

How reliable are Finite-Size Lyapunov Exponents (FSLE) for the assessment of ocean evolution?



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Abstract

Due to its inherent turbulent nature, ocean motion possess a great complexity. We can barely describe the main patterns of general circulation at large scale, but the extreme richness of circulation patterns at mesoscale and lower scales makes the assessment of ocean evolution quite complicated. These difficulties are specially relevant when one tries to study problems of Lagrangian character, such as mixing, dispersion and transport of oceanic properties. For that reason, the implementation of appropriate Lagrangian diagnostic tools are in order. A prominent Lagrangian technique which starts to be widely used in oceanography is that of Finite-Size Lyapunov Exponents (FSLE). FSLE is a local measure of particle dispersion is obtained at each point, which serves to characterize Lagrangian structures. Although mathematically appealing, it is rather unclear how robust are FSLE analyses when confronted to real data, that is, data affected by noise and with limited scale sampling. In this study, we analyze the effect of finite scale samplings and of diverse types of noise on FSLE diagnostics. Both effects should be accounted to determine which part of the diagnostics is reliable. Most importantly, scale dependence of FSLE reveals the emergence of a cascade-like hierarchy in Lagrangian structures, which can be used to improve diagnostics and to better understand ocean dynamics.

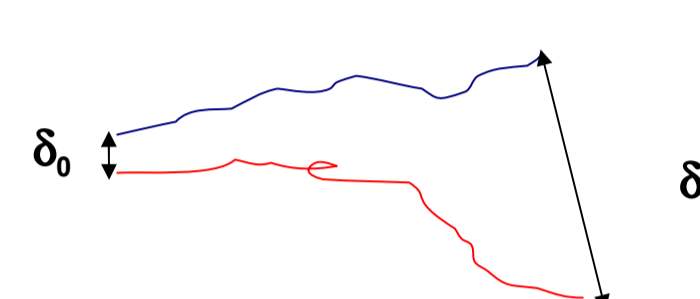
Multifractal character and scale invariance properties of FSLE

FSLE at different spatial scales and at the same resolution of velocity data:

The FSLEs are calculated by the formula

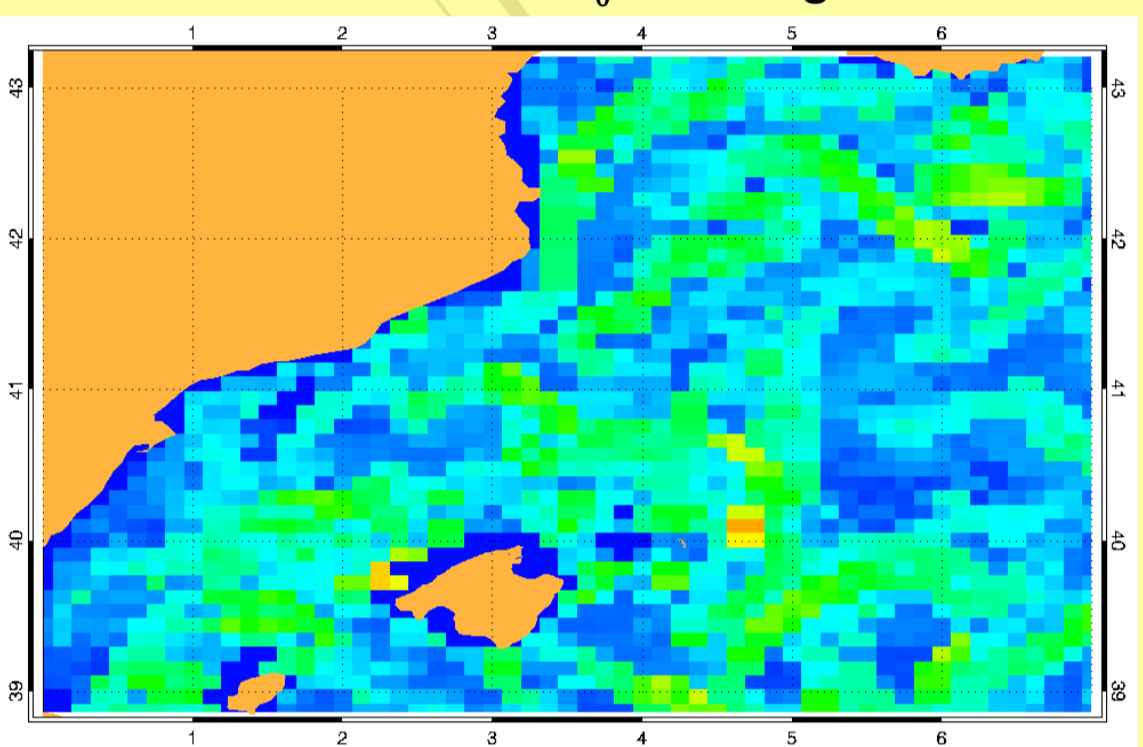
$$\Lambda(x, t_0, \delta_0, \delta_f) = \frac{1}{|\mathbf{r}|} \log \frac{\delta_f}{\delta_0}$$

δ_0 is the initial separation
 δ_f is the final separation
 t is the time needed for two particles initially separated δ_0 to get separate δ_f
 x_i are the initial coordinates
 t time

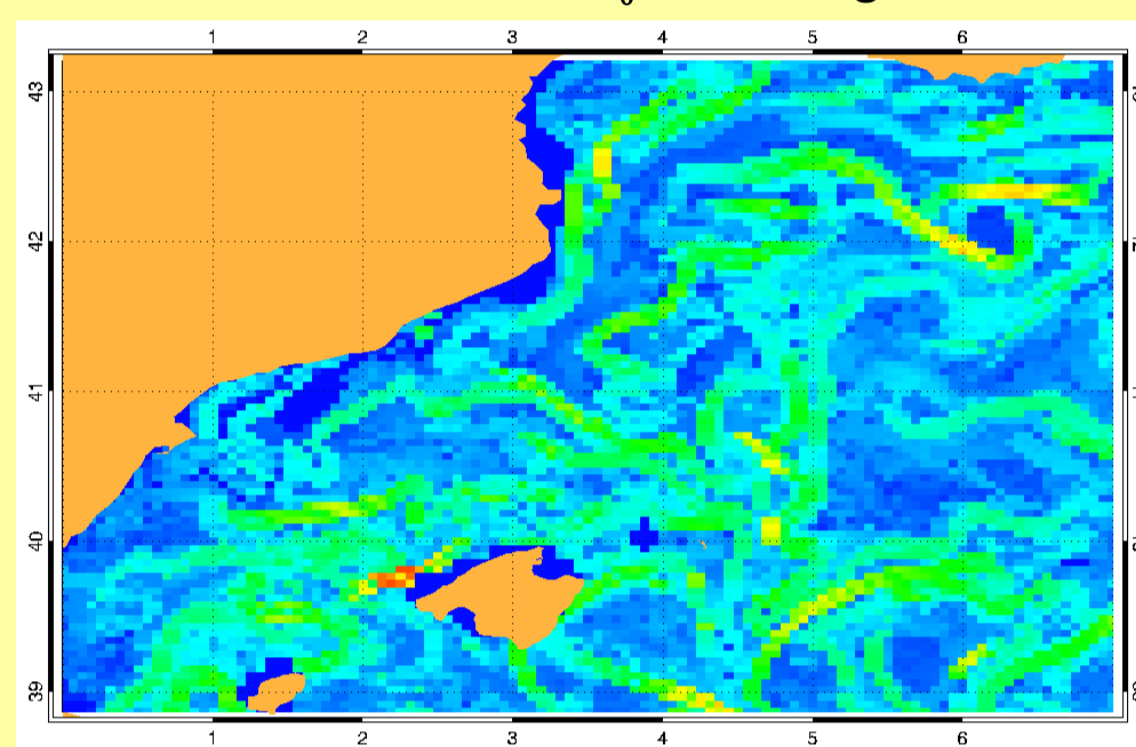


The FSLEs are computed using daily surface velocity data of DieçAST model applied to the Balearic Sea. In this model the resolution for velocity data is 1/8 degree. We modify the spatial scales at the FSLEs field changing the initial resolution δ_0 . The δ_0 used for this computation are: 1/8 degree, 1/16 degree, 1/32 degree and 1/64 degree.

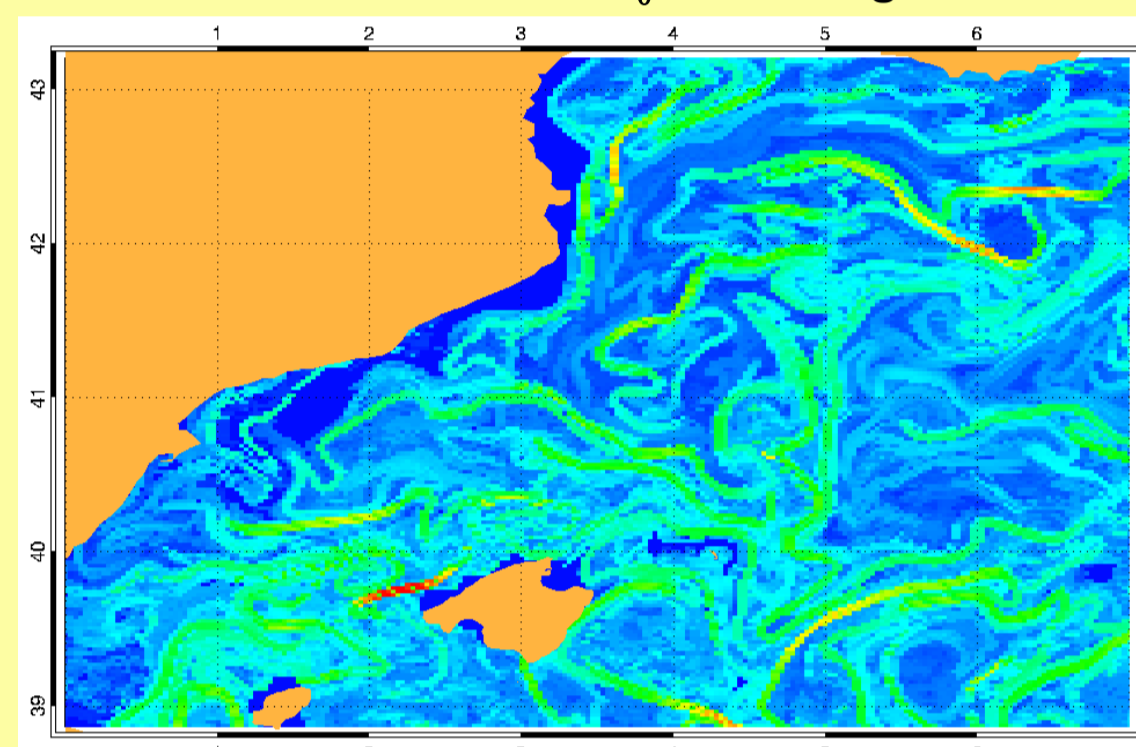
FSLEs $\delta_0 = 1/8$ degree



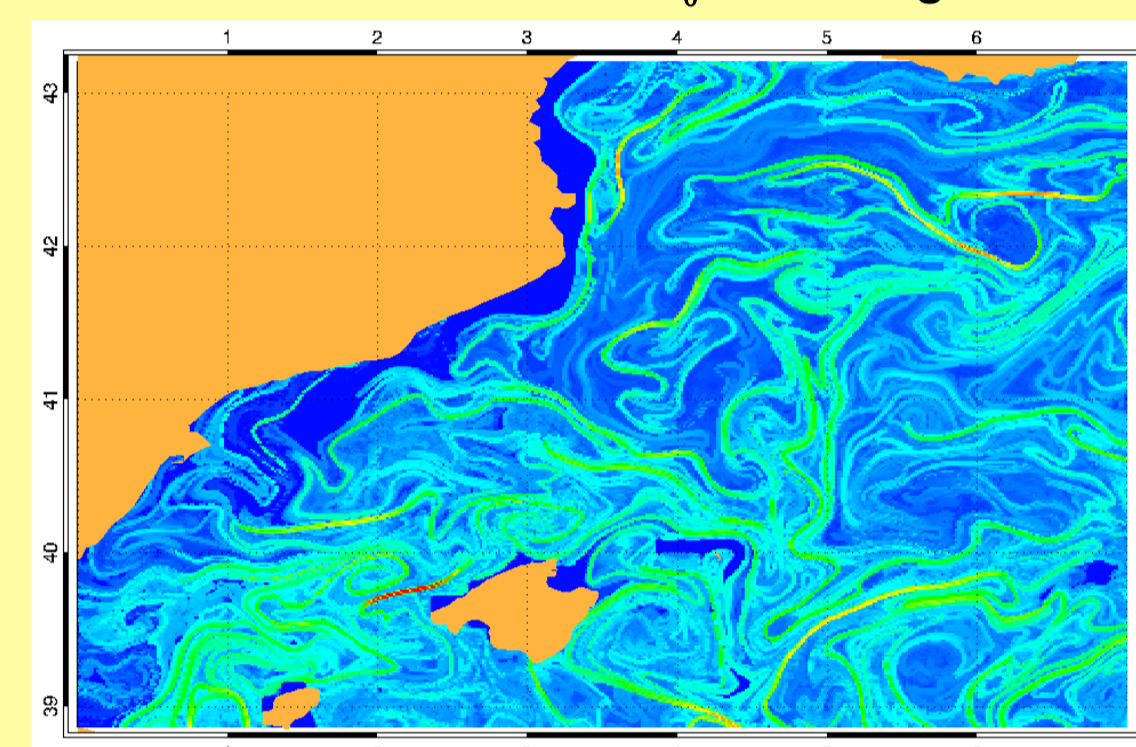
FSLEs $\delta_0 = 1/16$ degree



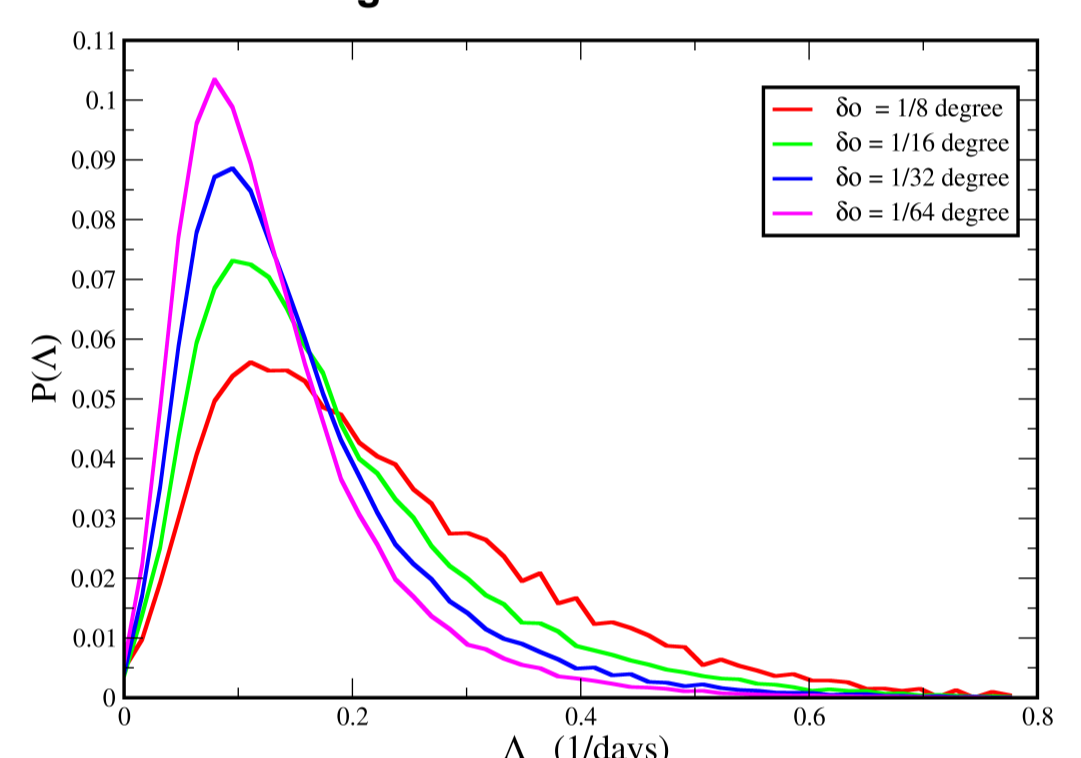
FSLEs $\delta_0 = 1/32$ degree



FSLEs $\delta_0 = 1/64$ degree



Histograms of FSLEs at different scales



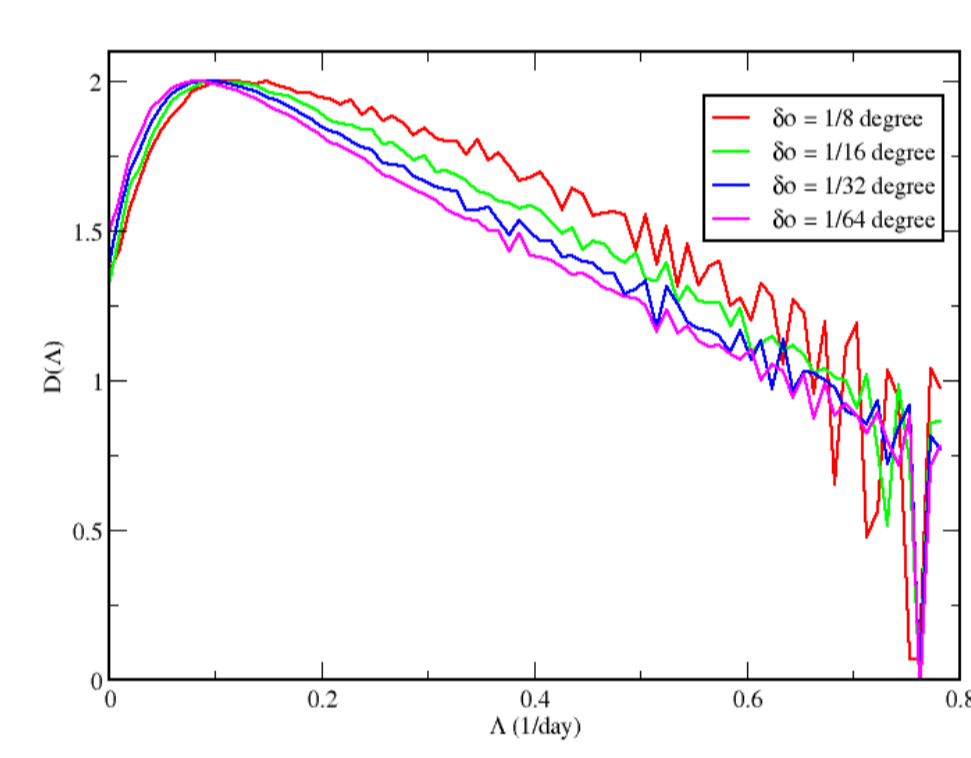
We calculated the probability distribution, $P(\delta_0, \Lambda)$, of the FSLEs at the different spatial resolutions δ_0 (1/8, 1/16, 1/32, 1/64 degrees). The histograms have been normalized to the same area, by dividing by the respective maximum values of the FSLEs. The plots shows that when the resolution is finer (smaller δ_0), the probability distribution $P(\delta_0, \Lambda)$ narrows and the values of the peak increase, in a way consistent with a multifractal character of the FSLEs.

The scale behaviour of these histograms indicates that the distribution of the FSLEs at a scale δ_0 is given by:

$$P(\delta_0, \Lambda) = P(\delta_0, \Lambda_c) \delta_0^{d-D(\Lambda)}$$

$P(\delta_0, \Lambda_c)$ is the maximum value of the probability distribution
 d in this case is the surface dimension = 2
 $D(\Lambda)$ is the fractal dimension of the set of initial conditions leading to FSLE
 δ_0 is the definition scale

Fractal dimension of FSLEs at different scales



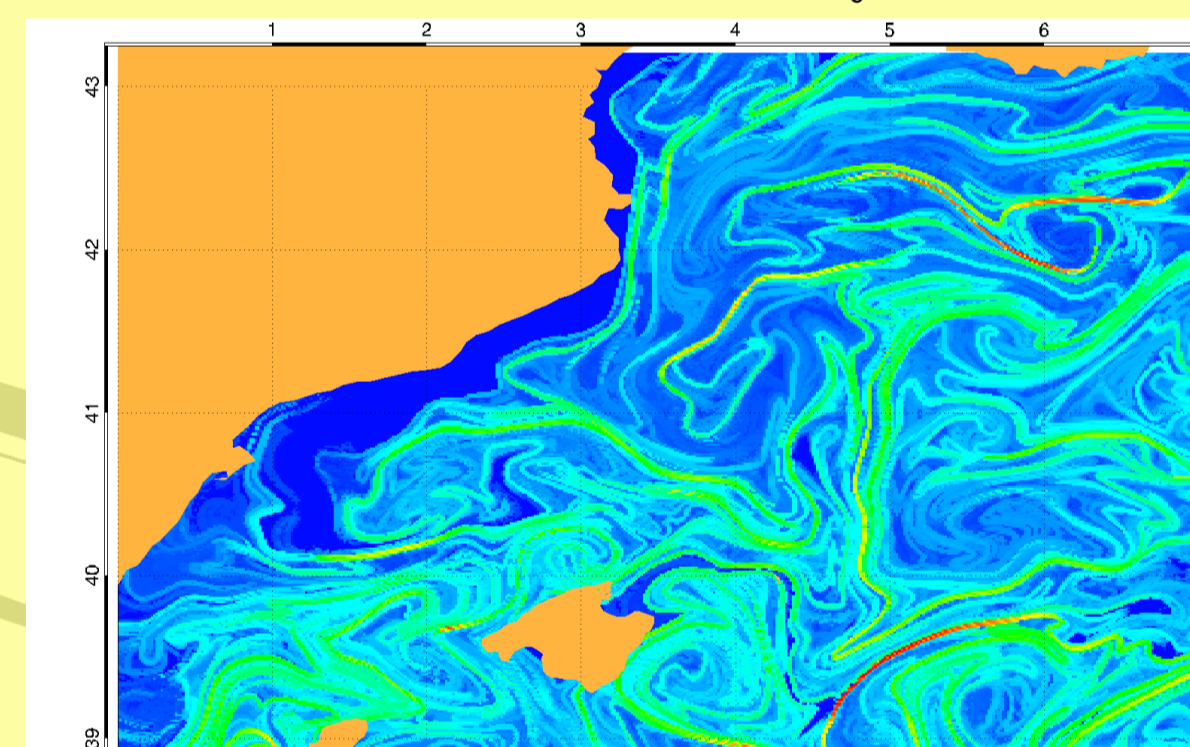
From Eq (1) one obtains a properly normalized expression to compute the fractal dimension at different scales

$$D(\Lambda) = d - \frac{\log \frac{P(\delta_0, \Lambda)}{P(\delta_0, \Lambda_c)}}{\log \delta_0} \quad (2)$$

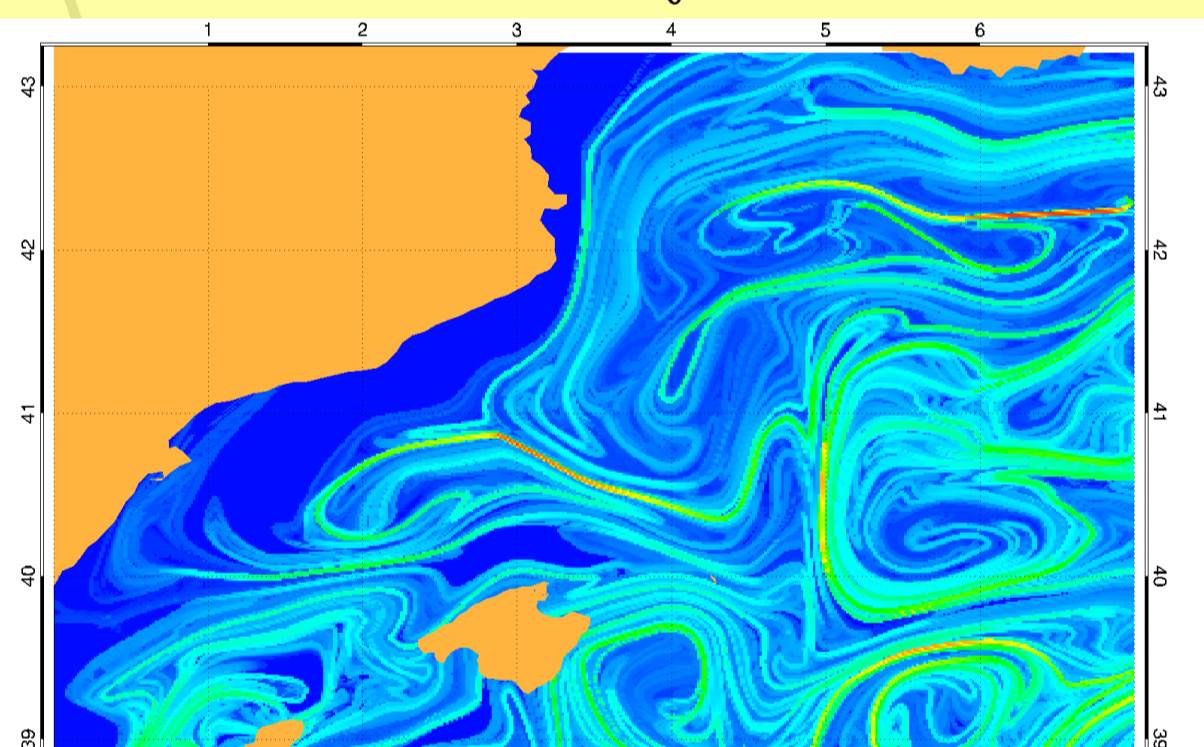
The plot of $D(\Lambda)$ shows a collapse of $D(\Lambda)$ at the different scales. Because of loss of translational invariance the collapse is not perfect. This effect happens when the resolution and the domain is small. (i.e. It has a few pixels)

FSLE at different spatial resolution of the velocity field, and at the same spatial scale of FSLE

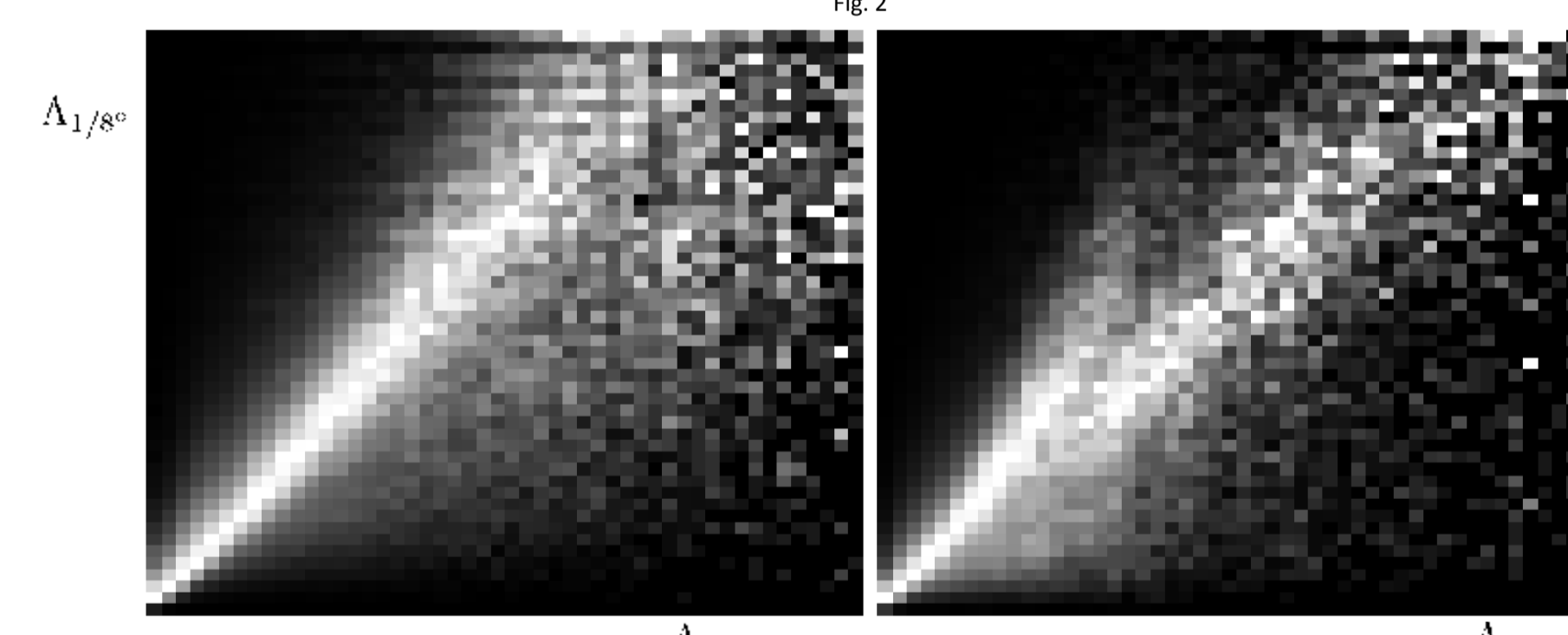
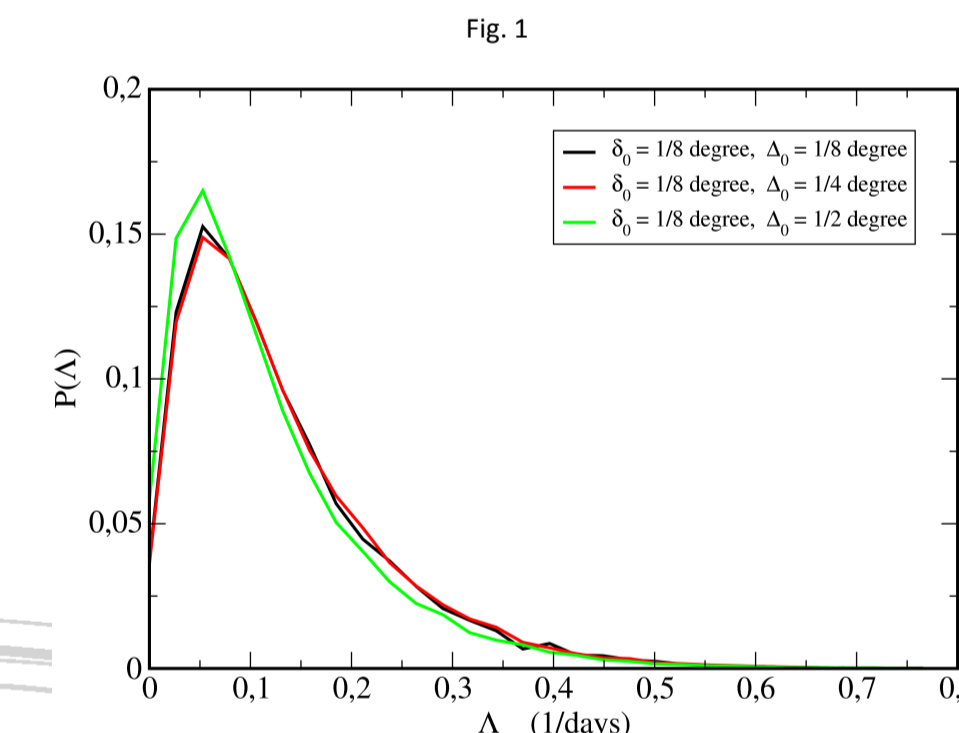
FSLEs $\Delta_v = 1/4$ degree



FSLEs $\Delta_v = 1/2$ degree



We compute the histograms of the spatial distributions of the FSLEs at the same spatial scale δ_0 , 1/64 degree and a different spatial resolution of the velocity field: $\Delta_v = 1/8, 1/4$ and $1/2$ degree (Fig 1), and probability distributions (coded as grey level) of FSLEs derived at a coarse velocity resolution (Δ_v , vertical axis) conditioned by a finer velocity grid (Δ_v , horizontal axis) (Fig 2).



Linear regression fits:

$$\Lambda_f(\vec{x}) = \Lambda_c(\vec{x}) + \Delta \Lambda_{fc}(\vec{x})$$

$\Lambda_f(\vec{x})$ FSLE at finer resolution
 $\Lambda_c(\vec{x})$ FSLE at lower resolution
 $\Delta \Lambda_{fc}(\vec{x})$ Contribution not account by the lower resolution

$$\Lambda_{1/8^\circ} = 1.08 \Lambda_{1/4^\circ} + 0.05$$

$$\Lambda_{1/4^\circ} = 0.99 \Lambda_{1/2^\circ} + 0.04$$

Even decreasing the spatial resolution of the velocity data, the structures remain.

This behaviour suggest a relation of scale between FSLE at different velocity resolutions. Linear regression analysis shows a correspondence one to one between values of FSLEs at different velocity resolutions.

Robustness of FSLEs

Error in the data

To study the robustness of the FSLEs, we compute the relative error of FSLE introducing a velocity data perturbed with a small random error, respect to FSLE without perturbation.

$$V \rightarrow V'(\text{perturbed}) = V + \text{error} = V + \varepsilon \cdot \eta \cdot V = V(1 + \varepsilon \cdot \eta)$$

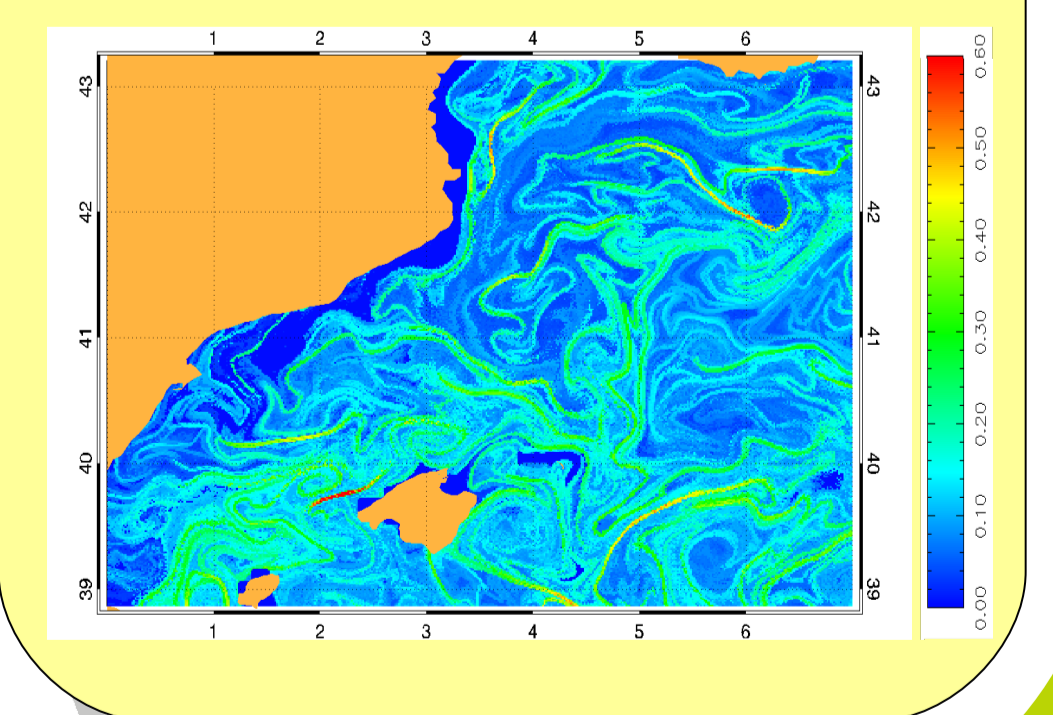
$\varepsilon = \frac{\text{error}}{V}$ is the % of error.
 η is a Gaussian distributed random number: mean = 0, variance = 1.
 V is the velocity data. V' is the perturbed velocity data

We computed the relative error of FSLEs by the formula:

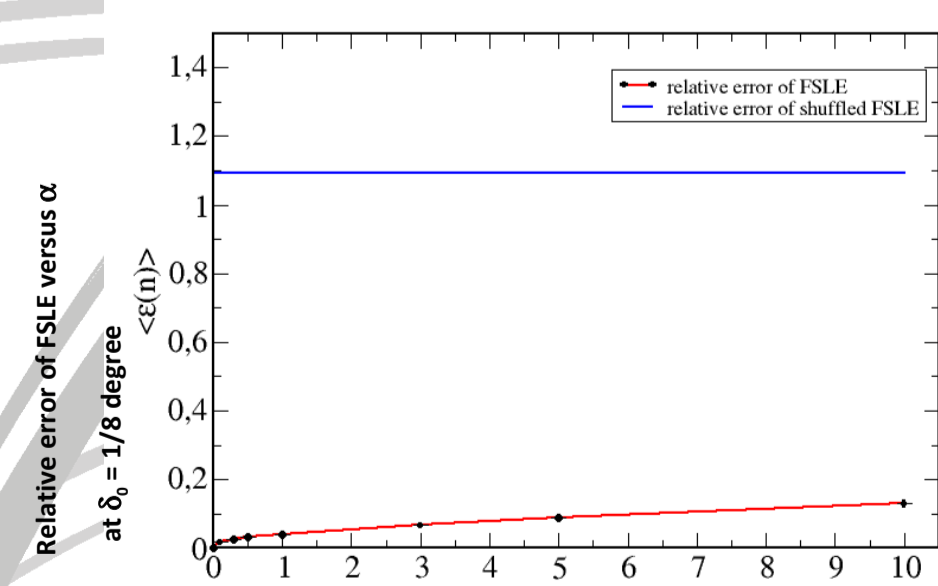
$$\epsilon(n) = \sqrt{\frac{1}{N} \sum_{i=1}^N \frac{|\Lambda_i^*(n) - \Lambda_i(n)|^2}{|\Lambda_i(n)|^2}}$$

Λ_i are the values of FSLEs without error in the velocity data.
 Λ_i^* are the values of FSLEs with error in the velocity data.
 N is the total number of points in the FSLEs field.

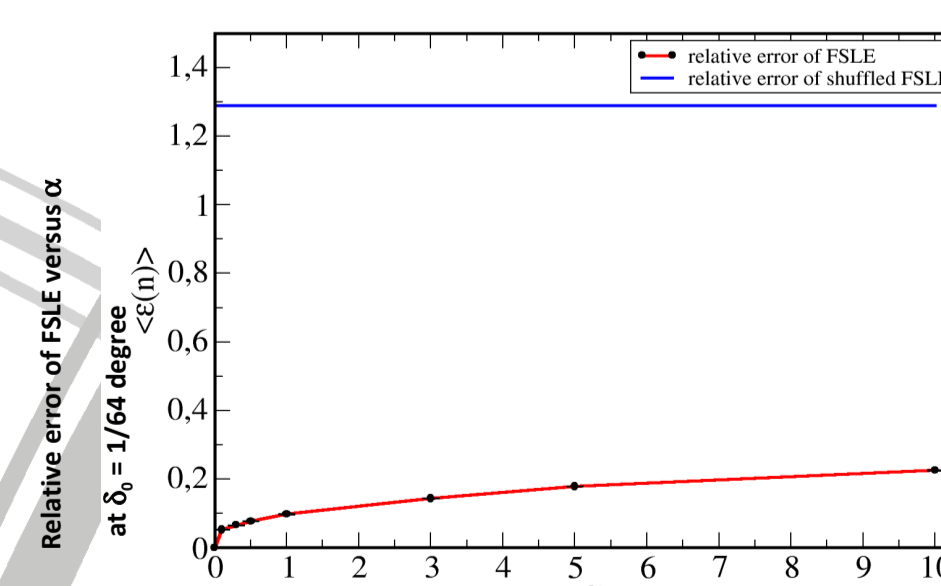
FSLEs at $\delta_0 = 1/64$ degree with 20% of perturbation in the velocity data



The mesoscales structures are maintained



Even with a perturbation 10 times the velocity data, the error remains smaller than 0.2



Noise in the particle's trajectories

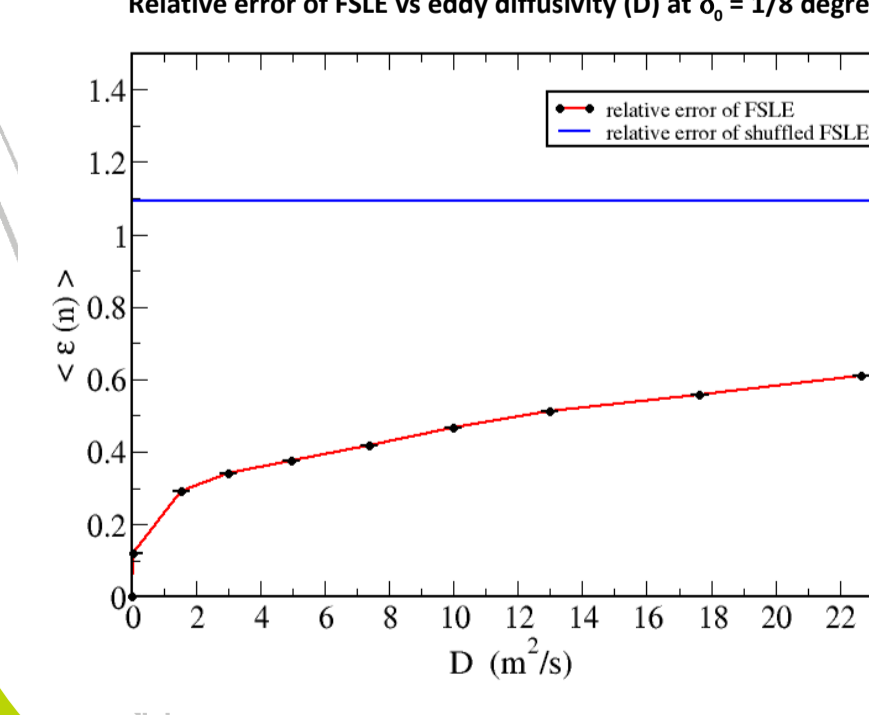
We have computed the FSLEs introducing white noise in the particles trajectories. We have solved the following equations:

$$\frac{d\phi}{dt} = \frac{u(\phi, \lambda, t)}{R \cos(\lambda)} + \frac{\sqrt{2D}}{\cos(\lambda)} \xi_1(t)$$

$$\frac{d\lambda}{dt} = \frac{v(\phi, \lambda, t)}{R \cos(\lambda)} + \sqrt{2D} \xi_2(t)$$

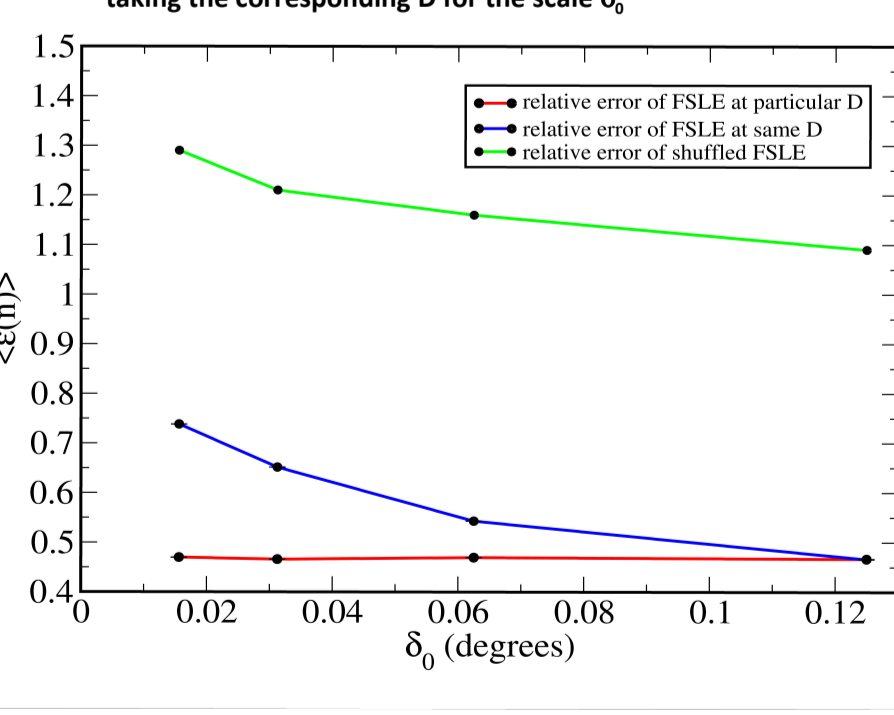
u, v are the components of the velocity.
 ϕ is the longitudinal coordinate.
 λ is the latitudinal coordinate.
 D is the eddy diffusivity computed by the Okubo's formula (Okubo, 1971)
 $\xi(t)$ is white noise $\langle \xi(t) \xi(t') \rangle = \delta(t-t') \quad \langle \xi(t) \rangle = 0$

Relative error of FSLE vs eddy diffusivity (D) at $\delta_0 = 1/8$ degree

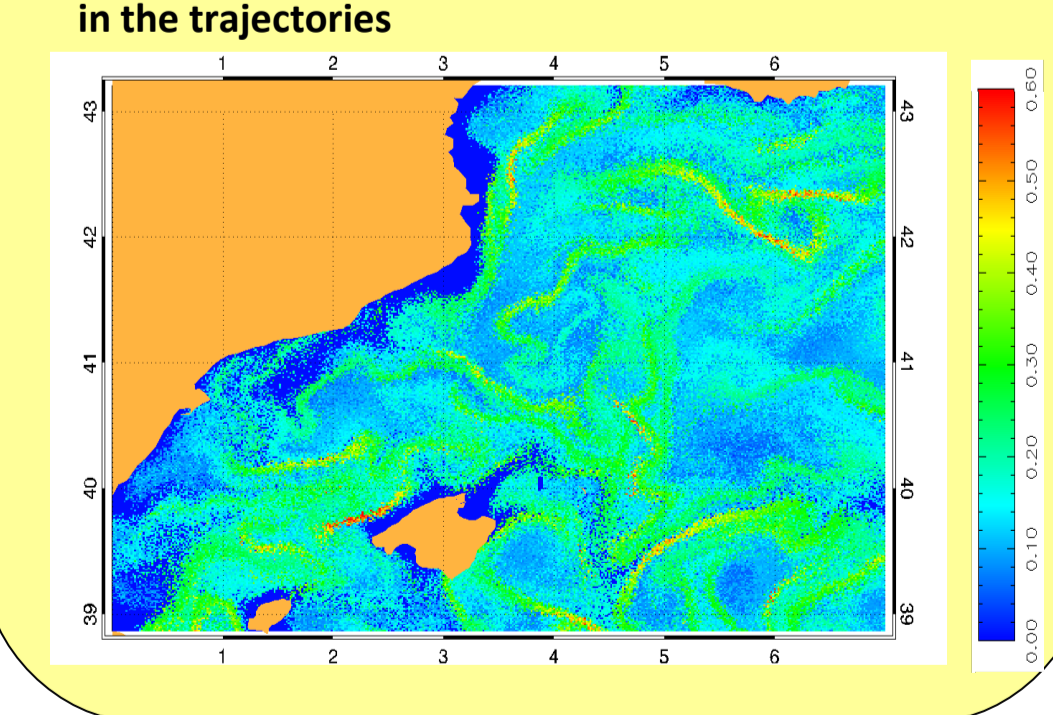


Even with 10m²/s of diffusion, the relative error remains smaller y than 0.5

Relative error of FSLE vs delta_0. One plot using constant D and another taking the corresponding D for the scale delta_0



FSLEs $\delta_0 = 1/64$ degree with the diffusion D=0.9m²/s in the trajectories



The mesoscales structures are maintained with eddy diffusivity

Conclusions

- Increasing the spatial resolution of FSLEs we improve the identification of surface mesoscale structures.
- The surface mesoscales structures in the ocean remain when the spatial resolutions of the velocity decreases
- All the dependence of FSLEs on scale parameters reveal a multifractal structure. What is diagnosed at the coarser scales is still valid when scale is refined.
- The FSLEs are rather robust. The relative error, even for an error of 10% of the velocity data, is smaller than 20 %.
- Mesoscale structures are maintained when the eddy diffusion is included.