x) + \ln(\sqrt{2\pi}) + (m + \tfrac{1}{2})\ln(m) - m,$$

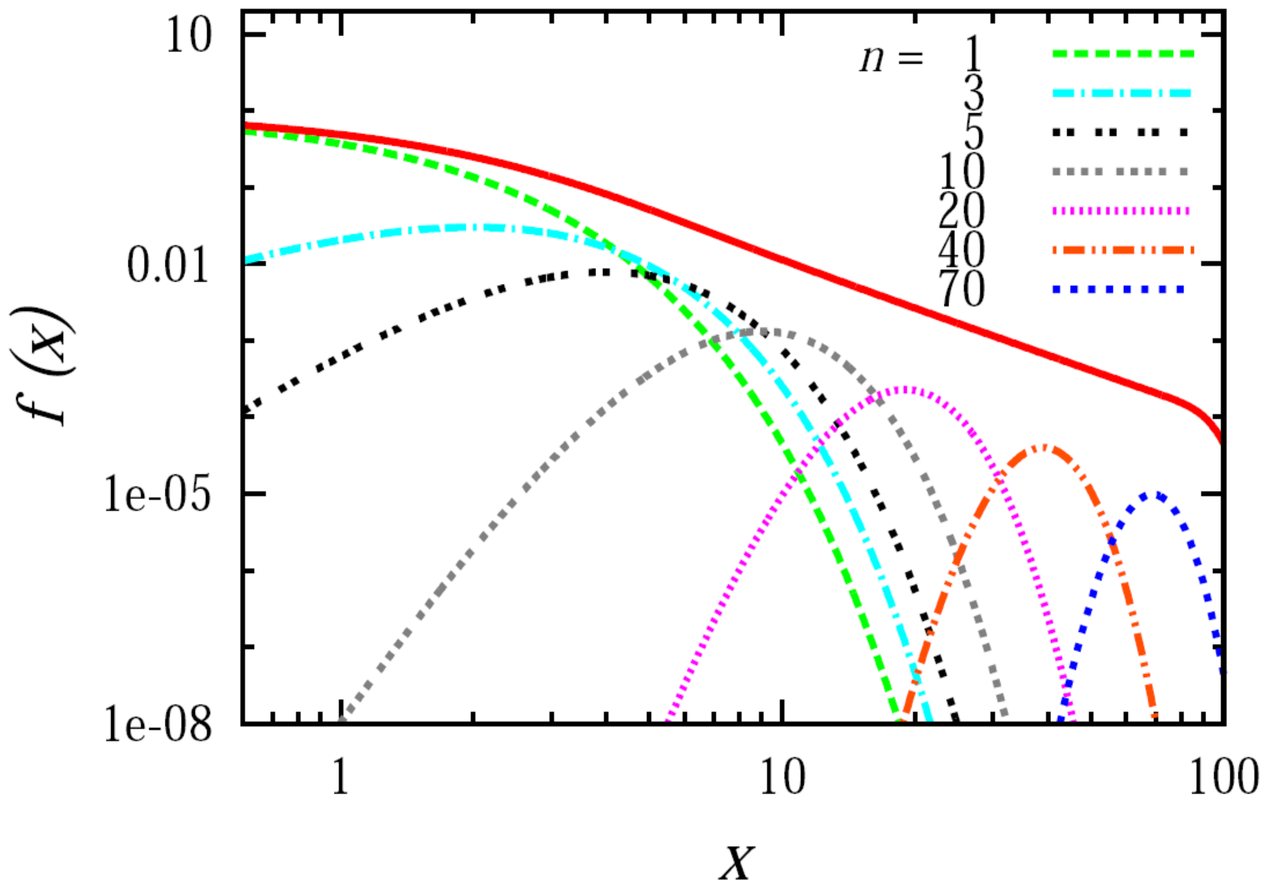
Using the saddle-point approximation, $f(x) \approx \beta \exp[-\beta x - \phi(m_0)]$

$$\times \int_{-\infty}^{+\infty} d\epsilon \exp[-\phi''(m_0)\epsilon^2/2]$$

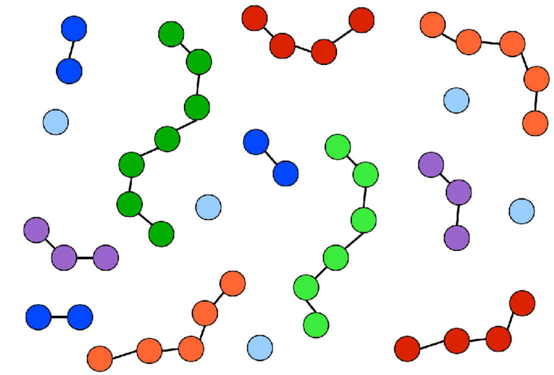
$$= \beta \sqrt{\frac{2\pi}{\phi''(m_0)}} \exp[-\beta x - \phi(m_0)].$$

The asymptotic result is $$f(x \gg \beta^{-1}) \equiv f_2(x) = \beta P(1 + \beta x).$$

Dimensional decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$



Aggregate distribution of dimensionally heterogeneous systems



Gas in *D* dimensions. For a given dimension D, the equipartition theorem provides an average kinetic energy

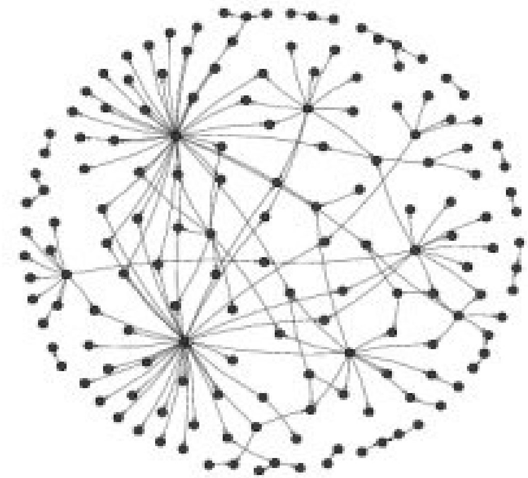
$$x(D) = D\, k_B T/2 \sim D,$$

where T is the temperature of the system.
If $P(D)$ is the dimension density of a heterogeneous system, then for probability conservation, i.e. $f(x)\, dx = P(D)\, dD$, one has

$$f(x) = P(D)\frac{dD}{dx} = \bar{x}^{-1} P(x/\bar{x})$$

$$\bar{x} = k_B T / ۲$$

Complex Networks. In a complex network with degree distribution $P(k)$, the average equilibrium load for the simplest case of free diffusion is



$$x(k) = x_0 k \sim k,$$

where x_0 is a constant (average flux per link and direction).
Then from probability conservation, $f(x)\, dx = P(k)\, dk$, it follows that

$$f(x) = P(k)\frac{dk}{dx} = x_{\cdot}^{-1} P(x/x_{\cdot})$$

In particular, **scale-free networks have a power law load distribution** in the stationary state, $f(x) \sim 1 / x^{\alpha}$.

Zipf's law from the random walk on the semantic network of language

Written text (or spoken language) can be conceived as a walk in the special space of concepts which can be represented by nouns, verbs, etc, the **semantic space.**

A. P. MASUCCI AND G. J. RODGERS

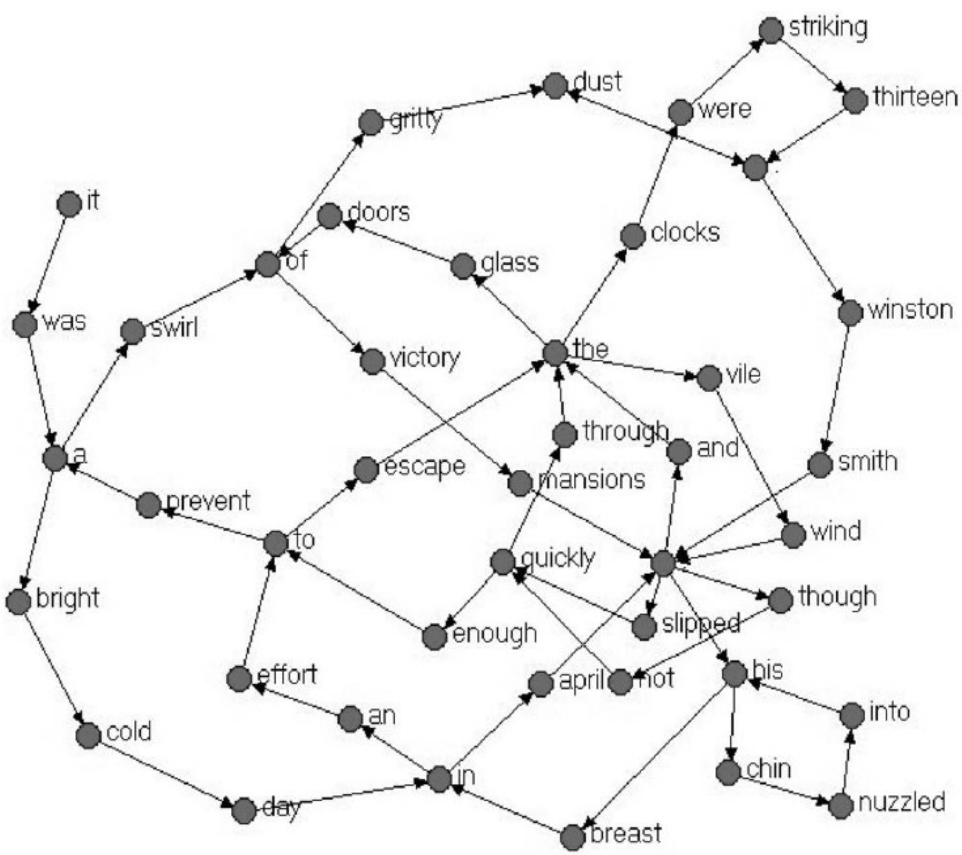


FIG. 1. Illustration of the language network for the first 60 words of Orwell's *1984*.

After writing a long text (e.g. a novel) or speaking a long speech, what is the expected rank distribution of words?

This depends obviously on the correlations between subsequent words, i.e. on the probability that, given a word *w*, another word *w*' will follow.

← From:

A.P. Masucci and G.J. Rodgers, Phys. Rev. E 74, 026102 (2006)

Measure of Zipf's law on "1984".

(a) The dashed line is a power law with slope -1.1, $x \sim r^{-1.1}$. If N is the total number of words, then

$Y = r / N = F(x) =$ cumulative distribution

$\rightarrow F(x) \sim x^{-1/1.1} \sim x^{-0.91}$

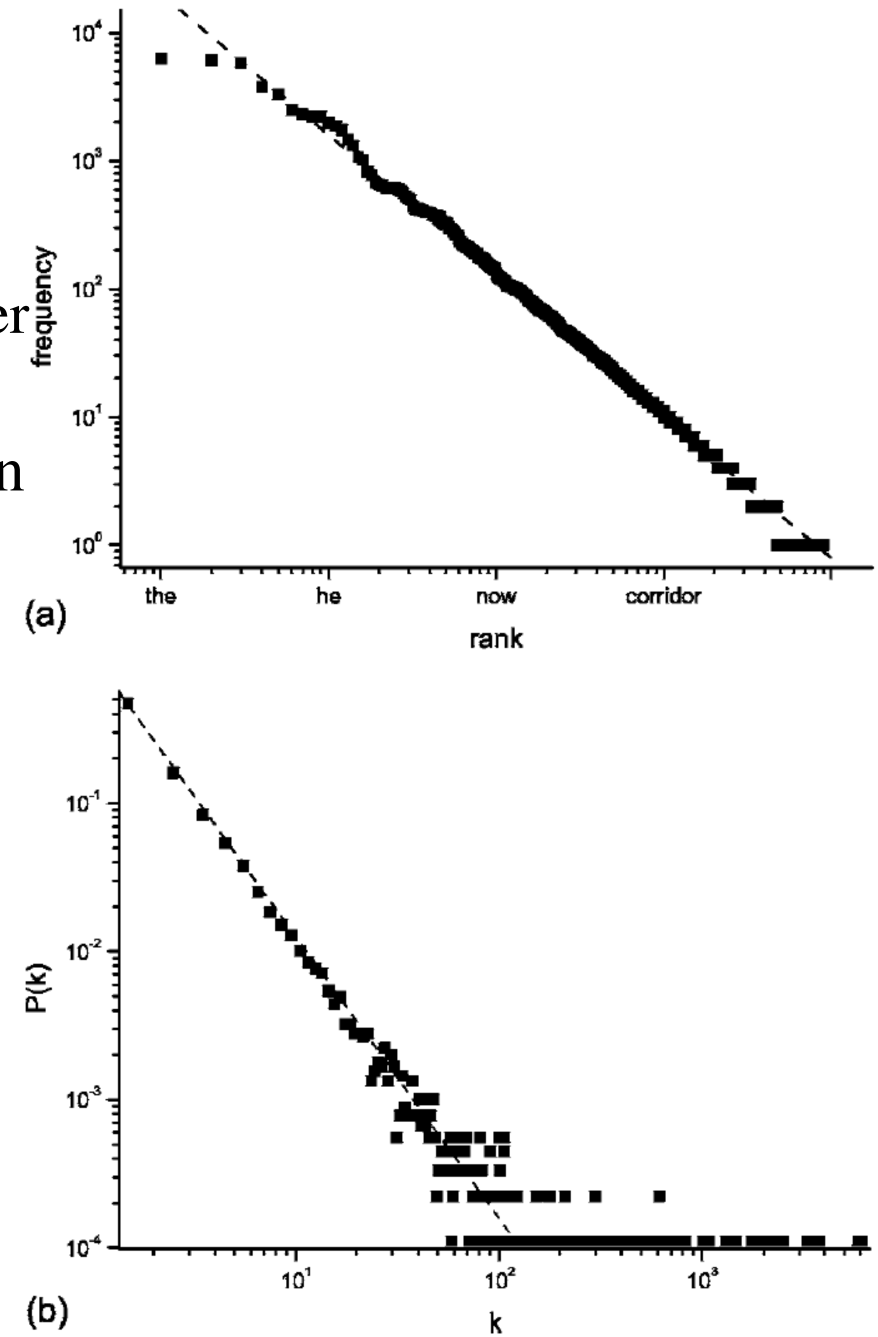
$\rightarrow f(x) \sim x^{-1.91}$

(b) The degree distribution $P(k)$ measured on the same novel.

The slope found is -1.9.

From ⟵

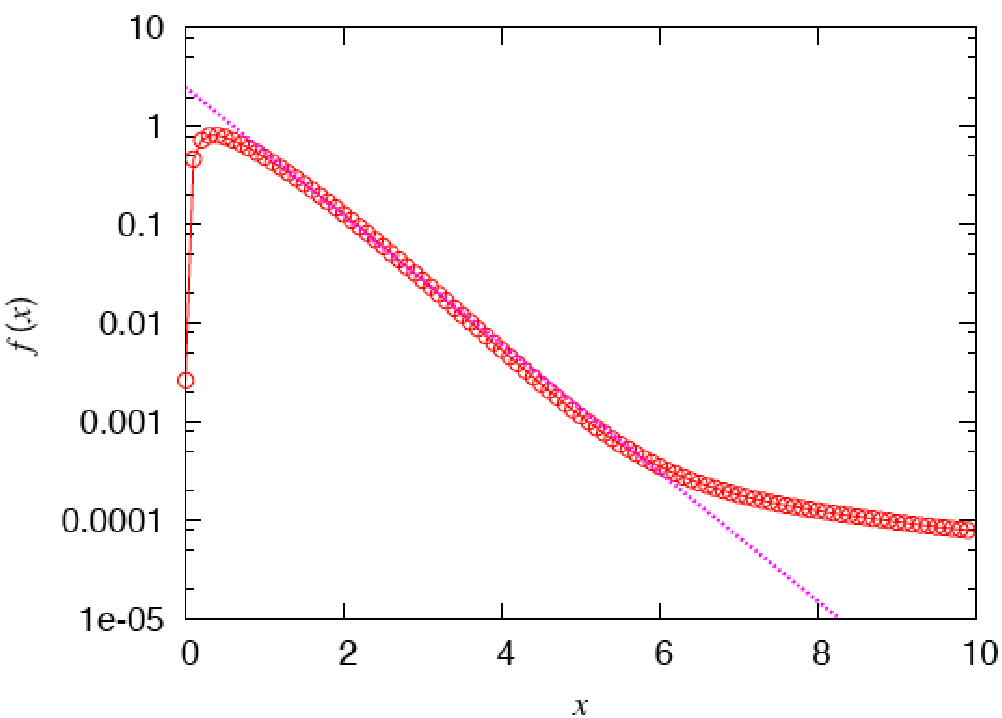
A.P. Masucci and G.J. Rodgers, Phys. Rev. E 74, 026102 (2006)

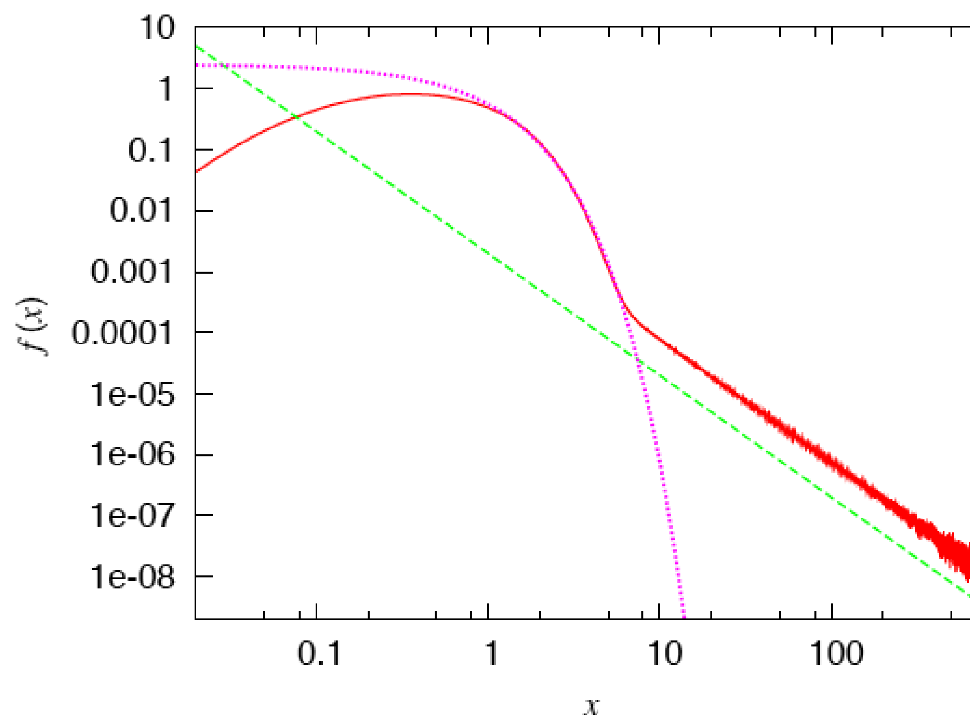


Heterogeneous Kinetic Exchange Models

- The analogy with dimensionally heterogeneous systems is based on the similarities discussed above between the models.

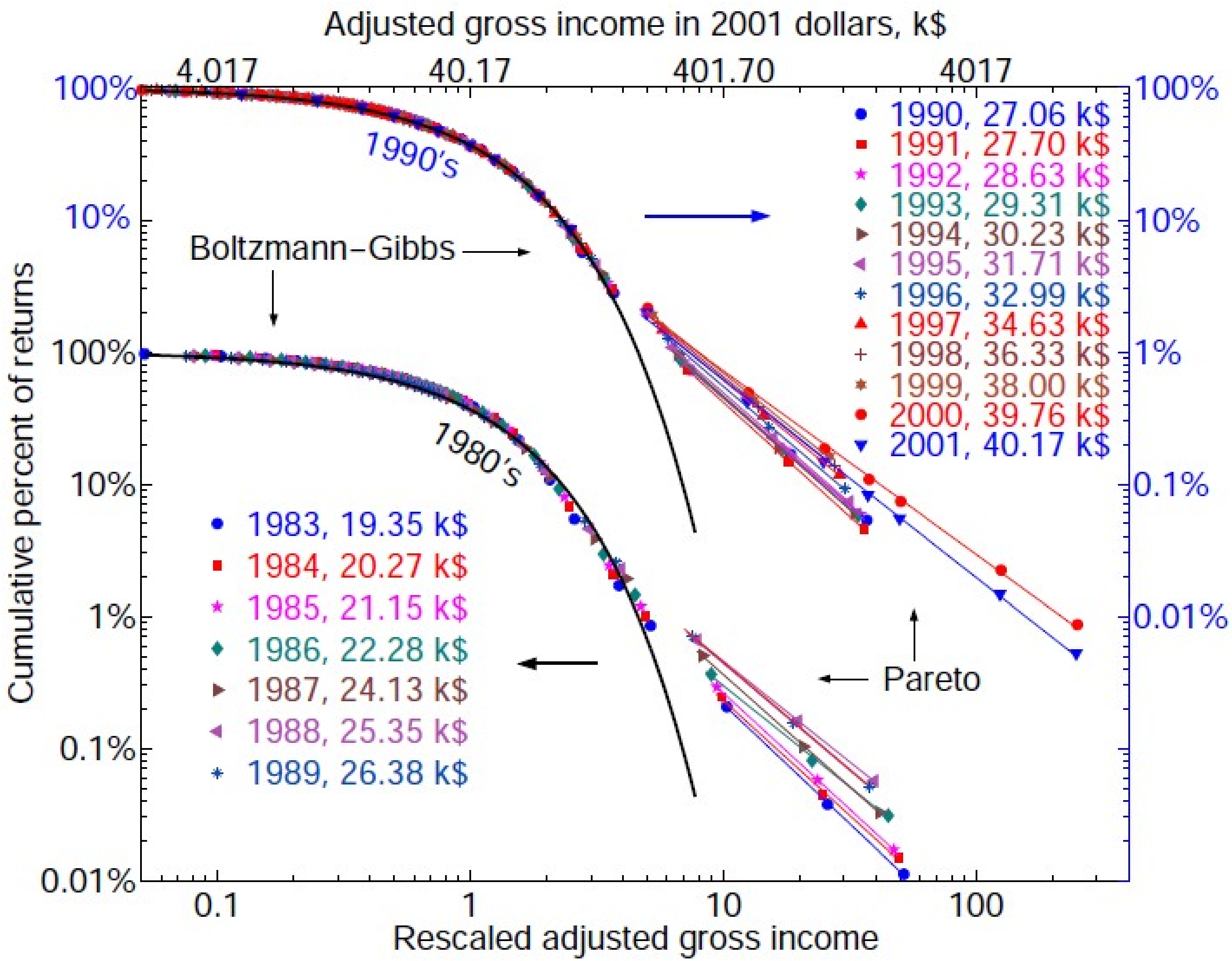
- **Example**: If the saving propensities of the *N* agents (λ_1, λ_2, ..., λ_N) are for 1% distributed uniformly λ in (0,1),





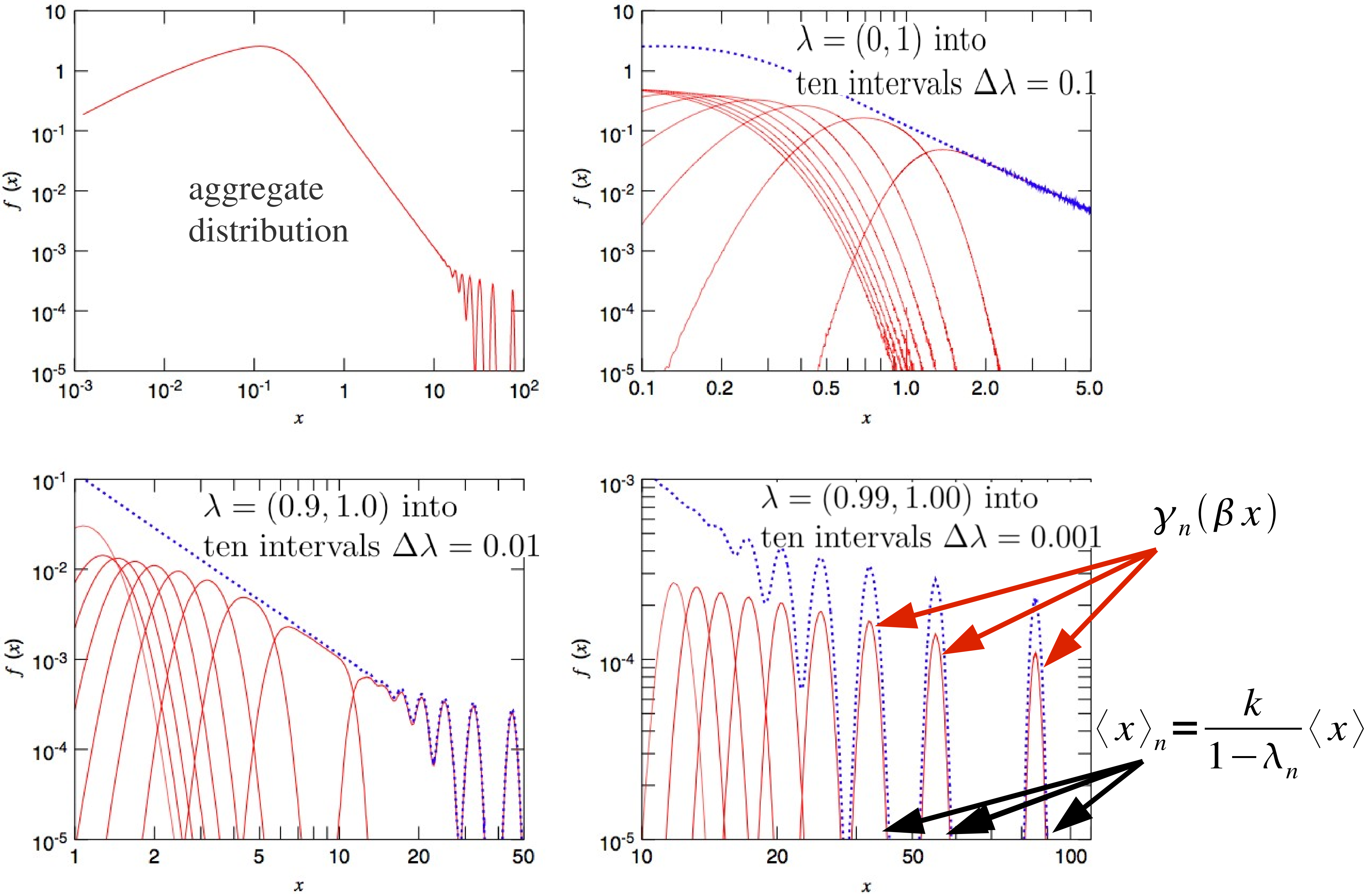
Compare with real data →

Income data overview



Colloquium: Statistical mechanics of money, wealth, and income [arXiv:0905.1518]
Victor M. Yakovenko, J. Barkley Rosser

Decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$ for λ's in (0,1)



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  Gibb's versus non-Gibb's distributions in money dynamics
  Physica A 340 (2004) 334 [arXiv:cond-mat/0312167]

Additional material

Variational principle for one degree of freedom

Variational principle approaches based on the variation of an entropy functional find a natural application in the study of social and economic processes.

Entropy	$S[f] = \int dq\, f(q) \ln[f(q)]$
Probability conservation	$I[f] = \int dq\, f(q)$
Wealth conservation	$X_{tot}[f] = \int dq\, f(q)\, X(q)$

Lagrange method:

$$\delta S_{eff}[f] = \delta\{S[f] + \mu I[f] + \beta X_{tot}[f]\}$$

$$= \delta \int dq\, f(q) \{\ln[f(q)] + \mu + \beta X(q)\} = \bullet \quad \rightarrow \quad f(x) = \frac{\exp(-\beta x)}{\langle x \rangle}$$

Variational principle for N degrees of freedom (dimensions)

Functional $S[f] = \int dq_1 dq_2 \ldots f(q_1, q_2, \ldots)\{\ln[f(q_1, q_2, \ldots)] + \mu + \beta X(q_1, q_2, \ldots)\}$

Energy in N-dimensions: $X(q) = \frac{1}{2}\left[q_1^2 + \ldots + q_N^2\right]$ (independent particles)

Integrate $N-1$ angular variables: $S[f_1] = \int dq\, f_1(q)\left\{\ln\left[\frac{f_1(q)}{\sigma_N q^{N-1}}\right] + \mu + \beta X(q)\right\}$

$(N-1)$-dimensional surface: $\sigma_N = 2\pi^{N/2}/\Gamma(N/٢)$

Reduced density in q $f_1(q) = f_N(q)/\sigma_N q^{N-1}$

Move to energy variable $x = X(q^2)$ and apply Lagrange method:

$$\delta S[f] = \delta \int dx\, f(x)\left\{\ln\left[\frac{f(x)}{\sigma_N x^{N/٢-1}}\right] + \mu + \beta x\right\} = ٠ \quad \rightarrow \quad f(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} e^{-\beta x}$$

Example: KWEM Aggregate distribution *f* (*x*) for distributed λ with density Φ(λ)

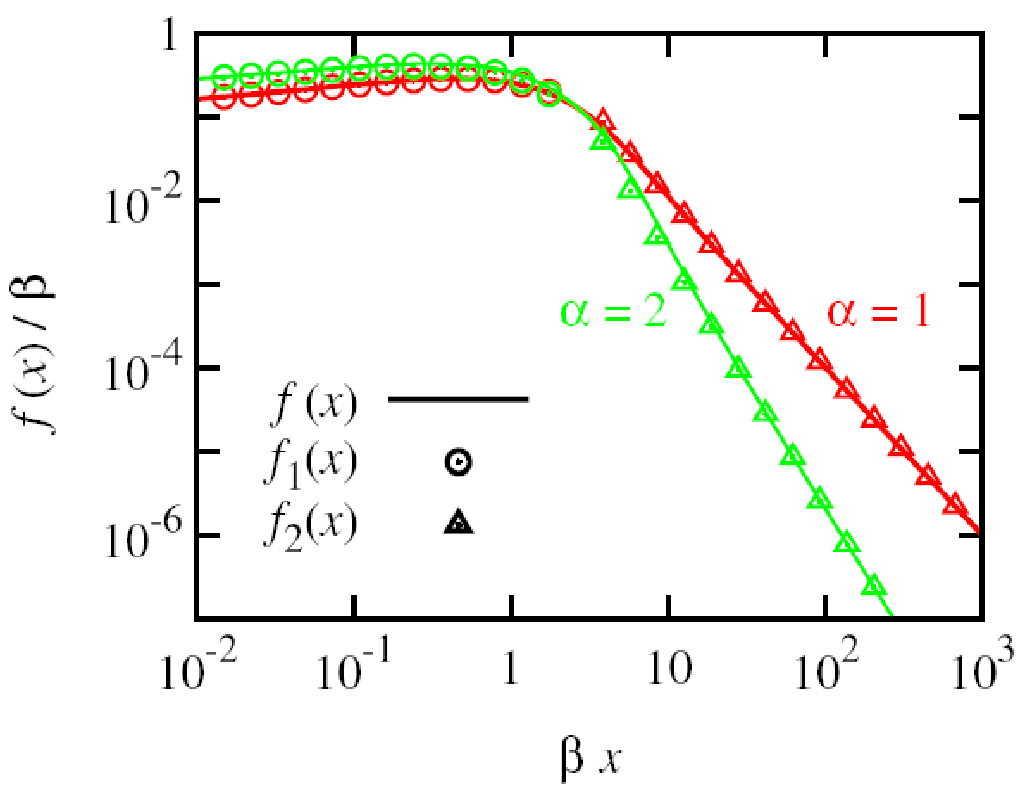
Simple example: the agents have different saving propensities λ_i with a uniform λ-density Φ(λ).

$$\phi(\lambda)=1 \quad , \qquad ٠<\lambda<1$$

$$\phi(\lambda)=0 \qquad \text{otherwise}$$

Corresponding form of the *n*-density

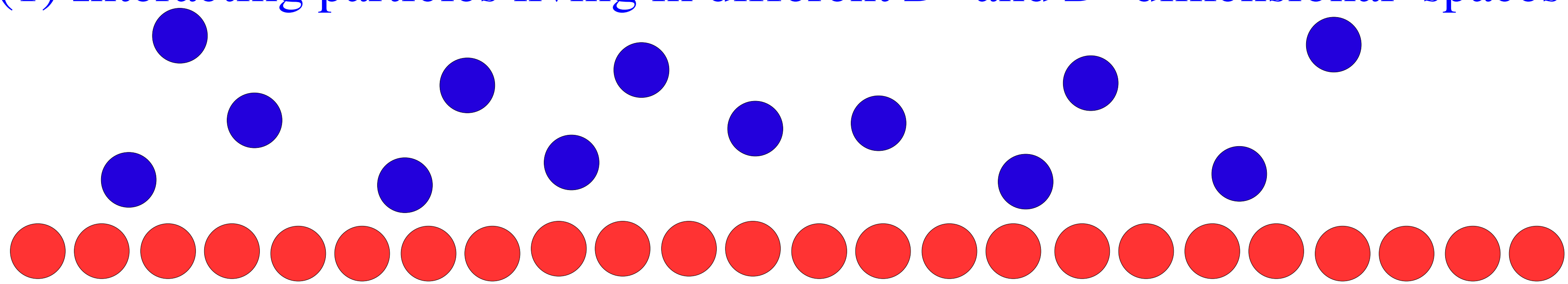
$$P(n) = \frac{d\lambda(n)}{dn}\phi(\lambda(n)) = \frac{٣}{(n+٢)^2}$$



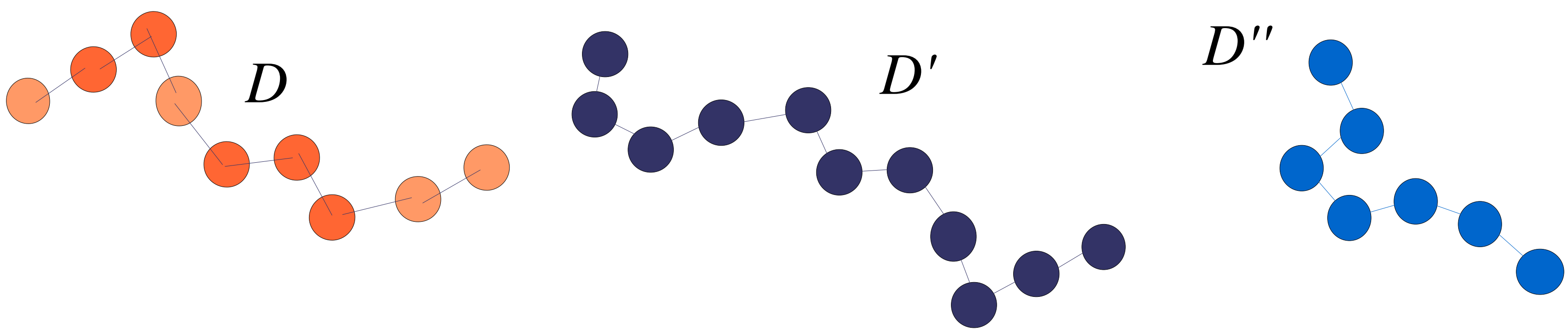
$$f(x)=\int_1^{\infty} dn\, P(n)\, \beta\, \gamma_n(\beta x)$$

Examples of dimensionally heterogeneous systems

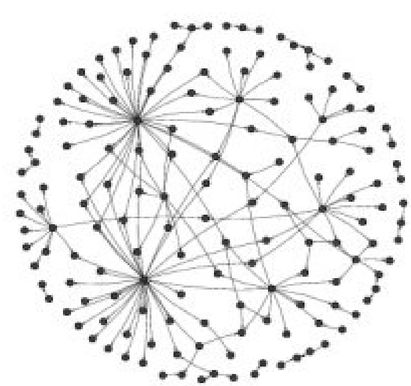
(1) Interacting particles living in different D- and D'-dimensional spaces



(2) Interacting polymers with different numbers of harmonic degrees of freedom



(3) Heterogeneous Networks



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Anirban Chakraborti and Marco Patriarca, PRL 103, 228701 (2009)

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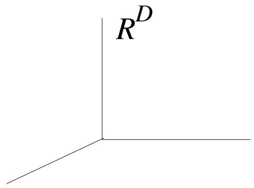

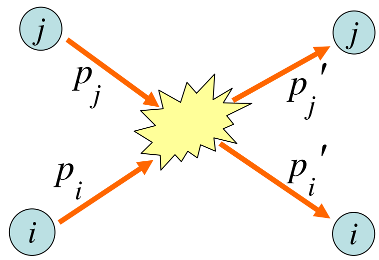

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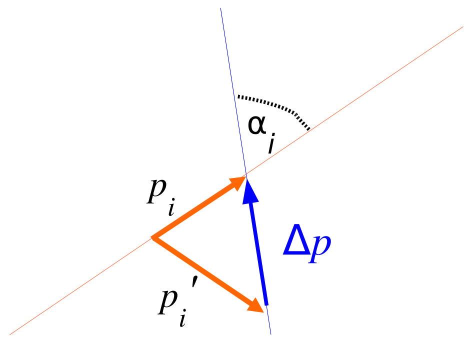

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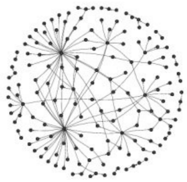
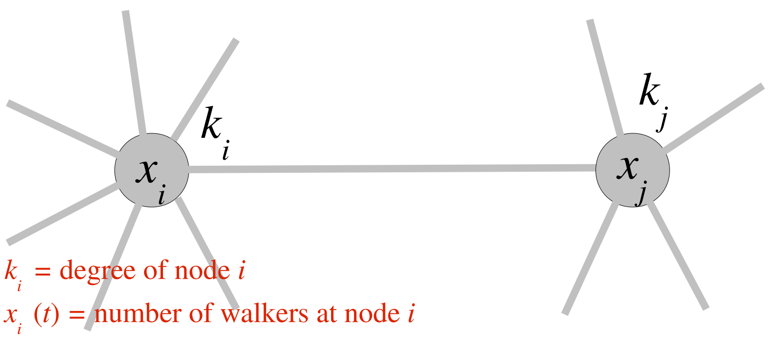
$$\langle x_i \rangle \;=\; D\,k_B T/\text{٢} \;\sim\; D$$

2. Random Walk across a network



Consider N walkers moving across a network.
A generic node i has k_i links and $x_i(t)$ walkers at time t.
The update rules for the flow between nodes i and j, assuming homogeneous diffusion, is

$$x_i(t+1) = x_i(t) - \tilde{\omega}_i x_i(t) + \tilde{\omega}_j x_j(t)$$
$$x_j(t+1) = x_j(t) + \tilde{\omega}_i x_i(t) - \tilde{\omega}_j x_j(t)$$



k_i = degree of node i
x_i (t) = number of walkers at node i

Here the ω's are random coefficients in the range

$$O \leq \tilde{\varpi}_i \leq 1$$

$$O \leq \tilde{\varpi}_j \leq 1$$

The average values are

$$\langle \tilde{\varpi}_i \rangle = 1/k_i$$

$$\langle \tilde{\varpi}_j \rangle = 1/k_j$$

It can be shown that in the stationary state

$$\langle x_i \rangle \sim k_i$$

3. Kinetic wealth exchange model with saving propensity (*)

Definition of the model

- N agents interacting randomly in pairs, characterized by the saving parameters $(\lambda_1, \lambda_2, ..., \lambda_N)$ with $0 < \lambda_i < 1$.
- The state of the system is specified through the agent wealths $(x_1, x_2, ..., x_N)$.
- At each time step t two agents i and j are extracted randomly and exchange wealth according to

$$x_i' = \lambda_i x_i + \epsilon_1 (1-\lambda_i) x_i + \epsilon_2 (1-\lambda_j) x_j$$
$$x_j' = \lambda_j x_j + (1-\epsilon_1)(1-\lambda_i) x_i + (1-\epsilon_2)(1-\lambda_j) x_j$$

(*) J. Angle, Social Forces 65, 293 (1986),
A. Chakraborti and B.K. Chakrabarti, Eur. Phys. J. B 17, 167 (2000).

Here ε_1 and ε_2 are uniform random numbers in (0,1), independent or possibly the same random number, depending on the model.

The update rule can be rewritten as

$$x_i(t+1) = x_i(t) - \tilde{\omega}_i x_i(t) + \tilde{\omega}_j x_j(t)$$
$$x_j(t+1) = x_j(t) + \tilde{\omega}_i x_i(t) - \tilde{\omega}_j x_j(t)$$

with

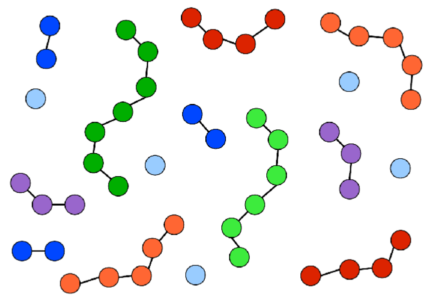
$$\tilde{\omega}_i = (1-\epsilon_1)(1-\lambda_i) \equiv (1-\epsilon_1)\omega_i$$
$$\tilde{\omega}_j = \epsilon_2(1-\lambda_j) \equiv \epsilon_2 \omega_j$$

In a heterogeneous model the average value is $\langle x_i \rangle \sim 1/(1-\lambda_i)$

Model system of a perfect gas with heterogeneous dimensions

The model system can represent a perfect gas with heterogeneous dimensions (each particle lives in a space with a different dimension) or a heterogeneous mixture of polymers, each polymer having a different number of degree of freedoms.

The heterogeneity is described by the probability $P(n)$ that a sub-system has a certain number $D = 2\,n$ of degrees of freedom.



For a fixed n, the equilibrium probability density of a D-dimensional harmonic oscillator is the gamma-distribution of order n,

$$f(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} \mathrm{e}^{-\beta x}$$

Then, for a general $P(n)$, the equilibrium distribution is the aggregate density,

$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x) = \int_1^\infty dn\, P(n) \frac{\beta^n}{\Gamma(n)} x^n e^{-\beta x}$$

This can be obtained by varying the Boltzmann entropy of the heterogeneous system.

Variational principle for heterogeneous dimensions

Given the dimension density $P(n)$, $1 < n < \infty$,
one can define the entropy functional as follows.

Entropy Functional $S[f] = \int dn\, P(n) \int dx\, f_n(x) \left\{ \ln \left[\frac{f_n(x)}{x^{n-1}} \right] + \mu_n + \beta x \right\}$

Constraints on probability conservation $I[f] = \int_{.}^{\infty} dx\, f_n(x) = 1$

(Single) constraint on energy conservation $X_{tot}[f] = \int dn\, P(n) \int_{.}^{\infty} dx\, x\, f_n(x) = 1$

By variation of S, one obtains the aggregate density, i.e. the probability density to obtain a certain value *x* of the energy, independently of the corresponding number 2*n* of degrees of freedom,

$$f(x) = \int_1^{\infty} dn\, P(n)\, \beta\, \gamma_n(\beta x) = \int_1^{\infty} dn\, P(n) \frac{\beta^n}{\Gamma(n)} x^n e^{-\beta x}$$

Result for the aggregate distribution

The aggregate density can be rewritten as

$$f(x) = \int dn\, P(n) \beta \gamma_n(\beta x) = \beta \exp(-\beta x) \int dm \exp(-\phi(m))$$

where $m = n - 1$. The integrand function has a maximum at $\beta x \sim 1$.
Then using the Stirling approximation, one can write

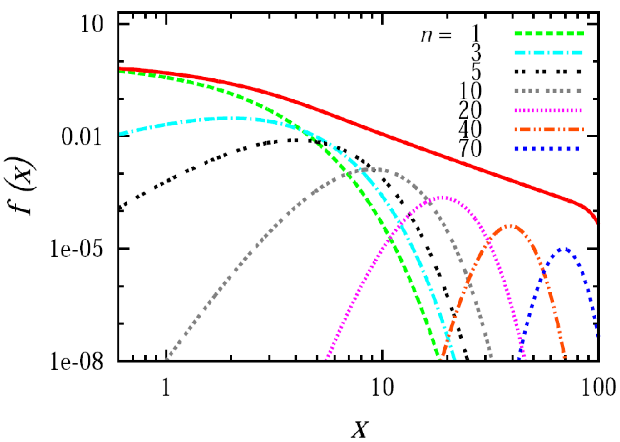
$$\phi(m) \approx -\ln[P(m+1)] - m\ln(\beta x) + \ln(\sqrt{2\pi}) + (m + \tfrac{1}{2})\ln(m) - m,$$

Using the saddle-point approximation,

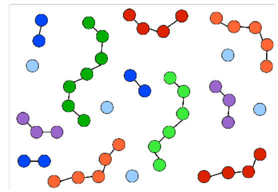
$$f(x) \approx \beta \exp[-\beta x - \phi(m_0)] \times \int_{-\infty}^{+\infty} d\epsilon \exp[-\phi''(m_0)\epsilon^2/2] = \beta\sqrt{\frac{2\pi}{\phi''(m_0)}}\exp[-\beta x - \phi(m_0)].$$

The asymptotic result is $f(x \gg \beta^{-1}) \equiv f_2(x) = \beta P(1 + \beta x).$

Dimensional decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$



Aggregate distribution of dimensionally heterogeneous systems



Gas in *D* dimensions. For a given dimension *D*, the equipartition theorem provides an average kinetic energy

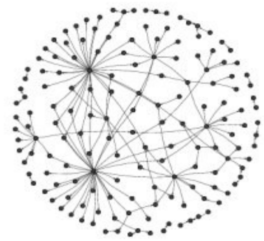
$$x(D) = D\, k_B T/2 \sim D,$$

where *T* is the temperature of the system.
If $P(D)$ is the dimension density of a heterogeneous system, then for probability conservation, i.e. $f(x)\, dx = P(D)\, dD$, one has

$$f(x) = P(D)\frac{dD}{dx} = \bar{x}^{-1} P(x/\bar{x})$$

$$\bar{x} = k_B T / \text{۲}$$

Complex Networks. In a complex network with degree distribution $P(k)$, the average equilibrium load for the simplest case of free diffusion is



$$x(k) = x_0 k \sim k,$$

where x_0 is a constant (average flux per link and direction).
Then from probability conservation, $f(x)\, dx = P(k)\, dk$, it follows that

$$f(x) = P(k)\frac{dk}{dx} = x_0^{-1} P(x/x_0)$$

In particular, **scale-free networks have a power law load distribution** in the stationary state, $f(x) \sim 1 / x^{\alpha}$.

Zipf's law from the random walk on the semantic network of language

Written text (or spoken language) can be conceived as a walk in the special space of concepts which can be represented by nouns, verbs, etc, the **semantic space.**

A. P. MASUCCI AND G. J. RODGERS

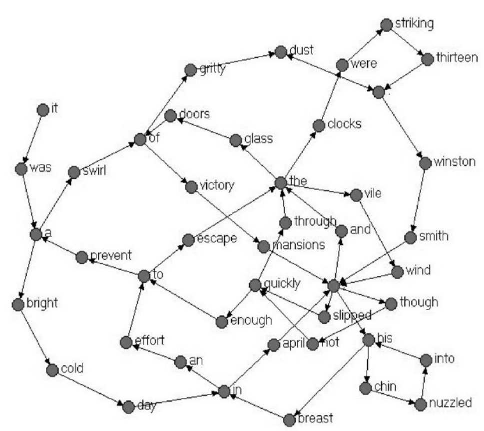


FIG. 1. Illustration of the language network for the first 60 words of Orwell's *1984*.

After writing a long text (e.g. a novel) or speaking a long speech, what is the expected rank distribution of words?

This depends obviously on the correlations between subsequent words, i.e. on the probability that, given a word *w*, another word *w*' will follow.

⟵ From:

A.P. Masucci and G.J. Rodgers, Phys. Rev. E 74, 026102 (2006)

Measure of Zipf's law on "1984".

(a) The dashed line is a power law with slope −1.1, $x \sim r^{-1.1}$. If N is the total number of words, then

$Y = r / N = F(x)$ = cumulative distribution

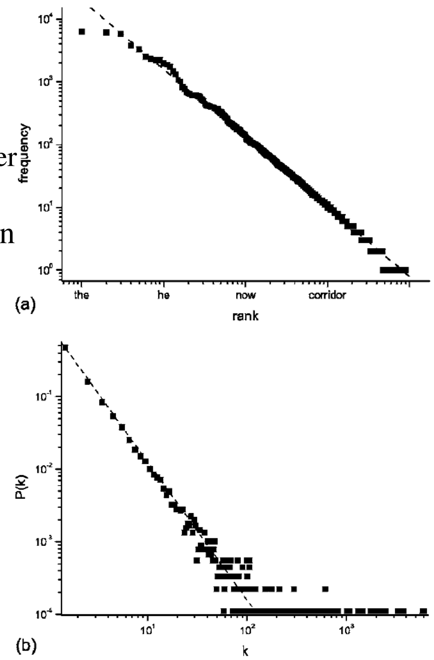
$\rightarrow F(x) \sim x^{-1/1.1} \sim x^{-0.91}$

$\rightarrow f(x) \sim x^{-1.91}$

(b) The degree distribution $P(k)$ measured on the same novel.
The slope found is −1.9.

From ⟵

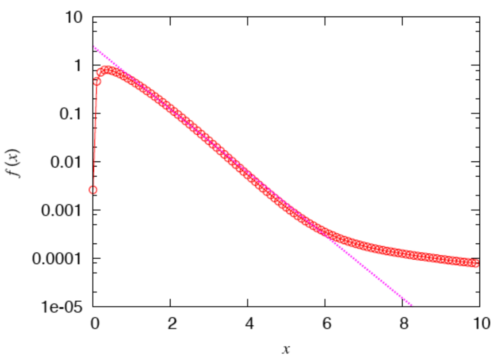
A.P. Masucci and G.J. Rodgers,
Phys. Rev. E 74, 026102 (2006)

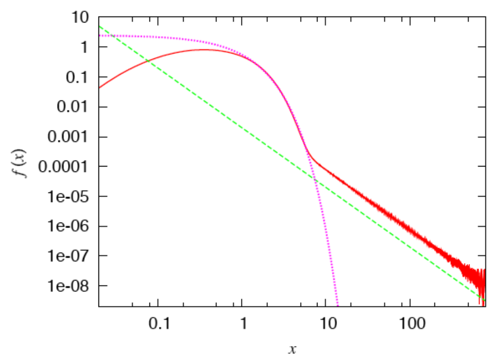


Heterogeneous Kinetic Exchange Models

- The analogy with dimensionally heterogeneous systems is based on the similarities discussed above between the models.

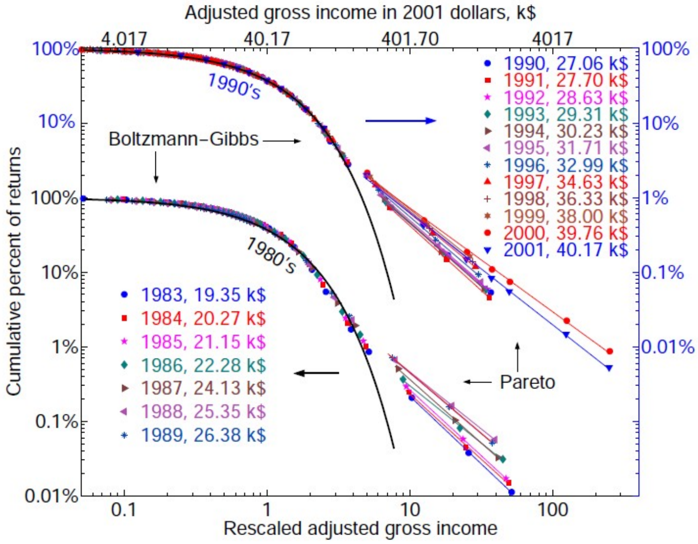
- **Example**: If the saving propensities of the *N* agents (λ_1, λ_2, ..., λ_N) are for 1% distributed uniformly λ in (0,1),





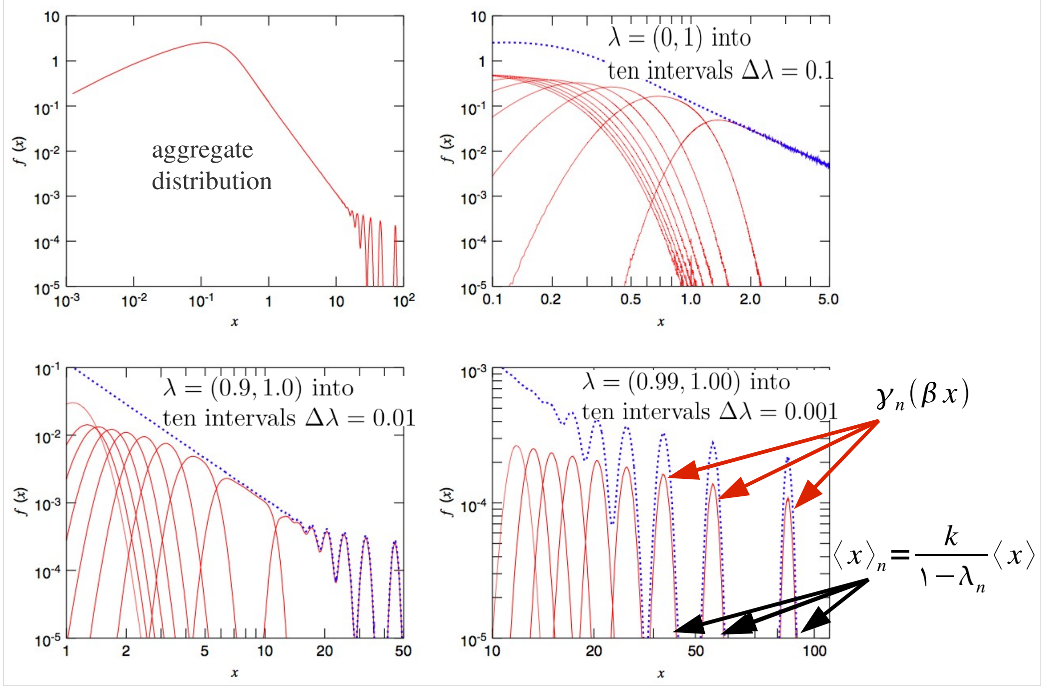
Compare with real data →

Income data overview



Colloquium: Statistical mechanics of money, wealth, and income [arXiv:0905.1518]
Victor M. Yakovenko, J. Barkley Rosser

Decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$ for λ's in (0,1)



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  Physica A 340 (2004) 334 [arXiv:cond-mat/0312167]

Additional material

Variational principle for one degree of freedom

Variational principle approaches based on the variation of an entropy functional find a natural application in the study of social and economic processes.

Entropy	$S[f] = \int dq\, f(q) \ln[f(q)]$
Probability conservation	$I[f] = \int dq\, f(q)$
Wealth conservation	$X_{tot}[f] = \int dq\, f(q)\, X(q)$

Lagrange method:

$$\delta S_{eff}[f] = \delta\{S[f] + \mu I[f] + \beta X_{tot}[f]\}$$

$$= \delta \int dq\, f(q)\{\ln[f(q)] + \mu + \beta X(q)\} = 0 \quad \rightarrow \quad f(x) = \frac{\exp(-\beta x)}{\langle x \rangle}$$

Variational principle for *N* degrees of freedom (dimensions)

Functional $S[f]=\int dq_1 dq_2 \ldots f(q_1, q_2, \ldots)\{\ln[f(q_1, q_2, \ldots)]+\mu+\beta X(q_1, q_2, \ldots)\}$

Energy in *N*-dimensions: $X(q)=\frac{1}{2}[q_1^2+\ldots+q_N^2]$ (independent particles)

Integrate $N-1$ angular variables: $S[f_1]=\int dq\, f_1(q)\left\{\ln\left[\frac{f_1(q)}{\sigma_N q^{N-1}}\right]+\mu+\beta X(q)\right\}$

$(N-1)$-dimensional surface: $\sigma_N=2\pi^{N/2}/\Gamma(N/2)$

Reduced density in q: $f_1(q)=f_N(q)/\sigma_N q^{N-1}$

Move to energy variable $x = X(q^2)$ and apply Lagrange method:

$$\delta S[f]=\delta\int dx\, f(x)\left\{\ln\left[\frac{f(x)}{\sigma_N x^{N/2-1}}\right]+\mu+\beta x\right\}=0 \quad \rightarrow \quad f(x)=\frac{\beta^n}{\Gamma(n)}x^{n-1}e^{-\beta x}$$

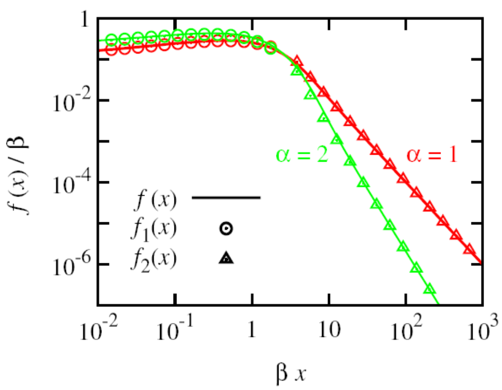
Example: KWEM Aggregate distribution $f(x)$ for distributed λ with density $\Phi(\lambda)$

Simple example: the agents have different saving propensities λ_i with a uniform λ-density $\Phi(\lambda)$.

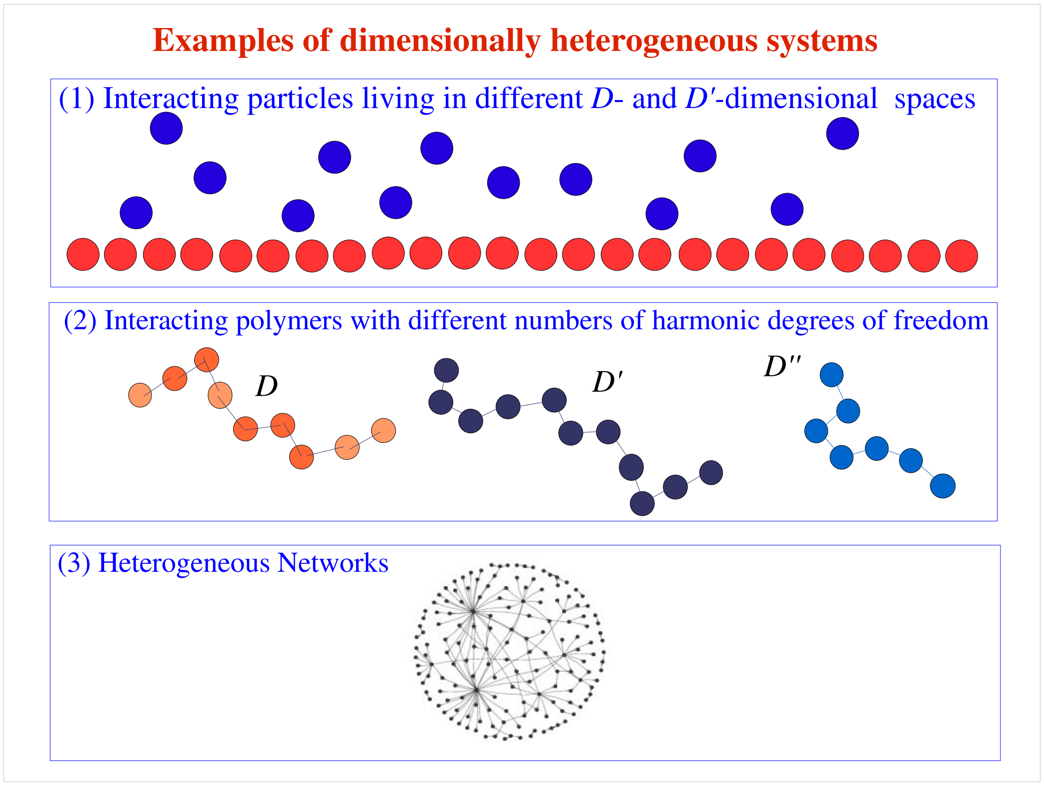
$$\phi(\lambda)=1 \;, \qquad 0<\lambda<1$$
$$\phi(\lambda)=0 \qquad \text{otherwise}$$

Corresponding form of the n-density

$$P(n) = \frac{d\lambda(n)}{dn}\phi(\lambda(n)) = \frac{3}{(n+2)^2}$$



$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x)$$


Examples of dimensionally heterogeneous systems
(1) Interacting particles living in different D- and D'-dimensional spaces
(2) Interacting polymers with different numbers of harmonic degrees of freedom
D
D'
D''
(3) Heterogeneous Networks